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Isotone projection cones in Euclidean spaces.

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0. Introduction.

The metric projections on closed convex sets in Hilbert or Banach spaces were deeply investigated (see for instance the monograph [18] and the papers [4-6], [13-16]).

A special case is the metric projection on a closed convex cone in a Hilbert space.

Although this subject, much studied by Zarantonello in [18], it seems that the relation between the projection operator and the ordering defined by cone was firstly considered in our paper [8].

The cited paper as well as [9], [10],[11] and [12] concern on various characterization of a cone K in a Hilbert space having the property that the metric projection P_K is isotone with respect to the order defined by K (called in this case isotone projection cone).

Besides its theoretical importance this property has interesting applications to the study and the solvability of the Complementarity Problem (important in Optimization, Mechanics, Game Theory,etc.), [9-15].

The aim of this paper is to place our investigations on isotone projection cones in Euclidean spaces in the recent literature which investigates some related problems.

More precisely, we intend to exploit from this point of view some recent results of Barker, Laidacker and Poole [1], to complete the existent characterzations of isotone projection cones with new ones, and finally, to simplify some earlier proofs and to present them in a concise and independent exposition.

1. Preliminaries and the main result.

For the following basic facts about cones we refer the reader to the book [17].

A subset K in the Euclidean space $\mathbb{R}^n$ is a <u>cone</u> if (i) $K+K\subset K$, (ii) $\lambda K\subset K$ whenever $\lambda\in\mathbb{R}_+$ and (iii) $K\cap(-K)=\{0\}$.

A cone is a convex set. We say that K is generating if $\mathbb{R}^n=K-K$. A cone in $\mathbb{R}^n$ is generating if and only if its interior is nonempty. The set

$$K^o=\{\, x\in\mathbb{R}^n \mid \langle x,y\rangle \leq 0,\ \forall y\in K \,\}$$

(where $\langle .,.\rangle$ is the inner product) is called the polar of K. If K is generating, then K^o is a closed cone. If K is closed then $K=(K^o)^o$.

If we put $x\leq y$ whenever $y-x\in K$, then we obtain an order relation (that is a reflexive, transitive and antisymmetric relation) compatible with the vector structure of $\mathbb{R}^n$. We say in this case that $(\mathbb{R}^n,K)$ is an ordered vector space and K is its positive cone. The order defined by K is called the order induced by K.

An upper bound of a set $A\subset\mathbb{R}^n$ is an element $b\in\mathbb{R}^n$ such that $a\leq b$ for every $a\in A$.

If there exists a least upper bound for A, it will be called the supremum of A and will be denoted by supA. Lower bounds and infima can be defined similarly.

If for any two elements $x,y\in\mathbb{R}^n$ it exists $\sup\{x,y\}$ (which will be denoted by $x\vee y$), then the ordered vector space is called a vector lattice and its positive cone K is said to be latticial (or minihedral).

We say that a subset F of the cone K is a <u>face</u> if it is a cone and if it satisfies the condition : from $x\in F$, $y\in K$ and $y\leq x$ it follows that $y\in F$.

A closed half-space of $\mathbb{R}^n$ having boundary point 0 is a subset of $\mathbb{R}^n$ of the form $\{x\in\mathbb{R}^n \mid \langle x,p\rangle\leq 0\}$ where $p\in\mathbb{R}^n$, $p\neq 0$.

A polyhedral cone in $\mathbb{R}^n$ is the intersection of finitely many closed half-spaces of $\mathbb{R}^n$ having boundary point 0.

A closed cone $K\subset\mathbb{R}^n$ is a polyhedral cone if and only if K is a finitely generated cone, that is there exists a finite subset $\{a_1, a_2, \ldots, a_k\}$ of $\mathbb{R}^n$, called a set of generators for K such that,

$$K = \{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_k a_k \mid \lambda_1 \lambda_2, \ldots\ldots, \lambda_k \geq 0\}.$$

A closed generating cone $K\subset\mathbb{R}^n$ is polyhedral if it has a finite number of proper faces having codimension one in $\mathbb{R}^n$ and every proper face of K is contained in some such face.

We shall use this last characterization for polyhedral cones.

If C is a closed conves set in $\mathbb{R}^n$, then for each $x\in\mathbb{R}^n$ there exists a unique point in C denoted by $P_C(x)$ such that $\|x-P_C(x)\| \leq \|x-y\|, \forall y\in C$. The operator P_C is called the projection (or metric projection) on C.[17].

The cone $K\subset\mathbb{R}^n$ is called correct if for every its face F it holds $P_{spF}(K)\subset F$, where spF denotes the linear span of the set F. Correct cones are called by Borwein and Wolkowicz projectionally exposed cones [2] and by Barker, Laidacker and Poole orthogonally projectionally exposed cones [1].

We have independently introduced this notion and called it correct by some analogy with the notion of perfect cones in which occur the additional condition $K=K^*$, where $K^*=-K^o$ (see [3].[7]).

We maintain this term here to be in keeping with our terminology in [9],[10],[11].

The closed cone $K\subset\mathbb{R}^n$ is called isotone projection cone if from $y-x\in K$ it follows that $P_K(y)-P_K(x)\in K$, for every $x,y\in\mathbb{R}^n$.

By using the order relation induced by K, this condition can be written in the form: $x \leq y \Rightarrow P_K(x) \leq P_K(y)$.

We are now ready to state our main result.

Theorem

Let K be a closed generating cone in $\mathbb{R}^n$.

Then the following assertions are equivalent:

(i): K is an isotone projection cone,

(ii): K is correct and latticial,

(iii): K is polyhedral and correct,

(iv): there exists a set of vectors $\{u_i \mid i \in I\}$ with the property that $\langle u_i, u_j \rangle \leq 0$, $\forall\, i,j \in I$, $i \neq j$ and such that $K = (\{u_i \mid i \in I\})^o$,

(v): K is latticial and $P_K(x) \leq x^+$ for every x in $\mathbb{R}^n$, where $x^+ = x \vee 0$. ■

The equivalence (i)$\Leftrightarrow$(iv) was proved in [8]. The equivalence (ii)$\Leftrightarrow$(iv) was independently established in [1] and [9] while (ii)$\Leftrightarrow$(iii) was established in [1].

In [9] was proved (i)$\Rightarrow$(ii) for a general Hilbert space.

We shall give in the sequel a complete proof of this theorem which we shall make as selfcontained as possible. The only facts we shall use regardless the ones in this section is the theorem of Youdine on latticial cones and some properties of the projection operator including Moreau's decomposition theorem with respect to mutually polar cones. The most part of the proofs are new.

The proof of (i)$\Rightarrow$(ii) is a simplified version of the similar result for Hilbert spaces proved in [9]. The most difficult steps are those which imply the operator P_K.

Hence one of the main reaches of the paper is the proof of (ii)$\Rightarrow$(i) presented in section 4 and which is much simpler as that of (iv)$\Rightarrow$(i) in [8].

Condition (v) constitutes a new characterization of the isotone projection cones in $\mathbb{R}^n$.

2. Preliminary results.

The following result of Youdine [19] will often used in our proofs.

Theorem [Youdine]

The cone $K \subset \mathbb{R}^n$ is latticial if and only if there exist n vectors linearly independent in $\mathbb{R}^n$, $u_1, u_2, \ldots, u_n$ such that

(2.1): $K=\{x \in \mathbb{R}^n \mid \langle x, u_i \rangle \leq 0, \quad i=1,2,\ldots,n\}$. ■

That is, K is latticial if and only if it is of form $K=(\{u_i \mid i=1,2,\ldots,n\})^o$, where $u_1, u_2, \ldots, u_n$ are linearly independent vectors.

Several technical corollaries follow from this result.

Let $A \subset \mathbb{R}^n$. The affine hull aff(A) of A is the smallest affine subset of $\mathbb{R}^n$ containing A. The *relative interior*, *rint(A)* of A is defined as the interior of A regarded as a subset of aff(A) (with the relative topology).

We remark that if $A \subset \mathbb{R}^n$ is nonempty and convex then rint(A) is nonempty and dim(rint(A))=dim(A).

Lemma 1

If K is of form (2.1) then for every subset $\{i_1,\ldots,i_k\} \subset \{1,2,\ldots,n\}$ (according to Youdine's Theorem) the set $F_{i_1,\ldots,i_k}=\{x \in K \mid \langle x, u_{i_j} \rangle = 0,\ j=1,\ldots,k\}$ is a face of K. If $i_h \neq i_l$ whenever $h \neq l$, then both $F_{i_1,\ldots,i_k}$ and $\mathrm{rint}(F_{i_1,\ldots,i_k})=\{x \in F_{i_1,\ldots,i_k} \mid \langle x, u_j \rangle < 0,\ j \in \{1,\ldots,n\} \setminus \{i_1,\ldots,i_k\}\}$ are for $k<n$ nonempty sets in $\mathbb{R}^n$ of codimension n-k.

Every face of K is of form $F_{i_1,\ldots,i_k}$ with some set $\{i_1,\ldots,i_k\} \subset \{1,2,\ldots,n\}$

Proof

The assertion that $F_{i_1,\ldots,i_k}$ and rint $(F_{i_1,\ldots,i_k})$ are nonempty and of codimension $n-k$ if $k<n$ is a routine exercise of linear algebra.

Suppose that $x \in F_{i_1,\ldots,i_k}$, $y \in K$ and $y \leqq x$.

Then $\langle x-y, u_{i_j} \rangle = -\langle y, u_{i_j} \rangle \leqq 0$, $j=1,2,\ldots,k$ since $x-y \in K$. Hence $\langle y, u_{i_j} \rangle = 0$; $j=1,2,\ldots,k$ because $y \in K$ and we know that $\langle y, u_j \rangle \leqq 0$; $j=1,2,\ldots,n$. Thus $y \in F_{i_1,\ldots,i_k}$ and this set is a face of K.

Suppose that F is an arbitrary proper face of K.

If for some $x \in F$ there would hold $\langle x, u_j \rangle < 0$, $j=1,\ldots,n$ then for arbitrary $y \in K$ it would exist some positive scalar t such taht $\langle x-ty, u_j \rangle \leqq 0$, $j=1,2,\ldots n$.

But then $x-ty \in K$, that is $ty \leqslant x$ and $ty \in K$ whence $ty \in F$ by the definition of F. Now, since F is a cone, it follows that $y \in F$ and y being arbitrary in K we obtain that $K \subset F$ contradicting the hypothesis that F is a proper face of K. Hence there exists some minimal set $\{i_1,\ldots,i_k\} \subset \{1,2,\ldots,n\}$, $k \geqq 1$ so that $\langle x, u_{i_j} \rangle = 0$, $j=1,2,\ldots,k$ for every $x \in F$. By the first part of the proof we have $F = F_{i_1,\ldots,i_k}$. ■

Lemma 2

If K is a latticial cone given by (2.1), then for $y,z \in \mathbb{R}^n$ the supremum $y \vee z$ is the solution of the following system in x:

$$(2.2): \begin{cases} \langle x, u_i \rangle = \min\{\langle y, u_i \rangle, \langle z, u_i \rangle\} \\ i=1,2,\ldots,n \end{cases}$$

In particular, if $v \in R^n$ and $\langle v, u_j \rangle = 0$ for some $j \in \{1,2,\ldots,n\}$ then $\langle v^+, u_j \rangle = 0$ where $v^+ = v \vee 0$.

Proof

Since $u_1, u_2, \ldots, u_n$ are linearly independent vectors

the system (2.2) has a unique solution x_o. Let us see that $x_o=z\vee y$. From the definition of x_o we have,
$\langle x_o-y,u_i\rangle=\langle x_o,u_i\rangle-\langle y,u_i\rangle=\min\{\langle y,u_i\rangle,\langle z,u_i\rangle\}-\langle y,u_i\rangle\leqq 0$,
$i=1,2,..,n$.

Hence $x_o-y\in K$, that is $y\leqq x_o$. Similarly we deduce that $z\leqq x_o$.

Suppose now that for some x in $\mathbb{R}^n$, $y\leqq x$ and $z\leqq x$. Then by the definition of K , $\langle x-y,u_i\rangle\leqq 0$ and $\langle x-z,u_i\rangle\leqq 0, i=1,2,..,n$ which imply $\langle x,u_i\rangle\leqq\min\{\langle y,u_i\rangle,\langle z,u_i\rangle\}=\langle x_o,u_i\rangle$, $i=1,2,...,n$.

Using again the definition of K we conclude that $x-x_o\in K$, i.e., $x_o\leqq x$. Thus we have $x_o=y\vee z$.

If for some $v\in\mathbb{R}^n$ and some $j\in\{1,2,..,n\}$ one has $\langle v,u_j\rangle=0$ we get $\langle v^+,u_j\rangle=\min\{\langle v,u_j\rangle,0\}=0$, since $v^+=v\vee 0$ is the solution of the system:

$$\begin{cases}\langle x,u_i\rangle=\min\{\langle v,u_i\rangle,\langle 0,u_i\rangle\}\\ i=1,2,..,n.\end{cases}$$

$\blacksquare$

Lemma 3

Suppose that K is a latiicial cone given by (2.1). Then there exist the linearly independent vectors $e_1,e_2,...,e_n$ in $\mathbb{R}^n$ with $\langle e_i,u_j\rangle=o$ if $i\neq j$ and $\langle e_i,u_i\rangle<0$, $i,j=1,2,..,n$, such that

(2.3): $K=\text{cone}\{e_1,...,e_n\}\left(=\left\{\sum_{i=1}^{n}\lambda_i e_i \,\middle|\, \lambda\geqq 0,\ i=1,2,..,n\right\}\right)$.

In particular, $K^o=\text{cone}\{u_i,u_2,..,u_n\}$ and every latticial cone has a representation of form (2.3) with some linearly independent vectors $e_1,e_2,..,e_n$.

Since $e_1,e_2,..e_n$ are linearly independent then every $y\in\mathbb{R}^n$ can be uniquely represented in the form, $y=c_1e_1+..+c_ne_n$; $c_1,c_1,...,c_n\in\mathbb{R}$.

If for another vector $z\in\mathbb{R}^n$ we have $z=d_1e_1+..+d_ne_n$; $d_1,d_2,...,d_n\in\mathbb{R}$ then $z\leqq y$ is equivalent with $d_i\leqslant c_i$, $i=1,2,..,n$.

Proof

Since $u_1,u_2,..,u_n$ are linearly independent, then $u_1,...,u_{j-1},u_{j+1},...,u_n$ span a hyperplan in $\mathbb{R}^n$. If e is a normal vector to this hyperplane then, since $u_j \notin sp\{u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_n\}$ it follows that $\langle e,u_j\rangle \neq 0$. Choose a normal e_j to this hyperplane so that $\langle e_j,u_j\rangle<0$. Obviously $\langle e_j,u_i\rangle=0$ if $i\neq j$ and hence $e_j\in K$.

Take $j=1,2,\ldots,n$ in order to obtain $e_1,e_2,\ldots,e_n$. By the biorthogonality of the systems $e_1, e_2,\ldots,e_n$ and $u_1,u_2,\ldots,u_n$, it can be easily deduced that $e_1, e_2, \ldots,e_n$ are linearly independent. We have obviously $\text{cone}\{e_1,e_2,\ldots,e_n\}\subset K$ To show the converse inclusion take $x=c_1e_1+\ldots+c_ne_n$ with $c_j<0$. By scalar multiplication with u_j it follows that $\langle x,u_j\rangle=c_j\langle e_j,u_j\rangle>0$ and hence $x\notin K$.

The last representation of the lemma follows directly from the representation (2.3) of K. ■

The next result is true for a well based closed convex cone in a reflexive Banach space but because in this paper K is in $\mathbb{R}^n$ we give this result with an elementary proof.

Lemma 4

If K is a closed cone in $\mathbb{R}^n$ then every K-increasing, K-order bounded sequence in $\mathbb{R}^n$ converges to its K-supremum.

Proof

Since K is a closed cone, we have $K=(K^o)^o$.

Hence K^o must be generating, since if K^o would be contained in some subspace of codimension one, then the orthogonal complement of this last space would be in $(K^o)^o=K$, contradicting the definition of K.

Let $u_1,u_2,\ldots,u_n$ be linearly independent vectors in K^o. Then $\text{cone}\{u_1,\ldots,u_n\}\subset K^o$ and hence $K\subset K_o$, where $K_o=(\{u_1,\ldots,u_n\})^o$..

By Lemma 3, K_o can be represented in the form, $K_o=\text{cone}\{e_1, e_2,\ldots,e_n\}$, $e_1,e_2,\ldots,e_n$ being linearly independent vectors in $\mathbb{R}^n$.

Consider now the sequence $\{x_m\}_{m\in N}$ in $\mathbb{R}^n$ such that,

$$x_1 \leq_K x_2 \leq_K \dots \leq_K x_m \leq_K \dots \leq_K u$$

for some $u \in \mathbb{R}^n$.

Since $K \subset K_o$ we have also

(2.4): $x_1 \leq_{K_o} x_2 \leq_{K_o} \dots \leq_{K_o} x_m \leq_{K_o} \dots \leq_{K_o} u$

Let us take the representations

$x_m = c_1^m e_1 + \dots + c_n^m e_n$; $m=1,2,\dots$; $u = u_1 e_1 + \dots + u_n e_n$

where $c_j^m, u_j \in \mathbb{R}$, $j=1,2,\dots,n$

Then according (2.4) and Lemma 3, every sequence of real numbers $\{c_j^m\}_{m\in N}$ $(j=1,2,\dots,n)$ is monotonically increasing and bounded by u_j, hence convergent. Denote

(2.5): $c_j^o = \lim\limits_{m\to\infty} c_j^m$; $(j=1,\dots,n)$

Then $\{x_m\}_{m\in N}$ is convergent to $x_o = c_1^o e_1 + \dots + c_n^o e_n$.

From the relations $x_p - x_q \in K$ for $q \leq p$ and $u - x_p \in K$ for each p, passing to the limit with $p\to\infty$ and taking into account that K is closed, we deduce that $x_q \leq_K x_o$ for each q and $x_o \leq_K u$, which completes the proof of the lemma. ∎

Before passing to some facts concerning correct cones, let us remember some results on projections maps. First of all we have that $P_C(x)$ is the nearest element in the closed convex set $C \subset \mathbb{R}^n$ to $x \in \mathbb{R}^n$, if and only if we have:

(2.6): $\langle x - P_C(x), P_C(x) - y \rangle \geq 0, \forall y \in C$ (see Lemma 1.1 in [18]).

We shall also use the fact that any x and y in $\mathbb{R}^n$ and every closed convex set $C \subset \mathbb{R}^n$ it holds

(2.7): $\|P_C(x) - P_C(y)\| \leq \|x-y\|$,

that is, P_C is nonexpansive and hence also continuous (see [18], formula (1.8)).

The characterization of projections on a cone and its polar is the object of the following result.

<u>Theorem [Moreau]</u>

<u>If K is a closed convex cone in $\mathbb{R}^n$ then the following</u>

assertions are equivalent:

(i): $x=u+v$, $u\in K$, $v\in K^{o}$ and $\langle u,v\rangle=0$

(ii): $u=P_K(x)$, $v=P_{K^o}(x)$.

Lemma 5.

If $K\subset R^n$ is a correct cone and if F is its face, then for every $x\in spF$ one has $P_K(x)=P_F(x)$.

Proof

Assume the contrary, that is, there exists some x in spF such that $P_K(x)\notin F$.

Since P_{spF} is nonexpansive (see (2.7)) we have

(2.8): $\| x-P_{spF}(P_K(x)\| = \|P_{spF}(x)-P_{spF}(P_K(x))\| \leqq \|x-P_K(x)\|$.

Since $P_{spF}(K)\subset F$ by the correctendess of K, we have $P_{spF}(P_K(x))\in F\subset K$.

By the unicity of the nearest element, we have by (2.8) that $P_{spF}(P_K(x))=P_K(x)$, whence $P_K(x)\in (spF)\cap K=F$ wich is impossible and the lemma is proved. ■

Let v be in K^o and consider the set

$$F_v=\{x\in K \mid \langle x,v\rangle=0\}.$$

Then a straightforward verification shows that F_v is a face of K.

Faces of the above kind are called exposed faces [18].

The vector v is said a normal to the face F_v.

Lemma 6

If K is a correct cone in $\mathbb{R}^n$ and if F is an exposed face of K having the codimensione one in $\mathbb{R}^n$, if v is a normal of F, then for any other normal v' to any other exposed face F' of K which is not contained in F one has $\langle v,v'\rangle\leqq 0$.

Proof

Suppose the contrary. So, we suppose that for some such normal v' we have $\langle v,v'\rangle>0$. Let $x\in F'\setminus F$.

Hence $\langle v,x\rangle<0$ (since $v\in K^{o}$) and we can determine a positive scalar t such that $\langle x+tv',v\rangle=0$.

But from Moreau's theorem we have $P_K(x+tv')=x$. Since F is of codimension one, its normal is v and $\langle x+tv',v\rangle=0$, necessarily we have $x+tv'\in spF$ and we have got a contradiction with Lemma 5.■

Proof of principal Theorem.

3. Proof of the implication (i)$\Rightarrow$(ii).

In proving that the isotone projection cone $K\subset\mathbb{R}^n$ is latticial we shall use the following assertion:

a) Let K be a closed and generating cone in $\mathbb{R}^n$ and u,v two wlements of $\mathbb{R}^n$.

If there exist $a\in u+K$, $b\in v+K$ with the properties $a=P_{u+K}(b)$ and $b=P_{v+K}(a)$, then $a=b\in(u+K)\cap(v+K)$.

Indeed, since K is generating the set $(u+K)\cap(v+K)$ is nonempty, that is, there exists some element w such that $u\leqq w$ amd $v\leqq w$. This follows by writing $u=u_1-u_2$, $v=v_1-v_2$, where u_1,u_2, $v_1,v_2,\in K$ and observing that we can consider $w=u_1+v_1$.

We have from the characterization (2.6) of the metric projections that,

(3.1): $\langle a-P_{v+K}(a),P_{v+K}(a)-w\rangle\geqq 0$ and

$$\langle b-P_{u+K}(b),P_{u+K}(b)-w\rangle\geqq 0.$$

Using the conditions in the assertion (a) the second relation becomes,

(3.2): $\langle P_{v+K}(a)-a,a-w\rangle\geqq 0$.

On the other hand we have,

$$\langle P_{v+K}(a)-a,a-w\rangle=\langle P_{v+K}(a)-a,(a-P_{v+K}(a))+(P_{v+K}(a)-w)\rangle=$$
$$=-(\|P_{v+K}(a)-a\|^2+\langle a-P_{v+K}(a),P_{v+K}(a)-w\rangle).$$

whence, taking into account (3.1) and (3.2) it follows

that, $\|P_{v+K}(a)-a\| = \|b-a\| = 0$,
and the assertion (a) is proved.

(b) Let us pass to the proof of the latticiality of K. Consider the arbitrary elements u and v in $\mathbb{R}^n$. We shall show, using the isotone projection property of K, that they admit a least upper bound $u \vee v$ by constructing effectively this element.

We can assume that u and v are not comparable.

Let w be an arbitrary upper bound of the set $\{u,v\}$, i.e. an abitrary element of the set $(u+K)\cap(v+K)$ which is not empty since K is generating by hypothesis.

Let us note next that if P_K is isotone, then for an arbitrary element y in $\mathbb{R}^n$ the operator P_{y+K} is isotone too. This follows from the relation $P_{y+K}(x)=P_K(x-y)+y$ which holds for an arbitrary x in $\mathbb{R}^n$ and which can be directly verified by using (2.6). Hence P_{u+K} and P_{v+K} are both isotone.

Since no one of the convex sets u+K and v+K is contained in the other, using assertion (a) we see that there cannot hold simultaneously the relations $u=P_{u+K}(v)$ and $v=P_{v+K}(u)$.

Suppose that $u \neq P_{u+K}(v) \in u+K$.

Then $u \leq P_{u+K}(v) \leq P_{u+K}(w)=w$, since P_{u+K} is isotone and $w \in u+K$.

Let us consider the operatores $Q=P_{v+K} \circ P_{u+K}$ and $R=P_{u+K} \circ P_{v+K}$. They are isotone since P_{u+K} and P_{v+K} are. Put $v_n=Q^n(v)$, $u_1=P_{u+K}(v)$ and $u_n=R^{n-1}(u_1)$. Then we have the following relations:

$$v \leq v_1 \leq \dots \leq v_n \leq \dots \leq w \text{ and}$$
$$u \leq u_1 \leq \dots \leq u_n \leq \dots \leq w.$$

since $u \leq u_1$, $v \leq v_1$, since P_{u+K}, Q and R are isontone, and since $P_{u+K}(w)=Q(w)=R(w)=w$.

(There hold obviously $P_{v+K} \circ P_{u+K}(v) \in v+K$, hence

$v \leqq P_{v+K} o P_{u+K}(v) = Q(v) = v_1$ and $u_1 = P_{u+K}(v) \leq P_{u+K} o P_{v+K} o P_{u+K}(v)$, that is, $u_1 \leqq R(u_1) = u_2$ etc)

We have further

(3.3): $v_n = Q^n(v) = (P_{v+K} o P_{u+K})^n(v) = P_{v+K} o (P_{u+K} o P_{v+K})^{n-1} o P_{u+K}(v) =$
$= P_{v+K} o R^{n-1}(u_1) = P_{v+K}(u_n)$ and

(3.4): $u_{n+1} = R(u_n) = P_{u+K} o P_{v+K}(u_n) = P_{u+K}(v_n)$.

Since the sequences $\{u_n\}$ and $\{v_n\}$ are increasing and upper bounded by w, we have (using Lemma 4) the following relations:

(3.5): $u_o = \lim_{n\to\infty} u_n$ and $v_o = \lim_{n\to\infty} v_n$

as well as

(3.6): $u \leqq u_o \leqq w$ and $v \leqq v_o \leqq w$.

From the continuity of the metric projections (see relation (2.7)) the formulas (3.3), (3.4) and (3.5) yield

$$v_o = P_{v+K}(u_o) \text{ and } u_o = P_{u+K}(v_o).$$

Using assertion (a) again we deduce that

$$u_o = v_o \in (u+K) \cap (v+K).$$

Since the upper bound w was arbitrary, from the relation (3.6) we obtain that indeed $u_o = v_o = u \vee v$ and the laticiality of K is proved.

To prove the correctness of K we begin by proving the following assertion.

c) <u>For every face F of the generating isontone projection cone K in $\mathbb{R}^n$ the subspace spF projects by P_K onto F and F is an isotone projection cone in the space spF.</u>

Consider $z \in spF$. Then $z = x - y$ with $x, y \in F \subset K$ whence $z \leqq x$.

Since P_K is isotone,one follows $0 \leqq P_K(z) \leqq P_K(x) = x \in F$. Hence $P_K(z) \in F$.

This relation shows that $P_F(z) = P_K(z)$ and implicitely that $P_F|_{spF}$ is isotone projection in spF and (c) is proved.

d) <u>We pass to the proof of correctness of the isotone projection cone K</u> by assuming the contrary, that is, we suppose that there exists a face F of K and an element k of K such that $z=P_{spF}(k)\notin K$.

Put $z_o=P_K(z)$. Since $z\in spF$, it follows from the assertion (c) that $z_o\in F$.

We shall show first that there can be determined a real number $t\in(0,1)$ such that the element w given by

(3.7): $w=tk+(1-t)z_o$

satisfies the relation

(3.8): $\langle z-w,k-z_o\rangle=0$

Indeed, we have,

$\langle z-tk-(1-t)z_o,k-z_o\rangle=\langle z-k+(1-t)(k-z_o),k-z_o\rangle=\langle z-k,k-z_o\rangle+$
$+(1-t)\|z_o-k\|^2=\langle z-k,k-z+z-z_o\rangle+(1-t)\|z_o-k\|^2=-\|z-k\|^2+(1-t)\|z_o-k\|^2$,

since $\langle z-k,z-z_o\rangle=0$ ($z-z_o\in spF$ and $z-k$ is orthogonal to spF).

Since $\|z-k\|<\|z_o-k\|$ by the definition of z and z_o, then putting

$$1-t=\frac{\|z-k\|^2}{\|z_o-k\|^2}<1,$$

we have (3.8) for w determined by (3.7).

Using the characterization (2.6) of the metric projections, we have

(3.9): $\langle z-z_o,z_o-k\rangle=\langle z-P_K(z),P_K(z)-k\rangle\geq 0$.

From the definition of w it follows on the other hand that

$$\langle z-z_o,z_o-k\rangle=\langle z-w+w-z_o,z_o-k\rangle=\langle w-z_o,z_o-k\rangle=\langle tk+(1-t)z_o-$$
$$-z_o,z_o-k\rangle=t\langle k-z_o,z_o-k\rangle<0.$$

This relation contradicts (3.9) and shows that our hypothesis that K is not correct, is false.

4. Proof of the implications (ii)$\Rightarrow$(iii)$\Rightarrow$(i).

Obviously, the implicaiton (ii)$\Rightarrow$(iii) is a consequence of Youdine's Theorem.

We shall prove (iii)$\Rightarrow$(i) by induction with respect to the dimension of the space.

For dimension one we have nothing to prove. We shall do the induction step for the sake of simplicity as follows.

Suppose that the implication

(4.1): $z \leq_F y \Rightarrow P_F(z) \leq P_F(y)$, $y, z \in spF$

holds for every face F of codimension one of K in $\mathbb{R}^n$ and prove it for F replaced by K.

(Observe that the hypothesis in (ii) hold for faces too since correctness and polyhedrality are both hereditary for faces)

Since K is polyhedral, there exists a finite set of unit vectors $\{u_i\}_{i=1}^m$, the normals to the maximal proper faces of K, such that $K=(\{u_i\}_{i=1}^m)^0$ and $F_i = K \cap \ker u_i$ is a face of codimension one for each i.

(a) Consider the elements y,z in $\mathbb{R}^n$ such that $z \leq y$. Let u_i be the normal to the face F of codimension one of K.

Then $\ker u_i = spF$ and let us denote $p = P_{spF}$. Since u_i is a unit vector we have,

$$p(y) = y - \langle y, u_i \rangle u_i \text{ and } p(z) = z - \langle z, u_i \rangle u_i .$$

Let us see that

(4.2): $p(z) \leq p(y)$

We have obviously $\langle p(y)-p(z), u_i \rangle = 0$.

Using the above expressions for p(y) and p(z) we have for $j \neq i$:

$$\langle p(y)-p(z), u_j \rangle = \langle y-z-\langle y-z, u_i \rangle u_j, u_j \rangle = \langle y-z, u_j \rangle - \langle y-z, u_i \rangle \langle u_i, u_j \rangle .$$

The first term in the last sum and the factor $\langle y-z, u_i \rangle$ in the second term are both non-positive since $y-z \in K$.

The correctness of K implies via Lemma 6 that $\langle u_i, u_j \rangle \leq 0$, whence the second term in the last sum of the above formula is also nonpositive.

According to the definition of K as $(\{u_j\}_{j=1}^m)^0$ the above conclusions prove (4.2), which can be written also in the form,

(4.3): $p(z) \leq_F p(y)$

since $p(z), p(y) \in spF$ and $F = spF \cap K$.

(b) Let us show next that, if conditions (iii) are satisfied then for every $x \in \mathbb{R}^n$ such that $\langle x,u_i\rangle \geqq 0$ for some i, one has

(4.4): $P_K(x)=P_F(p(x))$

with $F=(\ker u_i)\cap K$ and $p=P_{spF}$

Indeed, since K is correct, Lemma 5 implies,

$$P_F(p(x))=P_K(p(x)).$$

Hence, for an arbitrary $w \in R^n$ we have,

$\langle x-P_F(p(x)),P_F(p(x))-w\rangle=\langle x-p(x),P_K(p(x))-w\rangle+\langle p(x)-P_K(p(x)),$ $,P_K(p(x))-w\rangle$.

Let now w be an arbitrary element of K.

Then the second term in the last sum is non-negative according to the characterization (2.6) of the projection maps.

If $\langle x,u_i\rangle=0$, then $x=p(x)$ and the first term in the above sum is zero.

If $\langle x,u_i\rangle > 0$, then $x-p(x)$ is orthogonal to $spF=\ker u_i$.

Hence it is parallel with u_i and has its direction since $\langle x-p(x),u_i\rangle=\langle x,u_i\rangle > 0$ by hypothesis.

Whence $x-p(x)\in K^o$ and since $P_K(p(x))\in F\subset \ker u_i$, it follows that, $\langle x-p(x),P_K(p(x))-w\rangle=-\langle x-p(x),w\rangle\geqq 0$, for every $w\in K$.

In conclusion we have,

$$\langle x-P_F(p(x)),P_F(p(x))-w\rangle\geqq 0,\ \forall w\in K,$$

whence using again the characterization (2.6) of the projection, we conclude that the relation (4.4) holds.

(c) Let us consider again that $z\leqq y$ and suppose that $y\notin \operatorname{Int}K$. This condition is equivalent with the existence of some subscript i such that $\langle y,u_i\rangle\geqq 0$.

Since $y-z\in K$ we have $\langle y-z,u_i\rangle\leqq 0$ whence we have also $\langle z,u_i\rangle\geqq 0$.

If $F=(\ker u_i)\cap K$ and $p=P_{spF}$, then we have by the result proved in (a) (see relation (4.3)), that

(4.5): $p(z)\leqq_F p(y)$.

Use now the fact that both $\langle y,u_i\rangle$ and $\langle z,u_i\rangle$ are non-negative and the result proved in (b), formula (4.4) to conclude that

(4.6): $P_K(y)=P_F(p(y))$ and $P_K(z)=P_F(p(z))$.

Since $p(y)$ and $p(z)$ are in spF we have according to the induction hypothesis (4.1) via (4.5) that

$$P_F(p(z))\leqq P_F(p(y)).$$

Using now (4.6) we conclude that $P_K(z)\leqq_F P_K(y)$, whence $P_K(z)\leqq P_K(y)$.

(Particularly in this case it follows that both y and z project on the same proper face F)

(d) Suppose now that $y\in \text{Int}K$. If $z\in K$, then we have nothing to prove.

If $z\notin K$, then the line segment $\{y_t \mid t\in(0,1)\}$ with $y_t=tz+(1-t)y$ pierces the boundary of K at some point y_{t_o}, that is, we have $\langle y_{t_o},u_i\rangle=0$ for some subscript i and $\langle y_{t_o},u_j\rangle\leqq 0$ for $j\neq i$.

But $z\leqq y_{t_o}\leqq y$. From the result established by induction in the point (c) we have,

$$P_K(z)\leqq P_K(y_{t_o})=y_{t_o}.$$

Since $y_{t_o}\leqq y=P_K(y)$ the last two relations show that $P_K(z)\leqq P_K(y)$ also in this case.

Thus the proof of (iii)$\Rightarrow$(i) is complete.

Remark.

Putting together the results of the sections 3 and 4 we conclude that the assertions (i), (ii) and (iii) of our theorem are equivalent.

Hence we got in turn a new proof of the equivalence of (ii) and (iii) which was given in [1].

5. Proof of the implications (iii)$\Rightarrow$(iv)$\Rightarrow$(ii)

Suppose that (iii) holds. If we consider the normals u_i, $i=1,\dots,m$ to the maximal faces of the polyhedral cone K, then $K=(\{u_i\}_{i=1}^m)^o$ and using the correctness of K, we have by Lemma 6 that $\langle u_i,u_j\rangle\leqslant 0$, for $i\neq j$. Thus the implication (iii)$\Rightarrow$(iv) was established.

Suppose now that we have (iv) fulfilled.

We shall show first that the vectors u_i, $i \in I$ satisfying this condition are linearly independent.

Since K is a generating closed cone, in this set exist at least n linearly independent vectors (see the first part of the proof of Lemma 4)

Suppose that $u_1, u_2, \ldots, u_n$ are linearly independent vectors in this set and let us verify the assertion:

(a) Let $u_1, u_2, \ldots, u_n$ be linearly independent elements in $\mathbb{R}^n$ satisfying the conditions $\langle u_i, u_j \rangle \leq 0$, $i \neq j$, $i,j=1,2,\ldots,n$. If for some $v \in R^n$ one has $\langle v, u_i \rangle \leq 0$, $i=1,2,\ldots,n$. then

(5.1): $$v = c_1 u_1 + \ldots + c_n u_n \quad \text{with} \quad c_i \leq 0,\ i=1,2,\ldots,n.$$

We shall use in the proof a process, which yields an orthogonal basis $w_1, w_2, \ldots, w_n$, every w_i being a linear combination of elements u_j with non-negative coefficients.

Put $w_1 = u_1$ and suppose that $w_1, \ldots, w_{k-1}$ were determined, $\langle w_i, w_j \rangle = 0$, $i,j \leq k-1, i \neq j$ and each of them is a linear combination with non-negative coefficients of the vectors u_j with $j \leq k-1$.

Let be $w_k = t_1 w_1 + \ldots + t_{k-1} w_{k-1} + u_k$, where the real coefficients $t_1, \ldots, t_{k-1}$ will be determined.

According to the conditions on $w_1, \ldots, w_{k-1}$, we have $\langle w_j, u_k \rangle \leq 0$, $j \leq k-1$.

Hence we can determine $t_1, \ldots, t_{k-1}$ such that $t_j \geq 0$, $j \leq k-1$, from the relation

$$0 = \langle w_k, w_j \rangle = t_j \langle w_j, w_j \rangle + \langle u_k, w_j \rangle$$

This shows that w_k is a linear combination with non-negative coefficients of $u_1, u_2, \ldots, u_k$ and is orthogonal to w_j, $j \leq k-1$.

We have obviously that $w_1, \ldots, w_n$ are linearly independent.

Let us consider the representation,

(5.2): $v=d_1w_1+\ldots\ldots+d_nw_n$, $d_j\in\mathbb{R}$, $j=1,2,\ldots,n$.

Since $\langle v,u_i\rangle\leqq 0$, $i=1,2,\ldots,n$, by hypothesis and since w_k are combinations with non-negative coefficients of $u_1,\ldots,u_k$, $k=1,2,\ldots,n$, we have $\langle v,w_k\rangle\leqq 0$, $k=1,\ldots,n$.

Assume that we have in (5.2) $d_k>0$ for some k. Multiplying this relation with w_k we obtain,

$$0\geqq\langle v,w_k\rangle=d_k\langle w_k,w_k\rangle > 0$$

The obtained contradiction shows that $d_k\leqq 0$, $k=1,2,\ldots,n$.

Let we put in (5.2) the representations of w_k, $k=1,2,\ldots,n$ as linear combinations of u_j, $j=1,2,\ldots,n$. Since the coefficients in these representations are non-negative and d_k, $k=1,2,\ldots,n$ are non-positive, we get a representation of v as a linear combination of $u_1,\ldots,u_n$ with non-positive coefficients. But the resulting coefficients must be quite the coefficients $c_1,\ldots,c_n$ in (5.1) and the assertion (a) is proved.

(b) <u>Let $u_1,\ldots,u_n$ be linearly independent vectors in the set $\{u_i|i\in I\}$ considered in assertion (iv) of the theorem.</u>

<u>We shall show that they are the only nonzero vector of this set</u>.

Indeed, if v would be another nonzero vector in $\{u_i|i\in I\}$, then by the condition in (iv) and by assertion (a) it would follow the representation (5.1) with $c_i\leqq 0$.

But then $-v\in\text{cone}\{u_1,u_2,\ldots u_n\}\subset\text{cone}\{u_i|i\in I\}$, that is, v and -v would be both in $\text{cone}\{u_i|i\in I\}$ and hence K would be contained in the hyperplane perpendicular to v, contradicting the hypothesis on K to be generating. Thus we must have in fact that,

(5.3): $K=\{x\in\mathbb{R}^n|\langle x,u_i\rangle\leqq 0,\ i=1,\ldots,n;\ u_1,\ldots,u_n$ linearly independent$\}$,

relation which together with the theorem of Youdine shows that K is latticial.

(c) <u>To see that K is correct we shall prove first that if</u>

$F=(\ker u_n)\cap K$ then $P_{spF}(K)\subset F$.

From representation (5.3) deduced above and from Lemma 3, there exist the linearly independent vectors $e_1, e_2,\dots,e_n$ such that,

(5.4): $K=\text{cone}\{e_1, e_2,\dots,e_n\}, \langle e_i,u_j\rangle=0$ if $i\neq j$ and
$\langle e_i,u_i\rangle<0, i,j=1,2,\dots,n.$

Hence, we have that $\ker u_n=spF=sp\{e_1,\dots,e_{n-1}\}$.

The condition $P_{spF}(K)\subset F$ is then equivalent with $P_{spF}(e_n)\in F$, since for an arbitrary $x\in K$ we have,

$$x=c_1e_1+\dots+c_{n-1}e_{n-1}+c_ne_n$$

with $c_j\geqq 0$, $j=1,\dots,n$ and hence,

$$P_{spF}(x)=c_1e_1+\dots+c_{n-1}e_{n-1}+c_nP_{spF}(e_n)$$

by the linearity of P_{spF}.

We can suppose without loss the generality, that u_n is a unit vector and then,

$$P_{spF}(e_n)=e_n-\langle e_n,u_n\rangle u_n.$$

One has further,

$$\langle P_{spF}(e_n),u_j\rangle=-\langle e_n,u_n\rangle\langle u_n,u_j\rangle\leqq 0,$$

for $j=1,2,\dots,n-1$ (since $\langle e_n,u_n\rangle<0$ and $\langle u_n,u_j\rangle\leqq 0$ by hypothesis).

Since obviously $\langle P_{spF}(e_n),u_n\rangle=0$, it follows that

$$P_{SpF}(e_n)\in K\cap spF=F.$$

d) Let us see next that F has in spF the property similar to those of K in $\mathbb{R}^n$, that is,

(5.5): $F=\{x\in spF\mid \langle x,v_j\rangle\leqq 0,\ j=1,2,\dots,n-1;\ v\ ,\dots,v_{n-1}$ linearly independent in spF and $\langle v_i,v_j\rangle\leqq 0,\ i\neq j;\ i,j=1,2,\dots,n-1\}$.

Indeed, let us take

(5.6): $v_j=u_j-\langle u_j,u_n\rangle u_n;\ j=1,2,\dots,n-1.$

Then the vector v_j are obviously linearly independent and

$$\langle v_i,v_j\rangle=\langle u_i-\langle u_i,u_n\rangle u_n,u_j-\langle u_j,u_n\rangle u_n\rangle=\langle u_i,u_j\rangle-\langle u_j,u_n\rangle\langle u_i,u_n\rangle,$$

since $\|u_n\|=1$.

Because $\langle u_i,u_j\rangle$, $\langle u_i,u_n\rangle$ and $\langle u_j,u_n\rangle$ are all nonpositive if $i \neq j$, we conclude that in this case $\langle v_i, v_j \rangle \leqq 0$.

We have the representation,

$F = \{x \in \mathbb{R}^n \mid \langle x, u_n \rangle = 0$ and $\langle x, u_j \rangle \leqq 0,\ j = 1, 2, \ldots, n-1\}$.

and hence taking into account the representations (5.6) of v_j and the relations proved above, we arrive to (5.5)

e) Denote by G the face

$G = K \cap (\ker u_n) \cap (\ker u_{n-1})$.

Then G is a face of F of codimension one in spF and since (d) we can apply the assertion proved in (c) for K replaced by F and $\mathbb{R}^n$ replaced by spF.

Denote $p = P_{spF}$ and $q = P_{spG}$.

With these notations we have,

$(P_{spG} | spF)(F) = (q | spF)(F) \subset G$.

Let we show now that $q = (q | spF) \circ p$.

To verify it, consider an arbitrary element $x \in \mathbb{R}^n$ and put it in the form $x = u + v$ with $u \in spF$ and $v \in (spF)^o$.

Let be further $u = w + z$ with $w \in spG$ and $z \in (spG)^o \cap spF$. Then $x = w + z + v$.

Since $spG \subset spF$, it follows that $(spF)^o \subset (spG)^o$ and thus $z + v \in (spG)^o$, whence $q(x) = w$, $p(x) = w + z$ and $(q | spF)(w + z) = w$, that is $q(x) = (q | spF)(p(x))$.

If we apply twice (c) and use the above conclusions, it follows,

$q(K) = ((q | spF) \circ p)(K) \subset (q | spF)(F) \subset G$, that is $P_{spG}(K) \subset G$.

f) If H is an arbitrary face of K then we can include it in a chain $H \subset H_1 \subset \ldots \subset H_k$ such that $H_1, \ldots, H_k$ have the property in their spans similar to those of K in $\mathbb{R}^n$ stated at (iv) of our theorem and so that H is a face of codimension one of H_1, with respect to spH_1, H_i is a face of codimension one of H_{i+1} with respect to spH_{i+1} if $i \leqq k-1$ and H_k is a face of codimension one of K.

Repeting step by step the process just described in (c), (d) and (e) we conclude that $P_{spH}(K) \subset H$, that is K is correct.

The proof of implication (iv)$\Rightarrow$(ii) is hence completed.

6. Proof of the implications ((i) and (ii))$\Rightarrow$(v)$\Rightarrow$(iv)

If (ii) holds, then K is latticial.

Since $x\leqq x^+$ with $x^+=x\vee 0$, from (i) it follows that $P_K(x)\leqslant P_K(x^+)=x^+$ and we have (v).

We shall verify the implication (v)$\Rightarrow$(iv) by contradiction. That is, we assume that K is latticial, that is, it can be represented in the form

$$K=\{x\,|\,\langle x,u_i\rangle\leqq 0,\ i=1,2,..,n;\ u_2,..u_n \text{ linearly independent}\}.$$

and that for each x in $\mathbb{R}^n$ it holds the relation $P_K(x)\leqq x^+$, but there are some vectors, say u_1 and u_2 in the above representation such that $\langle u_1,u_2\rangle > 0$.

We shall suppose in what follows that $u_1, u_2,..,u_n$ are unit vectors.

a) If n=2, then we consider an element $x\in K$ with $\langle x,u_1\rangle=0$ $\langle x,u_2\rangle < 0$.

Since $-x\leqq 0$ we must have by (v) that $P_K(-x)\leqq(-x)^+=0$, that is $P_K(-x)=0$.

Consider now the vector $z=-x+\langle x,u_2\rangle u_2$.

Then $\langle z,u_2\rangle=0$ and

(6.1): $\langle z,u_1\rangle=\langle -x+\langle x,u_2\rangle u_2,u_1\rangle=\langle x,u_2\rangle\langle u_2,u_1\rangle < 0.$

Thus $z\in K$. We have further,

$\langle -x-z,z-w\rangle=\langle -x-(-x+\langle x,u_2\rangle u_2),(-x+\langle x,u_2\rangle u_2)-w\rangle =$

$=\langle -\langle x,u_2\rangle u_2,\langle x,u_2\rangle u_2-(x+w)\rangle=-\langle x,u_2\rangle^2+\langle x,u_2\rangle\langle u_2,x+w\rangle=$

$=\langle x,u_2\rangle\langle u_2,w\rangle\geqq 0,\ \forall\, w\in K$, since $\langle x,u_2\rangle<0$ and $\langle u_2,w\rangle\leqq 0$, $\forall\, w\in K$.

By the characterization (2.6) of the projection we have then that $P_K(-x)=z$. But by (6.1) it must be $z\neq 0$. The obtained contradiction shows that in this case we cannot have $\langle u_1,u_2\rangle > 0$.

b) Suppose that $n\geqq 3$. Let us show first that under the above hypotheses there exists an element w in $\mathbb{R}^n$ such that

(6.2): $\langle w,u_2\rangle=0,\langle w,u_j\rangle < 0$, $j\geqq 3$ and $P_K(w)\in\operatorname{rint} F$ where

$F=K\cap(\ker u_1)$.

Consider the cone,

$K_1=\{x\in\mathbb{R}^n \mid \langle x,u_1\rangle\geqq 0,\ \langle x,u_j\rangle\leqq 0,\ j=2,\dots,n\}$

By Lemma 1 there exist some elements y and z in K_1 such that

$\langle y,u_1\rangle > 0,\ \langle y,u_2\rangle=0,\ \langle y,u_j\rangle<0,\ j=3,\dots,n,$

and

$\langle z,u_1\rangle=\langle z,u_2\rangle=0, \langle z,u_j\rangle<0,\ j=3,\dots,n.$

Take $w_t=ty+(1-t)z$ with $t\in(0,1]$.

Then $w_t\in K_1$ and since $\|u_1\|=1$, we have

$$P_{spF}(w_t)=w_t-\langle w_t,u_1\rangle u_1,$$

with $spF=\ker u_1$. Let us see that for a sufficiently small t we have $P_{spF}(w_t)\in \operatorname{rint}F$.

We have for $j\geqq 3$ that.

$\langle w_t,u_j\rangle=t\langle y,u_j\rangle+(1-t)\langle z,u_j\rangle\leqslant\max\{\langle y,u_j\rangle,\langle z,u_j\rangle\}$.

Put $-\delta=\max\{\langle y,u_j\rangle,\langle z,u_j\rangle;\ j\geqq 3\}$.

Then $\delta>0$ and we can take t so small in (0,1] to have $0<\langle w_t,u_1\rangle<\delta$.

Then for $j\geqq 3$ one has,

$\langle P_{spF}(w_t),u_j\rangle=\langle w_t,u_j\rangle-\langle w_t,u_1\rangle\langle u_1,u_j\rangle\leqq\langle w_t,u_j\rangle+|\langle w_t,u_1\rangle$
$\langle u_1,u_j\rangle|\leqq-\delta+\langle w_t,u\rangle<-\delta+\delta=0$

and

$\langle P_{spF}(w_t),u_2\rangle=\langle w_t,u_2\rangle-\langle w_t,u_1\rangle\langle u_1,u_2\rangle=-\langle w_t,u_1\rangle\langle u_1,u_2\rangle<0,$

since $\langle w_t,u_2\rangle=0$, $\langle u_1,u_2\rangle>0$ and $\langle w_t,u_1\rangle>0$.

Since obviously, $\langle P_{spF}(w_t),u_1\rangle=0$, the obtained relation show that for a such t we have,

$$P_{spF}(w_t)\in \operatorname{rint}F,$$

whence it follows implicitely taht $P_{spF}(w_t)=P_K(w_t)$.

Take $w=w_t$ and observe that it satisfies the requirements in (6.2).

c) <u>We shall see next that w^+ is contained in the face $F_{1,2}$ of K given by</u>

<u>$F_{12}=\{x\in K \mid \langle x,u_1\rangle=\langle x,u_2\rangle=0\}$</u>.

Since $\langle w,u_2\rangle=0$ we have by Lemma 2 that $\langle w^+,u_2\rangle=0$.

Assuming that,

(6.3): $<w^+,u_1> < 0$

consider the element $v_t=tw^+ +(1-t)w$.

For any t in (0,1) one has

(6.4): $w<v_t<w^+$

(where $x<y$ means $x\leqq y$ and $x\neq y$).

This follows from conditions (6.2) which imply that $w<w^+$.

Since $w^+-w\in K$, we have $<w-w^+,u_j>\leqq 0$, that is, $<w^+,u_j>\leqq<w,u_j>$ whence $<w^+,u_j>\leqq 0$; for $j\geqq 2$ by the conditions in (6.2). Hence,

(6.5): $<v_t,u_j>=t<w^+,u_j>+(1-t)<w,u_j>\leqq 0$; $j\geqq 2$ for any $t\in(0,1)$.

From the hypothesis (6.3), taking into account that $<v_t,u_1>=t<w^+,u_1>+(1-t)<w,u_1>$ it follows that for t sufficiently close to 1 in (0,1) we have also $<v_t,u_1>\leqq 0$.

But this relation together with (6.5) show taht $v_t\in K$, taht is $v_t\geqq 0$.

Hence $w^+=w\vee 0\leqq v_t$ and we have got a contradiciton with (6.4).

Thus the assertion (c) was proved.

(d) Since $F_{1\,2}$ is a face of K, the relation $P_K(w)\leqq w^+$ would imply that $P_K(w)\in F_{1\,2}$, in contradiction with (6.2).

The obtained contradiction shows that the inequality $<u_1,u_2>>0$ cannot hold, that is, $<u_i,u_j>\leqq 0$ for $i\neq j$, $i,j=1,2,\ldots,n$. That is, we have the condition (iv) fulfilled. ■

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References

[1] Barker G.Ph.,M. Laidacker and G.Poole: Projectionally exposed cones.
SIAM J.Alg. Disc. Meth.8(1987), 100-105.

[2] Borwein J.M.and H.Wolkowicz: Regularizing the abstract convex program.
J.Math.Anal. Appl. 83(1981), 495-530.

[3] Barker G.Ph.: Perfect cones.
Linear Algebra Appl. 22(1978), 211-221.

[4] Haraux A.: How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities.
J.Math.Soc. Japan 29(1977), 615-631.

[5] Holmes R.B.: Smoothness of certain metric projections on Hilbert space.
Trans. Amer. Math. Soc. 184(1973), 87-100.

[6] Hiriart-Urruty J.B.: At what points is the projection mapping differentiable?
Amer. Math. Monthly 89(1982), 456-460.

[7] Jochum B.: Cônes autopolaires et algèbres de Jordan.
Lecture Notes in Math. Nr.1049
Springer-Verlag. Berlin, Heidelberg, New York (1984)

[8] Isac G. and A.B.Németh : Monotonicity of metric projections onto positive cones in ordered Euclidean spaces.
Arch. Math. 46 (1986), 568-576.

[9] " " " :Every generating isotone projection cone is latticial and correct.
To appear: J. Math. Anal.Appl.

[10] " " " : Ordered Hilbert spaces. (Preprint)

[11] " " " : Isotone projection cones in Hilbert spaces and the Complementarity Problem. (Preprint)

[12] " " " : Projection methods, isotone projection cones and the Complementarity Problem. (Preprint).

[13] Phelps R.R.: Convex sets and nearest points.
Proc.Amer. Math. Soc. 8(1957), 790-797; 9(1958), 867-873.

[14] Phelps R.R.: Metric projections and the gradient projection method in Banach spaces.
SIAM.J. Control and Optim. 23 Nr.6(1985),973-977.

[15] ——"——: The gradient projection methods using Curry's steplength.
SIAM J.Control Optim 24 Nr.4(1986), 692--699.

[16] Shapiro A: On differentiability of metric projections in R^n 1:Boundary case.
Proc.Amer. Math. Soc.99 Nr.1(1987), 123-128.

[17] Stoer J. and Witzgall C.: Convexity and Optimization in finite dimension I.
Springer-Verlag, Berlin (1970)

[18] Zarantonello E.H.: Projections on convex sets in Hilbert spaces and spectral theory.
Contributions to Nonlinear Functional Analysis, E. H. Zarantonello ed. Acad. Press, New York (1971), 237-424.

[19] Youdine A: Solution de deux problems de la théorie des espaces semi-ordonnés.
C.R.Acad. Sci. U.R.S.S. 27(1939), 418-422.