High-Order Compact Schemes and Incremental Unknowns for 1D Convection Diffusion Problems *

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Abstract

Induced hierarchical preconditioner are investigated for linear 1D convection-diffusion problems. In the context of high-order compact schemes we give a priori estimates and a uniform estimate of the condition number.

1 Introduction

The utilization of incremental unknowns (IU in short) with multilevel finite differences was proposed by R.Temam in [13] for the integration of elliptic partial differential equations, instead of the usual nodal unknowns. The idea, which stems from dynamical systems approach, consists in writing the approximate solution u_i in the form $u_i = y_i + z_i$, where z is a small increment. Passing from the nodal unknowns u_i to the IUs (y_i, z_i) amounts to a linear change of variables, that is to say, in the language of linear algebra to the construction of a preconditioner. Many numerical simulations have shown the efficiency of such induced preconditioners.

The incremental unknowns play the role of the small structures. Unfortunately they are not always "small", in certain practical cases. Indeed this situation arises typically when the number of points is not large enough and the discretized function has strong gradients (this is the case of convection diffusion problems). A small step size (or a large number of grid points) can be a limitation for practical applications since the dimension of the system will be increased artificially. The use of high-order schemes can be a solution to the above problem. Exist two main classes of high-order schemes: *explicit schemes* and *compact schemes*. Explicit schemes directly compute the numerical derivative by employing large computational stencils for accuracy. High-order schemes achieved in this manner always require non-compact stencils that utilize grid points located beyond those directly adjacent to the node about which the differences are taken. Compact scheme, proposed by Kreiss and Oliger[6] and which was later improved upon by Lele[7], use smaller stencils but requires a scalar tridiagonal or pentadiagonal matrix inversion. Another idea (proposed,

*2000 MSC Subject Classification: 65L10, 65L12, 65F35.

for convection diffusion problems, by MacKinnon, Carey, Johnson and Langerman [8],[9],[10],[11]) to obtain high-order compact schemes is to operate on the differential equation as an auxiliary relation to obtain expressions for higher-order derivatives in the truncation error. We will use this idea (also presented in [12]) to approximate the solution of the following (1D) convection-diffusion equation:

$$\begin{cases} -u''(x) + c(x)u'(x) = f(x), \; x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $f \in L^2(0,1)$ and $c(x) \in L^\infty(0,1)$.

This equation often appears in the description of transport phenomena. The magnitude of c(x) determines the ratio of the convection to diffusion. Of course the 1D problems are not difficult from a computational point of view; we consider them because the algebra is more transparent than for higher dimensions.

Numerical solution of a problem such as (1) using Incremental Unknowns has been considered in [3] and [4] but the IUs that have been used in these articles were connected to the second derivative only.

In [1], the authors propose a construction of IUs that are more adapted to the problem in the sense that they take into account and the convection term in the construction of the IUs.

We consider approximations to (1) on a uniform mesh $x_i = ih$ ($h = 1/2N$, N a positive integer, $i = 0, 1, ..., 2N$) having the form

$$\beta_i(u_i - u_{i-1}) + \alpha_i(u_i - u_{i+1}) = hf_i; \; i = 1, ..., 2N. \tag{2}$$

Here $u_i \simeq u(x_i)$, $f_i = f(x_i)$ and α_i and β_i are real numbers whose definition depends on the discretization scheme; we have indeed, for example the two following possibilities: the firs one consists in approaching the first derivative by a (centered type) three points scheme and the second one consists in using an uncentered type scheme for the approximation of u' at the grid points. For the moment we do not explicite the discretization scheme.

As usual when an IU method is implemented, two different kinds of unknowns must be distinguished: those associated with the coarse grid components and which are on G_c (coarse grid), and whose indices are even and those associated with complementary points (odd indices) which are on $G_f \backslash G_c$ (G_f is the fine grid),

$$. \; o \; \times \; o \; \times \; o \; . \; . \; . \; o \; \times \; o \; .$$

$Fig.1: \; \Omega = (0,1), \; x: \; points \; in \; G_c, \; o: \; points \; in \; G_f \backslash G_c,$

If we write the system (2) at the complementary points, we obtain

$$(\alpha_{2i+1} + \beta_{2i+1})u_{2i+1} - \beta_{2i+1}u_{2i} - \alpha_{2i+1}u_{2i+2} = hf_{2i+1}.$$

Hence, assuming that $\alpha_{2i+1} + \beta_{2i+1} \neq 0$, we have

$$u_{2i+1} = \frac{1}{\alpha_{2i+1} + \beta_{2i+1}}(\beta_{2i+1}u_{2i} + \alpha_{2i+1}u_{2i+2}) + \frac{1}{\alpha_{2i+1} + \beta_{2i+1}}hf_{2i+1}.$$

We note that u_{2i+1} is expressed as the sum of a convex combination of u_{2i} and u_{2i+2} which is nothing but a bilinear interpolation scheme, and a correction term whose order is connected to the order of the interpolation scheme. If we set

$$z_{2i+1} = u_{2i+1} - \frac{1}{\alpha_{2i+1} + \beta_{2i+1}} (\beta_{2i+1} u_{2i} + \alpha_{2i+1} u_{2i+2}) \tag{3}$$

then the system (1), at the complementary points, is reduced to

$$z_{2i+1} = \frac{1}{\alpha_{2i+1} + \beta_{2i+1}} h f_{2i+1}.$$

The numbers z_{2i+1} are the incremental unknowns attached to the (discrete) problem (2). They depend closely on the scheme used for the discretization.

We can obviously repeat recursively the process described above and if the coarsest grid is reduced to one point only, the preconditioned matrix becomes diagonal.

From the point of view of the matriceal framework, this construction can be summarized by the determination of two matrices S and ${}^tT$ under and upper triangular respectively such that ${}^tTAS$ is bloc diagonal, A being the discretization matrix.

We first consider two grid levels. The discretization matrix A is written with the hierarchical ordering (considering first the coarse grid unknowns and then the complementary ones) in the form

$$\widetilde{A} = \begin{pmatrix} \Lambda_1 & B_1 \\ B_2 & \Lambda_2 \end{pmatrix},$$

where Λ_i, $i = 1, 2$ are invertible diagonal matrices.

Construction of S

We want to construct a matrix S of the form:

$$S = \begin{pmatrix} I & 0 \\ G_1 & I \end{pmatrix},$$

and such that $\widetilde{A}S$ is upper triangular. We have

$$\widetilde{A}S = \begin{pmatrix} \Lambda_1 & B_1 \\ B_2 & \Lambda_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ G_1 & I \end{pmatrix} = \begin{pmatrix} \Lambda_1 + B_1 G_1 & B_1 \\ B_2 + \Lambda_2 G_1 & \Lambda_2 \end{pmatrix}.$$

Therefore the under-matrix G_1 satisfies $G_1 = -\Lambda_2^{-1} B_2$, hence

$$S = \begin{pmatrix} I & 0 \\ -\Lambda_2^{-1} B_2 & I \end{pmatrix}.$$

Construction of ${}^tT$

We now want to construct a matrix ${}^tT$ of the form:

$${}^tT = \begin{pmatrix} I & G_2 \\ 0 & I \end{pmatrix},$$

and such that ${}^t T\widetilde{A}S$ is bloc diagonal. We have

$$\hat{A} ={}^t T\widetilde{A}S = \begin{pmatrix} \Lambda_1 + B_1 G_1 & B_1 + G_2\Lambda_2 \\ 0 & \Lambda_2 \end{pmatrix},$$

and then G_2 must satisfy $B_1 + G_2\Lambda_2 = 0$.

Thus

$${}^t T = \begin{pmatrix} I & -B_1\Lambda_2^{-1} \\ 0 & I \end{pmatrix},$$

and then $\hat{A}$ can be written in the form

$$\hat{A} ={}^t T\widetilde{A}S = \begin{pmatrix} \Lambda_1 + B_1 G_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}.$$

We note that since the linear system is non-symmetric, these IUs lead to a non-symmetric hierarchical preconditioner.

The first diagonal bloc of $\hat{A}$ is still tridiagonal and we can repeat recursively the reduction procedure described above by using d levels of IUs.

2 A Priori Estimates

We can obtain (see [1])a priori energy estimates that show that the (induced) IUs are indeed small structures as expected.

We multiply (2) by $\delta_i u_i$, δ_i to be fixed later, and summing these expressions on all indices i, we obtain

$$\sum_{i=1}^{2N-1} \beta_i\delta_i(u_i - u_{i-1})u_i - \sum_{i=1}^{2N-1} \alpha_i\delta_i(u_{i+1} - u_i)u_i = \sum_{i=1}^{2N-1} hf_i\delta_i u_i.$$

Hence,

$$\sum_{i=0}^{2N-1} (\beta_{i+1}\delta_{i+1}u_{i+1} - \alpha_i\delta_i u_i)(u_{i+1} - u_i) = \sum_{i=1}^{2N-1} hf_i\delta_i u_i. \tag{4}$$

We now choose δ_i such that

$$\delta_0 = 1,\ \delta_{i+1} = \frac{\alpha_i}{\beta_{i+1}}\delta_i,$$

and we make the following hypothesis [1]:

Hypothesis(H)

i. There exists two strictly positive real numbers α and β such that

$$\alpha \leq \delta_i \leq \beta \ \forall i.$$

ii. α_i and β_i are strictly positive numbers, $\forall i = 0, ..., 2N$.

iii. There exists a strictly positive real number γ such that

$$\alpha_i \geq \frac{\gamma}{h} \; \forall i.$$

We shall see later that (H) is not too restrictive for practical cases.
From (4) we infer

$$\sum_{i=0}^{2N-1} \alpha_i \delta_i (u_{i+1} - u_i)^2 = \sum_{i=1}^{2N-1} h f_i \delta_i u_i. \tag{5}$$

We recall the discrete Poincaré inequality (see e.g. [2]):

Lemma 1 *Let $u_i, i = 0, ..., 2N$ be a sequence of real numbers such that $u_0 = u_{2N} = 0$. Then we have*

$$\sum_{i=1}^{2N-1} h u_i^2 \leq \sum_{i=0}^{2N-1} \frac{1}{h} (u_{i+1} - u_i)^2$$

Using (H), Cauchy-Schwartz inequality and Poincaré inequality, we bound the right-hand side of (5) by

$$\sum_{i=1}^{2N-1} h f_i \delta_i u_i \leq \beta \sum_{i=1}^{2N-1} h f_i u_i \leq \beta \sqrt{\sum_{i=0}^{2N-1} h u_i^2} \sqrt{\sum_{i=1}^{2N-1} h f_i^2} \leq \beta \sqrt{\sum_{i=0}^{2N-1} \frac{1}{h} (u_{i+1} - u_i)^2} \sqrt{\sum_{i=1}^{2N-1} h f_i^2}. \tag{6}$$

Also, using (H) we can bound the left-hand side of (5) by

$$\sum_{i=0}^{2N-1} \alpha_i \delta_i (u_{i+1} - u_i)^2 \geq \alpha \sum_{i=0}^{2N-1} \alpha_i (u_{i+1} - u_i)^2 \geq \alpha\gamma \sum_{i=0}^{2N-1} \frac{1}{h} (u_{i+1} - u_i)^2. \tag{7}$$

From (6) and (7) we have:

$$\sum_{i=0}^{2N-1} \frac{1}{h} (u_{i+1} - u_i)^2 \leq \frac{\beta^2}{\alpha^2 \gamma^2} \sum_{i=1}^{2N-1} h f_i^2. \tag{8}$$

We now introduce the incremental unknowns defined by (3). Since

$$\sum_{i=0}^{2N-1} \frac{1}{h} (u_{i+1} - u_i)^2 = \sum_{i=0}^{N-1} \frac{1}{h} (u_{2i+1} - u_{2i})^2 + \sum_{i=0}^{N-1} \frac{1}{h} (u_{2i+2} - u_{2i+1})^2,$$

we find, setting $y_{2i} = u_{2i}$

$$\sum_{i=0}^{N-1} \frac{1}{h} \left(z_{2i+1} + \frac{1}{\alpha_{2i+1} + \beta_{2i+1}} (\beta_{2i+1} y_{2i} + \alpha_{2i+1} y_{2i+2}) - y_{2i} \right)^2$$

$$+\sum_{i=0}^{N-1}\frac{1}{h}\left(y_{2i+2}-z_{2i+1}-\frac{1}{\alpha_{2i+1}+\beta_{2i+1}}(\beta_{2i+1}y_{2i}+\alpha_{2i+1}y_{2i+2})\right)^2\leq\frac{\beta^2}{\alpha^2\gamma^2}\sum_{i=1}^{2N-1}hf_i^2,$$

which yields

$$\sum_{i=0}^{N-1}\left(z_{2i+1}+\frac{\alpha_{2i+1}}{\alpha_{2i+1}+\beta_{2i+1}}(y_{2i+2}-y_{2i})\right)^2+$$

$$+\sum_{i=0}^{N-1}\left(z_{2i+1}-\frac{\beta_{2i+1}}{\alpha_{2i+1}+\beta_{2i+1}}(y_{2i+2}-y_{2i})\right)^2\leq\frac{\beta^2}{\alpha^2\gamma^2}\sum_{i=1}^{2N-1}h^2f_i^2.$$

Replacing $f_i=f(x_i)$ by $f_i=\frac{1}{2h}\int_{x_{i-1}}^{x_{i+1}}f(x)dx$ and using Hölder inequality we have:

$$\sum_{i=0}^{N-1}\left(z_{2i+1}+\frac{\alpha_{2i+1}}{\alpha_{2i+1}+\beta_{2i+1}}(y_{2i+2}-y_{2i})\right)^2+$$

$$+\sum_{i=0}^{N-1}\left(z_{2i+1}-\frac{\beta_{2i+1}}{\alpha_{2i+1}+\beta_{2i+1}}(y_{2i+2}-y_{2i})\right)^2\leq\frac{\beta^2}{4\alpha^2\gamma^2}\sum_{i=1}^{2N-1}\left[\int_{x_{i-1}}^{x_{i+1}}f(x)dx\right]^2\leq$$

$$\leq\frac{\beta^2}{4\alpha^2\gamma^2}\sum_{i=1}^{2N-1}\left(\int_{x_{i-1}}^{x_{i+1}}f^2(x)dx\right)\left(\int_{x_{i-1}}^{x_{i+1}}dx\right)=\frac{\beta^2}{2\alpha^2\gamma^2}h\sum_{i=1}^{2N-1}\left(\int_{x_{i-1}}^{x_{i+1}}f^2(x)dx\right)\leq\frac{\beta^2}{\alpha^2\gamma^2}h\left\|f\right\|^2$$

where $\|f\|=|f|_{L^2(0,1)}$.

We now develop the left-hand side of this inequality and we obtain:

$$A=2\sum_{i=0}^{N-1}z_{2i+1}^2+\sum_{i=0}^{N-1}\frac{\alpha_{2i+1}^2+\beta_{2i+1}^2}{(\alpha_{2i+1}+\beta_{2i+1})^2}(y_{2i+2}-y_{2i})^2+2\sum_{i=0}^{N-1}\frac{\alpha_{2i+1}-\beta_{2i+1}}{\alpha_{2i+1}+\beta_{2i+1}}z_{2i+1}(y_{2i+2}-y_{2i}).$$

Using Young inequality, we then find

$$2\sum_{i=0}^{N-1}\frac{\alpha_{2i+1}-\beta_{2i+1}}{\alpha_{2i+1}+\beta_{2i+1}}z_{2i+1}(y_{2i+2}-y_{2i})\geq-\varepsilon\sum_{i=0}^{N-1}z_{2i+1}^2-\frac{1}{\varepsilon}\sum_{i=0}^{N-1}\frac{(\alpha_{2i+1}-\beta_{2i+1})^2}{(\alpha_{2i+1}+\beta_{2i+1})^2}(y_{2i+2}-y_{2i})^2.$$

Here ε is a strictly positive real number which will be fixed later. Hence

$$A\geq(2-\varepsilon)\sum_{i=0}^{N-1}z_{2i+1}^2+\sum_{i=0}^{N-1}\left(\frac{\alpha_{2i+1}^2+\beta_{2i+1}^2}{(\alpha_{2i+1}+\beta_{2i+1})^2}-\frac{1}{\varepsilon}\frac{(\alpha_{2i+1}-\beta_{2i+1})^2}{(\alpha_{2i+1}+\beta_{2i+1})^2}\right)(y_{2i+2}-y_{2i})^2.$$

Since $\alpha_i>0$ and $\beta_i>0$, $\forall i$, we have

$$\frac{1}{(\alpha_{2i+1}+\beta_{2i+1})^2}\left(\alpha_{2i+1}^2+\beta_{2i+1}^2-\frac{1}{\varepsilon}(\alpha_{2i+1}-\beta_{2i+1})^2\right)\geq\left(1-\frac{1}{\varepsilon}\right)\frac{\alpha_{2i+1}^2+\beta_{2i+1}^2}{(\alpha_{2i+1}+\beta_{2i+1})^2}.$$

We set $\xi_i=\frac{\alpha_i}{\beta_i}$ and we introduce the function $\phi(\xi)=\frac{1+\xi^2}{(1+\xi)^2}$ for $\xi>0$. It is easy to check that $\phi(\xi)\geq\frac{1}{2}$, $\forall\xi>0$. Hence

$$\frac{\alpha_{2i+1}^2+\beta_{2i+1}^2}{(\alpha_{2i+1}+\beta_{2i+1})^2}=\phi(\xi_{2i+1})\geq\frac{1}{2},\ \forall i=0,...,N,$$

and we then have

$$\frac{1}{(\alpha_{2i+1}+\beta_{2i+1})^2}\left(\alpha_{2i+1}^2+\beta_{2i+1}^2-\frac{1}{\varepsilon}(\alpha_{2i+1}-\beta_{2i+1})^2\right)\geq\frac{1}{2}\left(1-\frac{1}{\varepsilon}\right).$$

Thus, taking $\varepsilon=\frac{3}{2}$, we obtain, after some simplifications

$$A\geq\frac{1}{6}\sum_{i=0}^{N-1}z_{2i+1}^2+\frac{1}{6}\sum_{i=0}^{N-1}(y_{2i+2}-y_{2i})^2.$$

We have then the following result:

Proposition 2 *[1]Under the hypothesis (H), the incremental unknowns defined by (3) satisfy the following a priori estimates:*

$$\sum_{i=0}^{N-1}z_{2i+1}^2\leq\frac{6\beta^2}{\alpha^2\beta^2}h,$$

$$\sum_{i=0}^{N-1}(y_{2i+2}-y_{2i})^2\leq\frac{6\beta^2}{\alpha^2\beta^2}h.$$

Proposition is valid for general definition of the IUs given in (3), under Hypothesis (H). In particular, the scheme used for the discretization of the convective term is not specified.

i. Centered Convection-Diffusion IUs

We have

$$\alpha_i=\frac{1}{h}\left(1-\frac{c_ih}{2}\right),\ \ \beta_i=\frac{1}{h}\left(1+\frac{c_ih}{2}\right).$$

Hence, if there exists a strictly positive real number γ such that

$$1-\max_i\frac{|c_i|\,h}{2}\geq\gamma,$$

then the asumptions i., ii. and iii. of (H) are satisfied. This condition can be satisfied by *taking h small enough.*

ii. Uncentered Convection-Diffusion IUs

• If $c_i\geq 0$ then

$$\alpha_i=\alpha_i^+=\frac{1}{h},\ \ \beta_i=\beta_i^+=\frac{1}{h}+c_i.$$

Hence, α_i and β_i are stricly positive real numbers.

• If $c_i\leq 0$ then

$$\alpha_i=\alpha_i^-=\frac{1}{h}-c_i,\ \ \beta_i=\beta_i^-=\frac{1}{h}.$$

Here again, α_i and β_i are strictly positive real numbers.

In conclusion, and without any asumption, the asumptions i., ii. and iii. of (H) are satisfied.

3 High-order compact schemes (HOC)

The real advantage of (HOC) lies not in the increased accuracy (although this is sometimes important), but rather in the fact that problems requiring fine grid can be solved on coarse grids. This implies that less memory and thus less time is required to solve the same problem to the same accuracy. Since "time is money", (HOC) schemes directly reduce the expense of approximating a differential equation numerically.

High-order compact (HOC) schemes of the type studied here increase the accuracy of the standard central difference approximation from $O(h^2)$ to $O(h^4)$ by including compact approximation to the leading truncation error terms. The idea is to operate on the differential equation as an auxiliary relation to obtain expressions for higher-order derivatives in the truncation error.

We define $\delta_x^n u_i$, $n = 1, 2$ to be the standard central difference operator for the $n-th$ derivative of u at point i on a uniform grid of mesh size h,

$$\delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2h}, \ \delta_x^2 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

Central differencing (1) yields

$$-\delta_x^2 u_i + c_i \delta_x u_i - \tau_i = f_i, \tag{9}$$

where τ_i is the local truncation error at node i,

$$\tau_i = \frac{h^2}{12}\left[2c\frac{d^3u}{dx^3} - \frac{d^4u}{dx^4}\right]_i + O(h^4). \tag{10}$$

We seek to approximate the leading term in (10) and include it in the difference formulation to yield an $O(h^4)$ method. Assuming the solution is sufficiently regular, we may accomplish this by differentiating (1) to yield

$$\left.\frac{d^3u}{dx^3}\right|_i = \left[c\frac{d^2u}{dx^2} + \frac{dc}{dx}\frac{du}{dx} - \frac{df}{dx}\right]_i,$$

which can be approximated compactly as

$$\left.\frac{d^3u}{dx^3}\right|_i = c_i\delta_x^2 u_i + \delta_x c_i \delta_x u_i - \delta_x f_i + O(h^2), \tag{11}$$

and similarly

$$\left.\frac{d^4u}{dx^4}\right|_i = \left[c\frac{d^3u}{dx^3} + 2\frac{dc}{dx}\frac{d^2u}{dx^2} + \frac{d^2c}{dx^2}\frac{du}{dx} - \frac{d^2f}{dx^2}\right]_i, \tag{12}$$

$$= c_i \left.\frac{d^3u}{dx^3}\right|_i + 2\delta_x c_i \delta_x^2 u_i + \delta_x^2 c_i \delta_x c_i - \delta_x^2 f_i + O(h^2).$$

Relations (11) and (12) can be combined with (10) to yield the new truncation error expression:

$$\tau_i = \frac{h^2}{12}\left[(c_i^2 - 2\delta_x c_i)\delta_x^2 u_i + (c_i\delta_x c_i - \delta_x^2 c_i)\delta_x u_i - c_i\delta_x f_i + \delta_x^2 f_i\right] + O(h^4),$$

which can be use to increase the accuracy of the approximation scheme. The resulting high-order compact scheme is

$$-A_i\delta_x^2 u_i + C_i\delta_x u_i = F_i + O(h^4), \tag{13}$$

where

$$A_i = 1 + \frac{h^2}{12}(c_i^2 - 2\delta_x c_i), \tag{14}$$

$$C_i = c_i + \frac{h^2}{12}(\delta_x^2 c_i - c_i\delta_x c_i), \tag{15}$$

$$F_i = f_i + \frac{h^2}{12}(\delta_x^2 f_i - c_i\delta_x f_i). \tag{16}$$

We can write (13) in the form (2) where:

$$\alpha_i = \frac{A_i}{h} - \frac{C_i}{2},\ \beta_i = \frac{A_i}{h} + \frac{C_i}{2}\ ,\ f_i := F_i \text{ for } i = 1, ..., 2N,$$

and we can define IU in this case. If we consider 1D convection diffusion problem with constant coefficients then:

$$A_i = 1 + \frac{h^2c^2}{12},\ C_i = c \text{ and } F_i = f_i \text{ for } i = 1, ..., 2N.$$

In this case we can see that, without any asumptions, the hypothesis (H) are satisfied.

4 Condition Number Analysis

The condition number is defined by

$$\kappa = \frac{\max|\lambda_k|}{\min|\lambda_k|}, \tag{17}$$

where $\{\lambda_k\}$ is the set of eigenvalues of the real matrix we wish to solve.

The tridiagonal matrices generated by compact 1D numerical schemes lend themselves to theoretical eigenvalue analysis, and our hope is that we can draw some conclusions from such analysis that might apply to higher dimensions.

We will consider 1D convection diffusion problem with constant coefficients. For a tridiagonal Toeplitz matrix which has the form

$$T_N(c,d,e) = \begin{pmatrix} d & e & & & & \\ c & d & \cdot & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & e \\ & & & & c & d \end{pmatrix},$$

the eigenvalues are known explicitly (see [5]),

$$d + 2(ce)^{1/2}\cos(k\pi/(N+1)),\ k = 1, ..., N.$$

Using this result, we can obtain condition number for each of the 3 schemes (see [12]):

$$\kappa_{HOC} = \frac{c^2h^2 + 12 + \sqrt{(c^2h^2 + 6ch + 12)(c^2h^2 - 6ch + 12)}}{c^2h^2 + 12 - \sqrt{(c^2h^2 + 6ch + 12)(c^2h^2 - 6ch + 12)}}, \tag{18}$$

$$\kappa_{CDS} = \begin{cases} \frac{2+\sqrt{4-c^2h^2}}{2+\sqrt{4-c^2h^2}},\ |ch| \le 2 \\ \frac{|ch|}{2},\ |ch| > 2 \end{cases}, \tag{19}$$

$$\kappa_{CDS} = \begin{cases} \frac{2+\sqrt{4-c^2h^2}}{2+\sqrt{4-c^2h^2}},\ |ch| \le 2 \\ \frac{|ch|}{2},\ |ch| > 2 \end{cases}, \tag{19}$$

$$\kappa_{UDS} = \frac{|ch| + 2 + \sqrt{|ch| + 2}}{|ch| + 2 - \sqrt{|ch| + 2}} \tag{20}$$

For HOC scheme, as $ch \to 0$, the condition number behaves as $c^{-2}h^{-2}$ and for large ch, the condition number behaves as c^2h^2. This means that for large N and $|ch|$ very large or very small, the condition number is extremely large and may pose some difficulty for an iterative solver. This is one instance where CDS and UDS appear to compare favorably with HOC, except of course for the fact that for $ch > 2$, CDS is oscillatory and for large ch, the UDS models an overly diffusive problem.

Introducing IU, after using all the levels (what means L if $h = \frac{1}{2^L}$) we will obtain a diagonal matrix which have the elements,

$$\alpha_1 = \frac{c^2h^2 + 12}{6h^2}, \alpha_n = \frac{\left(\frac{c^2h^2+6ch+12}{12h^2}\right)^{2^{n-1}} + \left(\frac{c^2h^2-6ch+12}{12h^2}\right)^{2^{n-1}}}{\prod_{i=0}^{n-2}\left[\left(\frac{c^2h^2+6ch+12}{12h^2}\right)^{2^i} + \left(\frac{c^2h^2-6ch+12}{12h^2}\right)^{2^i}\right]},\ n = 2, ..., L,$$

and $\alpha_1 > \alpha_2 > ... > \alpha_L$. In conclusion, the condition number will be:

$$\kappa_{IUHOC} = 2\frac{(12 + c^2h^2)\prod_{i=0}^{L-2}\left[\left(c^2h^2 + 6ch + 12\right)^{2^i} + \left(c^2h^2 - 6ch + 12\right)^{2^i}\right]}{\left(c^2h^2 + 6ch + 12\right)^{2^{L-1}} + \left(c^2h^2 - 6ch + 12\right)^{2^{L-1}}} \tag{21}$$

In the next table we have a comparison between condition number of HOC scheme and condition number of HOC scheme preconditioned by IU method.

	ch=2		ch=5		ch=10		ch=20	
h	c	k	c	k	c	k	c	k
$\frac{1}{8}$	16	**4.1426** 1.5960 **1.3322**	40	**3.3550** 1.4133 **1.2330**	80	**8.0960** 2.5549 **1.8357**	160	**16.215** 4.5692 **2.8620**
$\frac{1}{16}$	32	**4.6936** 1.7267 1.3718 **1.3333**	80	**3.6955** 1.4915 1.2524 **1.2333**	160	**10.639** 3.1834 2.1057 **1.8664**	320	**31.394** 8.3567 4.6964 **3.3772**
$\frac{1}{32}$	64	**4.8523** 1.7646 1.3844 1.3341 **1.3333**	160	**3.7903** 1.5135 1.2584 1.2335 **1.2333**	320	**11.526** 3.4032 2.2051 1.8887 **1.8667**	640	**40.698** 10.680 5.8448 3.8699 **3.4329**

A more acurate bound for k_{IUHOC} can be given as follows:

Proposition 3 *Let k_{IUHOC} be the condition number given by (21), where $h = 1/2^L$ $(L \in N)$. Then*

$$k_{IUHOC} \leqslant 2^{L-1} = \frac{1}{2h},$$

for each $L = 2, 3, \ldots$.

Proof. Let

$$\kappa_{IUHOC} = 2\frac{(12 + c^2h^2)\prod_{i=0}^{L-2}\left[\left(c^2h^2 + 6ch + 12\right)^{2^i} + \left(c^2h^2 - 6ch + 12\right)^{2^i}\right]}{\left(c^2h^2 + 6ch + 12\right)^{2^{L-1}} + \left(c^2h^2 - 6ch + 12\right)^{2^{L-1}}}$$

be the condition number of HOC scheme preconditioned by IU method. Let us denote

$$A = c^2h^2 + 6ch + 12, \quad B = c^2h^2 - 6ch + 12.$$

Then $A + B = 2(12 + c^2h^2)$ and with this notations we shall have

$$k_{IUHOC} = \frac{(A+B)\prod\limits_{i=0}^{L-2}(A^{2^i} + B^{2^i})}{A^{2^{L-1}} + B^{2^{L-1}}} = \frac{(A+B)^2(A^2+B^2)(A^4+B^4)\ldots(A^{2^{L-2}} + B^{2^{L-2}})}{A^{2^{L-1}} + B^{2^{L-1}}}.$$

Now, using the inequality $(x+y)^2 \leqslant 2(x^2+y^2)$, for real numbers sequentially, we get

$$k_{IUHOC} \leqslant \frac{2(A^2+B^2)^2(A^4+B^4)\ldots(A^{2^{L-2}} + B^{2^{L-2}})}{A^{2^{L-1}} + B^{2^{L-1}}} \leqslant \frac{2^2(A^4+B^4)^2\ldots(A^{2^{L-2}} + B^{2^{L-2}})}{A^{2^{L-1}} + B^{2^{L-1}}} \leqslant \ldots$$

$$\ldots \leqslant \frac{2^{L-2}(A^{2^{L-2}} + B^{2^{L-2}})^2}{A^{2^{L-1}} + B^{2^{L-1}}} \leqslant \frac{2^{L-1}(A^{2^{L-1}} + B^{2^{L-1}})}{A^{2^{L-1}} + B^{2^{L-1}}} = 2^{L-1}.$$

■

Remark 4 *We notice that the upper bound of the condition number does not depent of the equation's coefficient c. From this point of view we can say that we have a uniform condition number estimate.*

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