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THE AXIOM OF ARCHIMEDES, HEMISPACES, HYPERCONES AND ε-EFFICIENCY

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It is investigated the relation of the various forms of the axiom of Archimedes with the hemispaces, hypercones, lexicographic ordering and ε-efficiency.

0. Introduction. Although not explicitely stated in the literature, a wide class of problems concerning ordered vector space theory is intimately related to the axiom of Archimedes. This note aims to be an illustration of this. It has in fact three origins. The first one is the extended literature concerning the so called hemispaces, which are in fact convex sets in vector spaces having convex complements. The laborious activity of I. Singer and Martinez-Legas (see [2] and the literature cited there) is a good illustration of this streem. The second source is the technique of the separation functional in the spirit it was introduced and used in vectorial optimization by Chr. Tammer and P. Weidner (see e.g. [1] and [8]). Finally, we reconsider more carefully some of our earlier results concerning ε-efficiency. ([4] and [5]). Some of the ideas we shall work out are implicite in the above cited literature. The novelty of our approach is the consistency in following the ordered vector space setting and of the various forms of the axiom of Archimedes. Thus, we introduce the notion of the anti Archimedian element of the positive cone of an ordered vector space and the anti

Archimedian subcone of its. We consider then the notion of the anti Archimedian subspace of an ordered vector space. These notions link naturally to !exicographic ordering and hemispaces, hypercones and to ε-efficiency, allowing a concise treatement of many aspects of these problems.

1. Terminology. Let X be a real vector space. We shall use in the sequel the following terminology:

Cone: a subset K of X satisfying the conditions: $K + K \subset K$; $tK \subset K$ if $t \in \mathbb{R}_+ = [0,\infty)$; and $K \cap (-K) = \{0\}$.

Ordering: a relation "$\leq$" in X which is reflexive ($x \leq x$ for every $x \in X$), transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$) and antisymmetrical ($x \leq y$ and $y \leq x$ imply $x = y$).

An ordering $\leq$ in X is:

(ti) *translation invariant* if $x \leq y$ implies $x + z \leq y + z$, $\forall\, z \in X$;

(mi) *invariant with respect to multiplication with non-negative reals* if: $x \leq y$ implies $tx \leq ty$, $\forall\, t \in \mathbb{R}_+$.

The following facts are direct consequences of the definitions:

1.1 *If K is a cone, then the relation $\leq_K$ in X defined by considering $u \leq_K v$ whenever $v - u \in K$, is an ordering which satisfies* (ti) *and* (mi).

1.2 *If $\leq$ is an ordering in X which satisfies* (ti) *and* (mi), *then $K = \{x \in X : x \geq 0\}$ is a cone and $\leq$ coincides with $\leq_K$ defined above. The cone K is called the* ***positive*** *cone of the ordering $\leq$.*

The pair $(X,\leq)$ with $\leq$ an ordering in X which possesses the properties (ti) and (mi) is called an *ordered vector space* (OVS). The remarks 1.1 and 1.2 above motivate the often used notation (X,K) for $(X,\leq)$, where K is the positive cone of $\leq$. Depending of the context we shall

sometime use also this notation. The cone K is called also the *positive cone of the* OVS.

The cone K *generates* X or it is *generator cone* if $X = K - K$.

The terms introduced in the sequel will be used for the ordering and for the positive cone as well.

The OVS $(X,\leq)$ is called *semi Archimedian* if for $x,y \in K$ and $x \leq ty$, $\forall t \in \mathbb{R}^{\circ}_{+} = (0,\infty)$, it follows that $x = 0$.

(Equivalent definitions: $nx \leq y$, $\forall n \in \mathbb{N}$, or $tx \leq y$, $\forall t \in \mathbb{R}^{\circ}_{+}$ imply that $x = 0$.)

The OVS $(X,\leq)$ is called *almost Archimedian* if for $x \in X$, $y \in K$ and $-ty \leq x \leq ty$, $\forall t \in \mathbb{R}^{\circ}_{+}$ it follows $x = 0$.

(Equivalent definitions: $-y \leq nx \leq y$, $\forall n \in \mathbb{N}$ or $-y \leq tx \leq y$, $\forall t \in \mathbb{R}^{\circ}_{+}$ imply $x = 0$.)

The OVS $(X,\leq)$ is called *Archimedian* if for $x \in X$, $y \in K$ and $x \leq ty$, $\forall t \in \mathbb{R}^{\circ}_{+}$ it follows that $x \leq 0$.

(Equivalent definitions: $nx \leq y$, $\forall n \in \mathbb{N}$ or $tx \leq y$, $\forall t \in \mathbb{R}^{\circ}_{+}$ imply that $x \leq 0$.)

2. Section subcones, faces. The subset K_o of the cone K is a *section subcone* if $K_o = (\mathrm{sp}K_o) \cap K$, where $\mathrm{sp}K_o$ is the vector space engendered by K_o. The set K_o is obviously a cone. If X_o is an arbitrary subspace of X, then $X_o \cap K$ is a section subcone of K.

The subset K_o of the cone K is called a *face* of K, if it is a cone and if $x \in K$, $y \in K_o$ and $x \leq_K y$ imply $x \in K_o$. The face K_o of K is *proper* if $K_o \neq \{0\}$ and $K_o \neq K$.

2.1 *Every face of the cone K is a section subcone of K.*

◂ Let K_o be a face of K and $x \in (\mathrm{sp}K_o) \cap K$. Then $x = u - v$ with $u,v \in K_o$. But then $x \leq u$ and hence $x \in K_o$. This shows that $(\mathrm{sp}K_o) \cap K \subset K_o$. The converse inclusion is obtained. ▸

2.2 *If K_o is a section subcone of K then for $u,v \in \mathrm{sp}K_o$ we have $u \leq_K v$ if and only if*

$u \leq_{K_o} v$.

◂ From $u \leq_{K_o} v$ obviously follows $u \leq v$.

Suppose that $u \leq_{K_o} v$, $u,v \in \mathrm{sp}K_o$. Then $v-u \in K$, $v-u \in \mathrm{sp}K_o$ and hence $v-u \in (\mathrm{sp}K_o) \cap K = K_o$ which implies $u \leq_{K_o} v$.▸

2.3 *A cone K in X is semi Archimedian, almost Archimedian or Archimedian respectively, if every section subcone of K of dimension two has the respective property in the subspace it spans.*

3. **The core of cone.** Let M be a subset of the vector space X. The element $x \in M$ is called a *core point* of M if for every $z \in X$ there exists a positive real number t_z such that $x + tz \in M$ whenever $|t| < t_z$.

The set of the core points of M is denoted by core$M$. The point $x \in M$ is called an *intrinsic core point* of M if 0 is a core point of $M - x$ in the subspace $\mathrm{sp}(M - x)$. The set of the intrinsic core points of M is denoted by icore$M$. Observe that if $0 \in M$ then icore$M$ is the core of M in *sp*M.

3.1. *Let K be a core in X. If $x \in$ core$K$ and $x \leq_{K_o} y$, then $y \in$ core$K$. If $t > 0$ then $tx \in$ core$K$ if and only if $x \in$ core$K$.*

3.2. *If K is a core, then* core$K \cup \{0\}$ (icore$K \cup \{0\}$) *is a subcone of K called the core cone (intrinsic core cone) of K.*

3.3. *The intrinsic core cone $K_1 =$ icore$K \cup \{0\}$ of K can be characterized by the following property:*

$x \in K_1 \setminus \{0\} \Leftrightarrow \forall\, y \in K, \exists\, t \in \mathbb{R}^*_+$ *such that* $ty \leq x$.

◂ The implication $\Rightarrow$) is obvious.

Let $y \in K$ be arbitrary and $t_y \in \mathbb{R}^*_+$ satisfying $t_y y \le x$. Let be $t \in \mathbb{R}$ with $|t| < t_y$. Then $-|t|y \le x$, $|t|y \le t_y y \le x$. Thus $x - |t|y \in K$ and $x + |t|y \in K$, that is, $x + ty \in K$. ▹

In the following assertions "core" can be throughout replaced by "icore".

3.4. A cone K with core$K \neq \emptyset$, *has no proper face if and only if it coincides with its core cone.*

◃ Suppose that $x \in K \setminus (\text{core}K \cup \{0\})$. Let be $K_o = \{y \in K: \exists\, t > 0$ with $y \le tx\}$. Then K_o is a proper face of K. Indeed, K_o is obviously a cone, $K_o \neq \{0\}$ since $x \in K_o$, and $K_o \neq K$, because K_o cannot contain any core point of K by 3.1. K_o is a face of K by its definition.

Suppose now, that $K = \text{core}K \cup \{0\}$ and that K_o is a face of K. If $x \in K_o \setminus \{0\}$, then $x \in \text{core}K$. Let $y \in K$ be arbitrary. Since x is a core point of K, there exists a positive t such that $x - ty \ge 0$. Hence $0 \le ty \le x$ and thus $ty \in K_o$. But then $y \in K_o$ which shows that $K_o = K$. ▹

3.5. *Let K be a cone in X and X_o a subspace of X. If $K_o = K \cap X_o$, and* $(\text{core}K) \cap X_o \neq \emptyset$, *then* $\text{icore}K_o = (\text{core}K) \cap X_o$.

◃ Obviously, $(\text{core}K) \cap X_o \subset \text{icore}K_o$. Suppose that $x \in \text{icore}K_o$ and that $x \in (\text{core}K) \cap X_o$. Then there exists $t > 0$ such that $x - ty \in K_o$. But then $ty \le_{K_o} x$ and since $\text{core}K$ we have by 3.1 that $x \in \text{core}K$. Thus $x \in (\text{core}K) \cap X_o$ ▹

4. The anti Archimedian subcone of a cone. Let K be the positive cone of $(X, \le)$. The element $x \le K$ is called *anti Archimedian* if there exists some $y \in K$ such that $nx \in y\ \forall\, n \in \mathbb{N}$. Let K_a denote the set of anti Archimedian elements of K. Then K_a is itself a cone which is called the *anti Archimedian subcone of K*. If $K = K_a$ then it is called *anti Archimedian cone.*

4.1. *Example* ([6] p.55) Let α be a fixed ordinal>0 and consider the space X of all

families $(x_\beta)_{\beta<\alpha}$ of real numbers x_β indexed by the set of all ordinals $< \alpha$. Under the ordering defined by $x \le y$ if $x_\beta < y_\beta$ for the first ordinal for which $x_\beta \neq y_\beta$, or $x = y$, $(X,\le)$ is an OVS. This ordering is called the *lexicographic ordering*. Then K generates X and it is anti Archimedian if and only if α is infinite (see also section 6). Observe also that $X = K \cup (-K)$ and that X is the disjoint union of K and $(-K)\setminus\{0\}$.

4.2. *The core K in X is semi Archimedian, if and only if $K_a = \{0\}$.*

4.3. *The anti Archimedian subcone K_a of K is a face of K.*

4.4. *Let K be a cone. Then no element of* core$K$ *can be anti Archimedian.*

◁ Suppose by contradiction that $x \in K_a \cap (\text{core}K)$ with K_a the anti Archimedian subcone of K. Let $y \in K$ be an element for which $nx \le y$ $\forall n \in \mathbb{N}$, or equivalently $tx \le y$ $\forall\ t \in \mathbb{R}^*_+$. This contradiction is also equivalent with $tx \le sy$, $\forall\ t, s \in \mathbb{R}^*_+$. Since x is an icore point of K, we have that $x - t(y - x) \in K$ for $t \in \mathbb{R}^*_+$ small enough. Hence $-ty+(t+1)x \in K$ for a such t. We have on the other hand $(t+2)x \le ty$, and hence $ty-(t+2)x \in K$. By adding the above considered two elements of K we get that $-x \in K$. Thus $x = 0 \in \text{core}K$ furnishes a contradiction. ▷

4.5. *If K is a core and K_a is its anti Archimedian subcone, then* core$K_a = \varnothing$.

4.6. *No finite dimensional cone $K \neq \{0\}$ can be anti Archimedian.*

◁ If K is finite dimensional, then core$K \neq \varnothing$ (see e.g. [7]). ▷

4.7. *If $K \subset X$ is an anti Archimedian core, $K \neq \{0\}$, then* $\dim(K-K) = \infty$.

4.8. *The core K defined in* 4.1 *is a generator, anti Archimedian cone for α infinite. Hence* core$K = \varnothing$.

5. The anti Archimedian subspace of an OVS. Let $(X,\le)$ be an OVS and K its positive cone. To simplify the exposition we shall suppose in this section that K is generator. The subset X_a of X such that for every $x \in X_a$ there exists an y in K with the property that

$-ty \leq x \leq ty \ \forall \ t \in \mathbb{R}^*$, is obviously a subspace of X. It will be called the *anti Archimedian subspace* of $(X,\leq)$.

5.1. *If K_a is the anti Archimedian subcone of K, then $K_a = K \cap X_a$.*

5.2. *X_a cannot contain any core point of K. This follows from* 5.1. *and* 4.4.

5.3. *$X_a = X$ if and only if the core K is Archimedian. In this case X must be infinite dimensional by* 4.7.

5.4. *The* OVS $(X,\leq)$ *is almost Archimedian if and only if X_a the anti Archimedian subspace of* $(X,\leq)$, *is* $\{0\}$.

5.5. *Let X_a be the anti Archimedian subspace of* $(X,\leq)$ *and let X_1 be an algebraic complement of X_a. If* core$K \neq \emptyset$ *then $K_1 = X_1 \cap K$ is a generator cone of X_1,* icore$K_1 \neq \emptyset$ *and* core$K$ = icore$K_1 + X_a$.

◂ Let be $x \in$ core$K$ and $x_o \in X_a$. We have $-ty \leq x_o \leq ty$ for some $y \in K$ and for any $t \in \mathbb{R}^*$. Since $x \in$ core$K$ there exists some $t > 0$ such that $x-ty \in$ core$K$. But then $x-ty \leq x+x_o$ and by 3.1., $x+x_o \in$ core$K$.

Let X_1 be an algebraic complement of X_a and let x be in core$K$. Then $x = x_1 + x_o$ with some $x_1 \in X_1$, $x_o \in X_a$. Also, $-x_o \in X_a$ and by the above observation, $x_1 = x-x_o \in$ core$K$. Thus $x_1 \in$ core$K \cap X_1$ = icoreK_1 by 3.5. This shows that core$K$ = icore$K_1 + X_a$.

Since K is generator by convention, for every $x_1 \in X_1$ there exists $u, v \in K$ such that $x_1 = u - v$. By 3.1 we can assume that $u, v \in$ core$K$. Using the above representation of core$K$ we have $u = u_1 + u_o$, $v = v_1 + v_o$ with $u_1, v_1 \in$ icoreK_1, $u_o, v_o \in X_a$. Then $x_1 = (u_1 - v_1) + (v_o - u_o)$. But from the definition of X_1 we must have $v_o - u_o = 0$. Hence $x_1 \in K_1 - K_1$, that is K_1 generates X_1. ▸

6. Total ordering in $\mathbb{R}^n$. The OVS $(X,\leq)$ is *totally ordered* if for every u,v in X either

$u \le v$ or $v \le u$

6.1 *If K is the positive cone of the* OVS $(X,\le)$, *then the ordering $\le$ is total if and only if $X = K \cup (-K)$. In this case X is the disjoint union of K and $-K \setminus \{0\}$.*

The ordering $\le$ in $\mathbb{R}^n$ is called *lexicographic* if there exists some basis $e_1,\dots,e_n$ in $\mathbb{R}^n$ such that $u = u^1e_1 + \dots + u^ne_n$, $u^i \in \mathbb{R}$ and $v = v^1e_1 + \dots + v^ne_n$, $v^i \in \mathbb{R}^n$ are in the relation $u \le v$ if and only if either $u = v$ or $u^k < v^k$ for the first upperscript k for which $u^k \ne v^k$. The positive cone of the lexicographically ordered vector space $(\mathbb{R}^n,\le)$ is a lexicographic cone in the sense of the example 4.1.

If $(\mathbb{R}^n,\le)$ is lexicographic, then for the basis $e_1, \dots, e_n$ in the above definition of its we shall say that it *realizes the lexicographic ordering*.

6.2. *Every lexicographically ordered vector space is totally ordered.*

6.3. *If $(\mathbb{R}^n,\le)$ is totally ordered with the positive cone K, then K_∞, the anti Archimedian subcone of K has the dimension $n - 1$.*

◁ Indeed, $\mathrm{int}K \ne \varnothing$ since K is generator. (Here int denotes the interior with respect to the Euclidian topology in $\mathbb{R}^n$). Let $b \in \mathrm{int}K$, and denote

$$K_0 = \{x \in K: x \le tb, \ \forall t \in \mathbb{R}^+\}.$$

We shall show that $\dim K_0 = n - 1$ and $K_0 = K_\infty$.

Let $k_1, \dots, k_{n-1}$ be elements in K which form with b a linearly independent set of vectors. Consider the open line segments $(-k_i,b)$ joining the points $-k_i$ and b, $i=1, \dots, n-1$. Each of them intersects the topological boundary ∂K of K. That is, there exist some reals $t_i \in (0,1)$ such that $y_i = t_ik_i + (1 - t_i)b \in \partial K$, $i = 1, \dots, n - 1$. The set of vectors $y_1, \dots, y_{n-1}$ is linearly independent. Since $K \cup (-K) = X$, for every i, y_i or $-y_i$ belongs to K. Denote $\{z_i\} = \{y_i, -y_i\} \cap K$, $i = 1, \dots, n - 1$. Then $z_1, \dots, z_{n-1}$ are linearly independent elements in K. Suppose that for some z_i, there exists $t > 0$ such that $tb \le z_i$. Then by 3.1 it would follow that $z_i \in \mathrm{core}K$

$= \text{int}K$ which is impossible. Since the space is totally ordered, we have that $z_i \le tb$, $\forall\ i = 1,\ldots, n-1$, $\forall\ t \in \mathbb{R}^*$. Hence $z_1,\ldots, z_{n-1} \in K_o$.

We have obviously $K_o \subset K_a$ and since $\dim K_o \ge n-1$ and $\dim K_a \le n-1$, it follows that $\dim K_a = n-1$. Every $y \in K_a$ can be written as $y = t_1z_1 + \ldots + t_{n-1}z_{n-1}$ with some $t_1, \ldots, t_{n-1} \in \mathbb{R}$, and hence $y \le tb$, $\forall\ t \in \mathbb{R}^*$. That is, $y \in K_o$, and thus $K_o = K_a$.▸

6.4. *Let K be a generator cone in X. Thus (X,K) is a totally ordered* OVS *if and only if for every section subcone K_o of K, $(\text{sp}K_o, K_o)$ is a totally ordered* OVS.

◂ The if part is obvious, since K is itself a section subcone.

Suppose that K_o is a section subcone. Then if the space is totally ordered, we have $K \cup (-K) = X$ and hence $\text{sp}K_o = (\text{sp}K_o) \cap (K \cup (-K)) = ((\text{sp}K_o) \cap K) \cup ((\text{sp}K_o) \cap (-K)) = K_o \cup (-K_o)$.▸

6.5. *If $(\mathbb{R}^n, \le)$ is totally ordered, then the ordering $\le$ is lexicographic.*

◂ We proceed by induction. For $n = 1$ every total ordering in $\mathbb{R}^1$ is lexicographic.

Suppose that each total ordering in $\mathbb{R}^{n-1}$ is lexicographic.

Let $(\mathbb{R}^n, \le)$ be a totally ordered OVS. Then by 6.3 we have that the anti Archimedian subcone K_a of the cone K is of dimension $n-1$. Since K_a is a face of K it is a section subcone of its, and hence by 6.4 the OVS $(\text{sp}K_a, K_a)$ is totally ordered. Let $e_1 \in K \setminus K_a$. Then every $x \in \mathbb{R}^n$ can be uniquely represented in the form $x = te_1 + y$ for $t \in \mathbb{R}$, $y \in \text{sp}K_a$. Then $x \in K$ if $t \ge 0$ and $x \in -K$ if $t < 0$. Indeed, $x \in K$ and $t < 0$ would mean that $-te_1 + x = y \in K$, $y \in \text{sp}K_a$ and $0 \le -te_1 \le y$. Now $y \in K_a$ and K_a being a face, it follows that $-te_1 \in K_a$ and thus $e_1 \in K_a$, which is impossible.

$(\text{sp}K_a, K_a)$ is totally ordered $n-1$ dimensional Euclidean space. By the induction hypothesis there exists a basis $e_2, \ldots e_n$ in $\text{sp}K_a$ which realizes the lexicographic ordering in $(\text{sp}K_a, K_a)$. But then from the above considerations it follows that $e_1, e_2, \ldots, e_n$ is a basis in $\mathbb{R}^n$

which realizes the ordering $\leq$ as a lexicographic ordering. ▸

Remarks. The assertion 6.5 is well known. It appears for instance in [6], as Theorem II.3.6. It follows from more general considerations in [2]. The above proof uses the idea of anti Archimedian subcone and its properties and even though exploits related geometric ideas, it differs from the known ones.

The same problem in more general algebraic structures was considered by Lassak and Prószyński [3].

7. Hypercones and hemispaces in $\mathbb{R}^n$. The subset H in X is called a *hemispace* if H and $X \setminus H$ are both non-empty convex subsets in X.

The cone K in X is called a *hypercone* if it is not contained in any other cone as a proper subcone.

7.1. *Let K be a cone in $\mathbb{R}^n$. The following statements are equivalent:*

(i) *K is a hemispace;*

(ii) *K is a lexicographic cone.*

(iii) *K is a hypercone.*

◃ (i) $\Rightarrow$ (ii). Suppose that $x \in \mathbb{R}^n \setminus K$. If we would have $-x \in \mathbb{R}^n \setminus K$, it would follow that $1/2x + 1/2(-x) \in 0 \in \mathbb{R}^n \setminus K$, since the last set is convex. But this is impossible. Thus $-x \in K$ and hence $\mathbb{R}^n = K \cup (-K)$. From 6.5, K is then lexicographic.

(ii) $\Rightarrow$ (iii). Every set containing K as a proper subset, must contain an element of form $-x$ with $x \in K \setminus \{0\}$. Hence there is no cone containing K as a proper subcone.

(iii) $\Rightarrow$ (i). Soppose that $X \setminus K$ is not convex. Then there exist $u,v \in X \setminus K$ and $t \in (0,1)$ such that $z = tu + (1-t)v \in K$. Then $u - z$ and $v - z$ are not in K. Put $y = u - z \neq 0$. Then the set $K_1 = \{x \in \mathbb{R}^n: x = w + ty, w \in K, t \in \mathbb{R}_+\}$ is a cone. We have obviously that $K_1 + K_1$

$\subset K_1$ and $sK_1 \subset K_1$ for $s \geq 0$. Let us check that $K_1 \cap (-K_1) = \{0\}$. Suppose that $x \in K_1 \cap (-K_1)$. Then $x = w_1 + t_1 y = -w_2 - t_2 y$ with $w_1, w_2 \in K$ and $t_1, t_2 \in \mathbb{R}_+$. If $t_1 = t_2 = 0$ we must have also $x = 0$. If not, $(t_1 + t_2)y = - w_1 - w_2 \in -K \setminus \{0\}$. Using the definition of z and y we have $0 = t(u-z) + (1-t)(v-z) = ty + (1-t)(v-z)$; $(t_1+t_2)ty \in K \setminus \{0\}$ and hence $-(t_1+t_2)(1-t)(v-z) \in -K \setminus \{0\}$, wherefrom, since $(t_1+t_2)(1-t) > 0$ it follows that $v-z \in K$. But $z \in K$, hence we have $v \in K$ which is a contradiction. Thus K_1 is a cone which contains K as a proper subcone. ▹

7.2 *K is a semi Archimedian hypercone in $\mathbb{R}^n$ if and only if it is of form $K = G \cup \{0\}$, where G is an open semispace determined by a hyperplane H through 0.*

◃ Suppose that $K = G \cup \{0\}$. Let $x_1 \in G$ and suppose that $x_2, \ldots, x_n$ is a basis in H. We suppose that the elements in $\mathbb{R}^n$ are represented in the basis $\{x_1, x_2, \ldots, x_n\}$. Let $u = (u^1, \ldots, u^n)$, $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$ and denote by $\leq$ the ordering induced by K. Then $u \leq v$ if and only if $u^1 \leq v^1$. The only element $u = (u^1, \ldots, u^n)$ in K with $u^1 = 0$ is $0 = (0, \ldots, 0)$. Suppose that $mu \leq v$ with $u, v \in K$ and any $n \in \mathbb{N}$. Then $mu^1 \leq v^1$ with $u^1, v^1 \in \mathbb{R}_+$ and any $n \in \mathbb{N}$. Hence $u^1 = 0$ and $u = 0$. This shows, that K is semi Archimedian.

Assume that K is contained in the coreK_1. Then K_1 and K have the common supporting hyperplane H. Suppose that $x = (x^1, \ldots, x^n) \in K_1 \setminus K$. Then $x^1 = 0$ and thus $x = (0, x^2, \ldots, x^n) \neq 0$. By the definition of K we have obviously

$$(1,0,\ldots,0) - n(0,x^2,\ldots,x^n) \in K \subset K_1, \ \forall\, n \in \mathbb{N}.$$

This shows that K_1 cannot be semi Archimedian. That is, $K = G \cup \{0\}$ is a semi Archimedian hypercone.

Suppose now that K is a semi Archimedian hypercone. Let H be the hyperplane through 0 which supports K. Let G be the open semispace determined by H which intersects K. Then $K_1 = G \cup \{0\}$ is by the above considerations a semi Archimedian cone. Assume that $x \in K \setminus K_1$. Then $x \in H \setminus \{0\}$. If $-x \in \partial K$ from standard reasonings on convex cones we have

that $-nx \in \partial K$, $\forall\, n \in \mathbb{N}$ and $-nx + y \in K$ for an arbitrary fixed $y \in \text{core}K$ and $\forall\, n \in \mathbb{N}$. This shows that $nx \leq y$, $\forall\, n \in \mathbb{N}$ with $x,y \in K$, $x \neq 0$. Thus K cannot be semi Archimedian. The obtained contradiction shows that $K \subset K_1$ and since K is a semi Archimedian hypercone and K_1 is semi Archimedian, we have in fact $K = K_1$.▸

7.3 *Let K be a cone in $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) *K is a semi Archimedian hypercone;*

(ii) *If K_a denotes the anti Archimedian X subcone of K and X_a the anti Archimedian subspace in (X,K), then $K_a = \{0\}$ and* $\dim X_a = n - 1$.

◂ (i) $\Rightarrow$ (ii). We have obviously $K_a = \{0\}$. We have by 7.2 that $K = G \cup \{0\}$, with G an open semispace determined by a hyperplane H through 0. Let $x_1 \in G$ and let $x_2,...,x_n$ be a basis in H. If we represent the elements in $\mathbb{R}^n$ in the basis $\{x_1,...,x_n\}$, then putting $u = (u^1,...,u^n)$, $v = (v^1,...,v^n) \in \mathbb{R}^n$ we have $u \leq v$ with $\leq$ the ordering induced by K, if and only if $u^1 \leq v^1$. Let $x \in H$. Then $x = (0,x^2,...,x^n)$ and taking $y = (1,0,...,0) \in K$, we have that $-ty \leq x \leq ty$ for any $t \in \mathbb{R}^*$. This shows that $H \subset X_a$. By 5.2 we have $\dim X_a \leq n - 1$. Hence $\dim X_a = n - 1$.

(ii) $\Rightarrow$ (i). Since K is convex $X_a \cap K = \{0\}$ and X_a is a hyperplane it follows that X_a supports K at $\{0\}$. Let $x \in X_a$ be arbitrary. Then we have some $y \in K$ such that $-ty \leq x \leq ty$ for every $t \in \mathbb{R}^*$. Let $z \in \text{core}K$. Then we have also $-t(y+z) \leq x \leq t(y+z)$ $\forall\ t \in \mathbb{R}^*$ with $y + z \in \text{core}K$. Further, if we denote $u = y + z$ then $tu + x \in K$ for every $t \in \mathbb{R}^*$. But $tu + x \to x$ as $t \to 0$, and hence $x \in \partial K$. Then from standard considerations on convex sets in $\mathbb{R}^n$ we have also that $tu + x \in \text{core}K$ $\forall\ t \in \mathbb{R}^*$ and hence $x \in \partial(\text{core}K)$. Thus $X_a \subset \partial(\text{core}K)$ and using a geometric reasoning similar with that in the proof of [9], Lemma I.2.14, we conclude that $\bar{K}$ coincides with the closed hyperplane determined by X_a.

7.4. *Let K be a generating core cone in $\mathbb{R}^n$ which is not a semi Archimedian hypercone*

and is not an almost Archimedian cone. Then there exist two section subcones K_0 *and* K_1 *in* K *with* $\dim K_0 \cap K_1 = 1$, K_0 *is a semi Archimedian hypercone in* $\mathrm{sp}K_0$ *and* K_1 *is almost Archimedian cone in* $\mathrm{sp}K_1$, *such that* $K = K_0 + K_1$.

◂ Since K is not almost Archimedian, the anti Archimedian subspace X_a of (X,K) is not $\{0\}$. Denote by X_1 an algebraic complement of X_a in $\mathbb{R}^n$ and let be $K_1 = K \cap X_1$. Since K is not a semi Archimedian hypercone in $\mathbb{R}^n$, $\dim X_a < n - 1$ by 7.3. Hence $\dim K_1 \geq 2$ and obviously $\mathrm{icore}K_1 \neq \varnothing$. By 5.5 we have $\mathrm{core}K = \mathrm{icore}K_1 + X_a$. Let $x \in \mathrm{icore}K_1$ and consider the subspace $X_0 = \mathrm{sp}\{X_a \cup \{x\}\}$. Then it can be easily verified that $K_0 = X_0 \cap K = \{y+tx: y \in X_0, t \in \mathbb{R}_+^*\} \cup \{0\}$.This representation of K_0 shows according 7.2 that K_0 is a semi Archimedian hypercone in X_0. We have also that $K_0 \cap K_1 = \{tx: t \in \mathbb{R}_+\}$. Let $z \in K$ be arbitrary. If $z = 0$ we have $z \in K_0 + K_1$. If $z \neq 0$ then $z \in \mathrm{core}K$ since K is a core cone and hence also $z \in \mathrm{icore}K_1 + X_a$. From the definition of X_1 it follows that there exist $x_0 \in X_a$, $x_1 \in X_1$ such that $z = x_0 + x_1$. Since $z \in \mathrm{core}K = \mathrm{icore}K_1 + X_a$ we must have $x_1 \in \mathrm{icore}K_1$. If x is the above defined element of $\mathrm{icore}K_1$, then for $t \in \mathbb{R}_+^*$ sufficiently small we have $x_1 - tx \in K_1$ and thus $z = (x_0+tx) + (x_1-tx)$ with $x_0+tx \in K_0$, $x_1-tx \in K_1$.

Observe finally that K_1 is almost Archimedian in X_1. Indeed if not that there would exist some element $u \in X_1 - \{0\}$ such that for some $v \in K_1$ it would hold $u+tv \in K_1$ and $tv-u \in K_1$ for each $t \in \mathbb{R}_+^*$. But this means that $-tv \leq u \leq tv$ for each $t \in \mathbb{R}_+^*$ and thus $u \in X_a$ contradicting the construction of X_1.▸

Examples in $\mathbb{R}^3$ show that a reprezentation as in 7.4 does not hold when K is not a core cone.

8. ε - efficient points. Let $(X,\leq)$ be an OVS and $M \subset X$. The element $b \in X$ is a *lower bound* of $M \neq \varnothing$, if $b \leq x$, $\forall\, x \in M$. Let K be the positive cone in $(X,\leq)$. If $(b-K) \cap M \neq \varnothing$

for some $y \in X$, then this set is called a *section* of M.

The point $x \in M$ is an *efficient point* of M if it holds $(x\text{-}K) \cap M = \{x\}$. Let be $\varepsilon \in K \setminus \{0\}$. The point $x \in M$ is called an ε-*efficient point* of M if $(x - \varepsilon - K) \cap M = \emptyset$.

8.1. *Every non empty lower bounded set in $(X,\leq)$ possesses ε-efficient points if and only if every set with lower bounded sections possesses ε-efficient points.*

8.2. *Let K be the positive cone of $(X,\leq)$ and let K_a be its anti Archimedian subcone. Let be $\varepsilon \in K \setminus \{0\}$. Then every non empty lower bounded set in X possesses ε-efficient points, if and only if $\varepsilon \notin K_a$.*

◃ Indeed, let M be lower bounded and suppose that it has no ε-efficient point. Then for arbitrary $x_0 \in M$ we must have $(x_0 - \varepsilon - K) \cap M \neq \emptyset$. Choose $x_1 \in (x_0 - \varepsilon - K) \cap M$, $x_2 \in (x_1 - \varepsilon - K) \cap M$ and in general $x_n \in (x_{n-1} - \varepsilon\text{-}K) \cap M$. Let b be a lower bound of M. Then $x_n \geq b$, $\forall\, n \in \mathbb{N}$, and since $x_n \in x_{n-1} - \varepsilon\, K$ it follows that we have $x_n = x_{n-1} - \varepsilon - k_n$ for some $k_n \in K$. Hence

$$\varepsilon \leq \varepsilon + k_n = x_{n-1} - x_n.$$

By summing the above type relations from $n = 1$ to $m \in \mathbb{N}$ we obtain:

$$m\, \varepsilon \leq x_0 - x_m \leq x_0 - b.$$

Thus $m\, \varepsilon \leq x_0 - b$, $\forall\, m \in \mathbb{N}$ and $x_0 - b \in K$ which shows that $\varepsilon \in K_a$, contradicting our assumption on ε. ▹

8.3. *The following two conditions are equivalent:*

(i) *Every non empty lower bounded set possesses in (X,K) ε-efficient points for every $\varepsilon \in K \setminus \{0\}$.*

(ii) *(X,K) is semi Archimedian.*

◃ According 8.2, (i) holds if and only if $K_a = \{0\}$, condition which characterizes the semi Archimedian cones. ▹

Remarks. The assertion 8.3 and related questions in ordered topological vector spaces occur in [4].

Fare going generalization of the idea in 8.2 for more general order relations is in fact the supporting functional method developed by P. Wiedner and Chr. Tammer in [1].

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