

AN EXTREMAL PROBLEM
FOR THE TRANSFINITE DIAMETER OF A CONTINUUM

by

PETRU T. MOCANU and DUMITRU RİPEANU

Cluj-Napoca

In this paper we solve an extremal problem connected with the transfinite diameter of a continuum by using SCHIFFER's variational method [2] and also the same simple geometric arguments as described in the paper of REICH and SCHIFFER [1]. As the matter of fact, our problem is quite similar to those solved in [1].

Let

$$(1) \quad \Phi(c_1, c_2, c_3) = |c_1 - c_2| + |c_2 - c_3| + |c_3 - c_1|,$$

where c_1, c_2, c_3 are complex numbers. It is obvious that the function (1), which represents the perimeter of the triangle (c_1, c_2, c_3) , is invariant under translations and rotations of the plane.

Let E be a continuum in the complex plane, and let c_1, c_2, c_3 be three arbitrary points belonging to E . Our problem is to find

$$(2) \quad \sup_{c_1, c_2, c_3 \in E} \frac{\Phi(c_1, c_2, c_3)}{d(E)},$$

where $d(E)$ is the transfinite diameter of E .

The result is the following:

THEOREM. If E is a continuum in the plane and c_1, c_2, c_3 belong to E , then

$$(3) \quad |c_1 - c_2| + |c_2 - c_3| + |c_3 - c_1| \leq 3^{3/2} 4^{1/3} d(E).$$

This inequality is sharp, equality being achieved if and only if E is the union of three segments of equal length making angles of $2\pi/3$ with each other, having a common initial point, and c_1, c_2, c_3 as endpoints.

Proof. It is well-known that if c_1, c_2 belong to E , then $|c_1 - c_2| \leq 4d(E)$, [3]. Hence $\Phi(c_1, c_2, c_3)/d(E) \leq 12$, which shows that (2) exists and is assumed. Let E, c_1, c_2, c_3 be extremal for (2), and let D be the complementary domain of E which contains the point at infinity. Consider the conformal mapping

$$(4) \quad w = f(z) = d(E) \left[z + a_0 + \frac{a_1}{z} + \dots \right],$$

of $1 < |z| < \infty$ onto D .

For $w_0 \in D$, we consider the variation

$$(5) \quad w^* = w + \frac{\lambda}{w - w_0},$$

where $|\lambda|$ is sufficiently small. Denote by c_1^*, c_2^*, c_3^*, E^* the images of c_1, c_2, c_3, E by (5).

If we set $\Phi = \Phi(c_1, c_2, c_3)$ and $\Phi^* = \Phi(c_1^*, c_2^*, c_3^*)$ then we have the following variational formulas

$$(6) \quad \log \Phi^* = \log \Phi - \operatorname{Re} \left\{ \frac{\lambda}{\Phi} A(c_1, c_2, c_3; w_0) \right\} + o(\lambda),$$

where

$$A(c_1, c_2, c_3; w_0) = \frac{|c_1 - c_2|}{(c_1 - w_0)(c_2 - w_0)} + \frac{|c_2 - c_3|}{(c_2 - w_0)(c_3 - w_0)} + \frac{|c_3 - c_1|}{(c_3 - w_0)(c_1 - w_0)}$$

and

$$(7) \quad \log d(E^*) = \log d(E) - \operatorname{Re} \left\{ \frac{\lambda}{z_0^2 f'(z_0)^2} \right\} + o(\lambda),$$

with $w_0 = f(z_0)$, [2].

Since $\Phi^*/d(E^*) \leq \Phi/d(E)$, we have

$$\log \Phi^* - \log d(E^*) \leq \log \Phi - \log d(E)$$

and by using (6) and (7) we obtain

$$\operatorname{Re} \left\{ \frac{\lambda}{z_0^2 f'(z_0)^2} - \frac{\lambda}{\Phi} A(c_1, c_2, c_3; w_0) \right\} + o(\lambda) \leq 0,$$

for all small enough values of $|\lambda|$. From this we conclude that if E, c_1, c_2, c_3 are extremal for (2), then the extremal function (4) satisfies the differential equation

$$(8) \quad \frac{w - \psi(c_1, c_2, c_3)}{(w - c_1)(w - c_2)(w - c_3)} = \frac{(dz)^2}{z^2(dw)^2},$$

where

$$\psi(c_1, c_2, c_3) = \frac{1}{\Phi} [c_1|c_2 - c_3| + c_2|c_3 - c_1| + c_3|c_1 - c_2|].$$

Since

$$\psi(c_1 + a, c_2 + a, c_3 + a) = \psi(c_1, c_2, c_3) + a,$$

and the extremal points are determined within an additive constant, we can suppose $\psi(c_1, c_2, c_3) = 0$, i.e.

$$(9) \quad c_1|c_2 - c_3| + c_2|c_3 - c_1| + c_3|c_1 - c_2| = 0.$$

The differential equation (8) becomes

$$(10) \quad \frac{w(dw)^2}{(w - c_1)(w - c_2)(w - c_3)} = \frac{(dz)^2}{z^2}, \quad |z| > 1.$$

As in [1] it is easy to show that the extremum continuum E is the set of values omitted by the extremal function f , and the range D of f has no exterior points.

The extremal points c_k are distinct from each other and distinct from 0. Indeed, if $c_1 = 0$, then from (9) we deduce $c_2|c_3| = -c_3|c_2|$ and we have $\Phi \leq 2 \max\{|c_2|, |c_3|\}$, and $\Phi/d(E) \leq 8$. If $c_1 = c_2 = a, c_3 = b$, then $\Phi = 2|b - a| \leq 8d(E)$. In each case the value of $\Phi/d(E)$ is not extremal. We remark that the extremal points c_1, c_2, c_3 can not be collinear, since in this case we also have $\Phi = 2|c_1 - c_2| \leq 8d(E)$, if we suppose that c_2 lies between c_1 and c_3 .

Since Φ is invariant under rotations we can suppose $c_1 > 0$.

The extremum continuum E consists of the union of three analytic arcs $\gamma_k, k = 1, 2, 3$, having 0 as the only common initial point and c_1, c_2, c_3 as endpoints. The three arcs γ_k meet 0 in equally spaced angles [1]. Using the same topological argument as in [1], we conclude that there exist numbers $t_k, 0 < t_k < 1$, such that

$$(11) \quad \operatorname{Im} \frac{c_{k+1} + c_{k+2}}{c_k} t_k = \operatorname{Im} \frac{c_{k+1} c_{k+2}}{c_k^2}, \quad k = 1, 2, 3,$$

where we denote $c_4 = c_1, c_5 = c_2$.

From (9) we obtain

$$\operatorname{Im} \frac{c_{k+2}}{c_k} = - \frac{|c_{k+2} - c_k|}{|c_{k+1} - c_k|} \operatorname{Im} \frac{c_{k+1}}{c_k}$$

and

$$\operatorname{Re} \frac{c_{k+2}}{c_k} = - \frac{|c_{k+2} - c_{k+1}|}{|c_{k+1} - c_k|} - \frac{|c_{k+2} - c_k|}{|c_{k+1} - c_k|} \operatorname{Re} \frac{c_{k+1}}{c_k}.$$

Hence

$$(12) \quad \operatorname{Im} \frac{c_{k+1} + c_{k+2}}{c_k} = \frac{|c_{k+1} - c_k| + |c_{k+2} - c_k|}{|c_{k+1} - c_k|} \operatorname{Im} \frac{c_{k+1}}{c_k}$$

$$(13) \quad \operatorname{Im} \frac{c_{k+1} c_{k+2}}{c_k^2} = - \left[\frac{|c_{k+2} - c_{k+1}|}{|c_{k+1} - c_k|} + 2 \frac{|c_{k+2} - c_k|}{|c_{k+1} - c_k|} \operatorname{Re} \frac{c_{k+1}}{c_k} \right] \operatorname{Im} \frac{c_{k+1}}{c_k}$$

Using (9), (11), (12) and (13) we find that the extremal points c_1, c_2, c_3 satisfy the following conditions

$$(14) \quad c_1 |c_2 - c_1| + c_2 |c_3 - c_1| + c_3 |c_1 - c_2| = 0,$$

$$(15) \quad (|c_3 - c_1| - |c_2 - c_1|) t_1 = |c_3 - c_2| + 2 |c_3 - c_1| \operatorname{Re} \frac{c_2}{c_1},$$

$$(16) \quad (|c_1 - c_2| - |c_3 - c_2|) t_2 = |c_1 - c_3| + 2 |c_1 - c_2| \operatorname{Re} \frac{c_3}{c_2},$$

$$(17) \quad (|c_3 - c_3| - |c_1 - c_3|) t_3 = |c_2 - c_1| + 2 |c_2 - c_3| \operatorname{Re} \frac{c_1}{c_3},$$

where $c_1 > 0$ and $t_k \in (0, 1)$.

We shall show that this conditions imply

$$(18) \quad |c_1 - c_2| = |c_2 - c_3| = |c_3 - c_1|.$$

If we let

$$(19) \quad c_2 - c_1 = d = re^{it}, \quad c_3 - c_1 = \delta = \rho e^{i\tau},$$

condition (14) becomes

$$(20) \quad |c_3 - c_2| = |\delta - d| = - \left[r + \rho + \frac{r\rho}{c_1} (e^{it} + e^{i\tau}) \right].$$

From (15) we obtain

$$(21) \quad (r - \rho)(1 - t_1) = \frac{r\rho}{c_1} [\cos t - \cos \tau - i (\sin t + \sin \tau)].$$

If $r\rho = 0$, then from (21) we deduce $r = \rho = 0$ and from (19) we get the trivial solution $c_1 = c_2 = c_3$ which is not possible. Thus $r\rho > 0$ and from (21) we obtain $\sin t + \sin \tau = 0$, which implies $\cos t = \pm \cos \tau$. Suppose $\cos \tau = -\cos t$. Then (20) becomes $|\delta - d| = -(r + \rho)$, that is, $r = \rho = 0$ which is not possible. Therefore we have only the case $\cos t - \cos \tau = \sin t + \sin \tau = 0$ and from (21) we obtain $r = \rho$. From (19) we deduce

$$(22) \quad c_2 = c_1 + re^{it}, \quad c_3 = c_1 + re^{-it}.$$

Employing (22) together with (16) and (17), we obtain

$$(23) \quad ax^2 + bx + c = 0, \quad a'x^2 + b'x + c' = 0,$$

where $x = r/c_1$ and

$$(24) \quad \begin{cases} a = -3 + 4 \sin^2 t + (1 - 2 |\sin t|) t_2 \\ b = 2 \cos t [(-3 + (1 - 2 |\sin t|) t_2)] \\ c = -3 + (1 - 2 |\sin t|) t_2 \\ a' = 1 + (1 - 2 |\sin t|) t_3 \\ b' = 2 \cos t [1 + 2 |\sin t| + (1 - 2 |\sin t|) t_3] \\ c' = 1 + 4 |\sin t| + (1 - 2 |\sin t|) t_3. \end{cases}$$

On the other hand, employing (22) together with (14) or (15) we get

$$(25) \quad x = - \frac{1 + |\sin t|}{\cos t}.$$

If $\sin t = 0$, then $x = r/c_1 = -1$, which is not possible. If $\sin t > 0$, then from (23), (24) and (25) we obtain

$$(1 - t_2) \sin^2 t (1 - 2 \sin t) = 0$$

and

$$(1 - t_3) \sin^2 t (1 - 2 \sin t) = 0,$$

hence $\sin t = 1/2$ and from (22) we get (18). In the case $\sin t < 0$ we obtain:

$$(1 - t_2) \sin^2 t (1 + 2 \sin t) = 0$$

$$(1 - t_3) \sin^2 t (1 + 2 \sin t) = 0,$$

hence $\sin t = -1/2$ and from (22) we also get (18).

We remark that (18) holds if we only suppose $t_k \neq 1$.

We conclude that the extremal points c_1, c_2, c_3 must satisfy (18), that is, (11) is of the form $0 \cdot t_k = 0$, for $k = 1, 2, 3$. As in [1] this means that the arc γ_k coincides with the segment from 0 to c_k , for $k = 1, 2, 3$. We have $c_2 = \omega c_1$, $c_3 = \omega^2 c_1$, where $\omega^3 = 1$ and the differential equation (10) becomes

$$\frac{z^2 w}{w^3 - c_1^3} \left(\frac{dw}{dz} \right)^2 = 1.$$

The extremal function will be

$$f(z) = d(E)z(1 + z^{-3})^{2/3},$$

where $d(E) = 4^{-1/3}c_1$. Moreover the extremal value of Φ is $\Phi = 3 \cdot 3^{1/2}c_1 = 3^{3/2}4^{1/3}d(E)$. This completes the proof of our Theorem.

COROLLARY. *If the function*

$$f(z) = z + a_0 + \frac{a_1}{z} + \dots$$

is regular and univalent in $1 < |z| < \infty$ and E is the complement of its range, then the perimeter of any triangle with vertices in E is less or equal to $3^{3/2}4^{1/3}$. The equality holds if and only if the function f is

$$f(z) = a_0 + z(1 + e^{it}z^{-3})^{2/3}$$

In this case E is the union of three segments of equal length $L = 4^{1/3}$, making angles of $2\pi/3$ with each other, having a common initial point. The vertices of the triangle are the endpoints of the three segments.

REFERENCES

- [1] Reich, E. and Schiffer M., *Estimates for the transfinite diameter of a continuum*, Math. Zeitschr. 85, 91–106 (1964).
- [2] Schiffer M., *Hadamard's formula and variation of domain functions*, Amer. J. Math., 68, 417–448 (1946).
- [3] Szegő G., *Jahresbericht Deutsche Math. Ver.* 31, problem section, p. 42 (1922); 32, problem section, p. 45 (1923).

*Faculty of Mathematics Babeş-Bolyai University
Cluj-Napoca, Romania*

Received 1. VII. 1975