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# EXISTENCE OF ALGEBRAIC SUBGRADIENTS FOR CONVEX MAPPINGS

by

A. B. NÉMETH

A mapping from a vector space to an ordered vector space is called *convex* if it satisfies the convexity inequality with respect to the order relation. By an analogy with the case of convex functionals here a *subgradient* is a linear operator between the two spaces which satisfy the subgradient inequality (with respect to the order relation in the adress space). The problem of the existence of the subgradients as well as the properties of the set of subgradients at a point (the set which is called the *subdifferential* at that point) of a convex mapping seems to be the central one in the analysis of these mappings developed till now by M. VALADIER [8], V. L. LEVIN [4], M. M. FEL'DMAN [1] and others. Here it can happen (even in the case of the space of continuous functions [4]) that there are no subgradients in some points. The existence „in general" is closely related to the order relation in the adress space. Therefore conditions are considered about this order relation which assure the existence of subgradients for any convex mapping (see [1] and [6]). Because these conditions are rather restrictive, the complete characterization of the situations when subgradients exist is of a real interest. In the present note we shall consider this problem without any topological assumptions. (However, we shall remark some conditions of topological character in the case of examples, to illustrate that even for some continuous convex mappings we meet situations of void algebraic subdifferentials at some points.)

**1. The convex mapping and its subdifferential.** Consider an ordered real vector space $Y$. Let $K$ be the cone of positive elements in $Y$. We shall suppose throughout that the cone $K$ is a strict one, i.e., that $x, -x \in K$ implies $x = 0$. Set

$$K^* = \{y' \in Y^* : \langle y', y\rangle \geqslant 0 \text{ for any } y \text{ in } K\},$$

where $Y^*$ stands for the algebraic dual of $Y$ (i.e., for the vector space of all the real linear functionals on $Y$). $K^*$ is said to be the *dual* of $K$. We say that $K$ is *dually supported by the subset* $\Gamma$ in $K^*$ if

$$K = \bigcap \{(\ker y')_+ : y' \in \Gamma\}, \tag{1}$$

where $(\ker y')_+ = \{y \in Y : \langle y', y\rangle \geqslant 0\}$.

We shall use in the sequel also the notation $\ker y' = \{y \in Y : \langle y', y\rangle = 0\}$.

If $K$ is dually supported by $\Gamma \subset K^*$ then it is dually supported by $K^*$ too and it is *lineally closed* in the sense that the interesection of $K$ with any straight line is a closed subset of the line.

From simple considerations which use some algebraic separation theorems for convex sets (see e.g. [7], Theorems 5 and 6 in § 8), it follows that if $K$ is lineally closed and has an algebraic interior point, then it is dually supported by $K^*$.

Let $U$ be a convex subset of the real vector space $X$. Then the mapping $F: U \to Y$ is said to be *convex* if for any $x_1$, $x_2$ in $U$ and any $t$ in the interval $[0, 1]$ of the real axis, it holds the relation

$$F(tx_1 + (1-t)x_2) \leqslant tF(x_1) + (1-t)F(x_2).$$

where $\leqslant$ is the ordering defined in $Y$.

Let us denote by $X + Y$ the direct sum of the vector spaces $X$ and $Y$ and let $U + Y = \{x + y \in X + Y : x \in U\}$. The set

$$\text{epi } F := \{x + y \in U + Y : F(x) \geqslant y\}$$

is called the *epigraph* of $F$. By a straightforward verification it can be shown that the mapping $F$ is convex if and only if its epigraph defined as above is a convex set in $X + Y$.

Let $G: U \to Y$ be a mapping from $U$ to $Y$. We shall use also the notation

$$\text{gr } G := \{x + y \in U + Y : y = G(x)\}.$$

gr $G$ is said to be the *graph* of $G$.

Denote by $\mathcal{H}(X, Y)$ the space of the linear operators from $X$ to $Y$ and suppose $z \in U$. The operator $A$ in $\mathcal{H}(X, Y)$ is called an (algebraic) *subgradient of F at z* if it holds the relation

$$Ax \leqslant F(z + x) - F(z)$$

whenever the right hand side of it is defined. The set $\partial F(z)$ of all subgradients of $F$ at $z$ is called the *subdifferential of F at z*.

If for some $x$ in $X$ there exists $\inf \{t^{-1}(F(z + tx) - F(z)) : t > 0\}$. then it is called the *directional derivative of F at z in the direction x* and is denoted by $F'(z; x)$.

There exist exemples of (continuous) convex operators without directional derivatives in some directions and without subgradients at some points. The respective examples furnish the non-existence of subdifferentials by a machinery in which this is in fact closely related to the non-existence of some directional derivatives (see, e.g. [4]). It isn't meaningless therefore to point aut that the above two questions (i.e., that of the existence of directional derivatives and that of existence of subgradients) are in general independent. We remark first that the example of Yu. E. LINKE in [5] can be interpreted as a case of a (continuous) convex mapping having directional derivatives at a point in any direction and without subgradients in that point. On the other hand we consider below an example of a (continuous) convex mapping which has subgradients at a point and for which there do not exist some directional derivatives at the same point.



*Example.* Consider the convex mapping $F: C[-1, 1] \to C[-1, 1]$ defined by the relation $F(x)(t) := g(x(t))$, where $x$ is in $C[-1, 1]$ and $g$ is the function defined by the relations

$$g(t) := \begin{cases} t, & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

Consider the element $z$ in $C[-1, 1]$ defined by the formula

$$z(t) := \begin{cases} 0, & 0 \leqslant t \leqslant 1, \\ t, & -1 \leqslant t < 0. \end{cases}$$

By a straightforward verification it can be seen that the directional derivative $F'(z; x)$, where $x(t) = 1$, $t \in [-1, 1]$ does not exist. We have also $F(z) = 0$ and $F(u) \geqslant 0$ for any $u$, if we consider $\geqslant$ as to be the pointwise ordering in $C[-1, 1]$. Therefore for any $v$ in $C[-1, 1]$ we have

$$F(z + v) - F(z) = F(z + v) \geqslant 0,$$

that is, the zero operator is in $\partial F(z)$. (We observe also that $F$ is continuous and the zero operator is continuous too.)

**1. The existence of the algebraic subgradients. A geometrical approach.**

1. Proposition. *Let Y be an ordered vector space with the positive cone K dually supported by $\Gamma \subset K^*$. Let $F: U \to Y$ be a convex mapping from the convex subset U of X to Y and suppose z is a point in the algebraic interior of U. Then $\partial F(z) \neq \emptyset$ if and only if for the mapping $G := F(z + \cdot) - F(z)$ there exists a mapping $u: \Gamma \to X^*$ with the properties:*

(i) ker $(u(y') + y')$ *is a supporting hyperplane for the convex set* epi $G$ *in* $X + Y$;

(ii) *if $p_1$ denotes the projection onto X in the space $X + Y$, then it holds*

$$p_1(\bigcap \{\ker(u(y') + y') : y' \in \Gamma\}) = X.$$

*There exists A in $\partial F(z)$ with $Ax_0 = y_0$ if and only if* (i) *and* (ii) *are satisfied and in plus it holds*

(iii) $x_0 + y_0 \in \ker\,(u(y') + y')$ *for any $y'$ in* $\Gamma$.

*Proof.* Without further remarks in all the relations that follow we shall consider only elements for which these relations have sense.

*The necessity.* We have $\partial F(z) = \partial G(0)$. Suppose that $A$ is in $\partial F(z)$ and let be $y' \in \Gamma$. Consider the element $u'$ in $X^*$ defined by the relation $\langle u', x\rangle = \langle y', -Ax\rangle$. Then $\langle u' + y', x + Ax\rangle = 0$, that is, gr $A \subset \ker\,(u' + y')$. Let $x + y$ be in epi $G$. Then $y - Ax \geqslant G(x) - Ax \geqslant 0$ and because $y' \in \Gamma \subset K^*$ we have $\langle y', y - Ax\rangle \geqslant 0$. But $\langle y', y - Ax\rangle = \langle u' + y', x + y\rangle$ and then ker $(u' + y')$ is a supporting hyperplane for epi $G$. Putting now $u(y') := u'$ we have the condition (i) of the proposition. It holds also the inclusion gr $A \subset \ker(u(y') + y')$ for any $y'$ in $\Gamma$. Then gr $A \subset \bigcap \{\ker\,(u(y') + y') : y' \in \Gamma\}$ and because $p_1(\text{gr } A) = X$ the condition (ii) also follows. If $Ax_0 = y_0$ then obviously $x_0 + y_0 \in \ker\,(u(y') + y')$ for any $y'$ in $\Gamma$ and we have (iii).

*The sufficience.* Assume that there exists $u : \Gamma \to X^*$ so as to have (i) and (ii). We shall check at once that the projection $p_1$ onto $X$ in the space $X + Y$ restricted to

$$V := \bigcap \{\ker (u(y') + y') : y' \in \Gamma\}$$

is a bijection onto $X$. Let us suppose for this that $z \in V \cap Y$. Then from (i) it follows that $\langle u(y') + y', z\rangle = \langle y', z\rangle$ for any $y'$ in $\Gamma$. Because $K$ is dually supported by $\Gamma$, this implies $z = 0$. Two elements in $V$ which have the same image $x$ by $p_1$ are of the forms $x + y_1$ and $x + y_2$ respectively, with $y_1, y_2$ in $Y$. $V$ is a subspace in $X + Y$ and then $(x + y_1) - (x + y_2) = y_1 - y_2 \in V$, and in accordance with the above observation it follows $y_1 = y_2$ because $y_1 - y_2 \in Y$. That is, $p_1|V$ is a bijection. Denote by $B$, $B : X \to V$ the inverse operator of $p_1|V$. We have then $p_1B = I_1$, where $I_1$ stands for the identity operator of $X$. Let $I$ be the identity operator of $X + Y$ and denote by $p_2$ the projection onto $Y$ in the direct sum space. Set

$$A := p_2B.$$

$A$ is obviously in $\mathcal{K}(X, Y)$. We shall show that gr $A \subset V$. Indeed, for any $x$ in $X$ it holds

$$x + Ax = x + p_2Bx = x + (I - p_1)Bx = x + Bx - p_1Bx = Bx \in V.$$

Let now suppose that $G(x) - Ax \notin K$. Then by the property of $K$ to be dually supported by $\Gamma$ we have

(2) $$\langle y', G(x) - Ax\rangle < 0$$

for an appropriate $y'$ in $\Gamma$. Because $\ker(u(y') + y')$ is a supporting hyperplane for epi $G$, we have

(3) $$\langle u(y') + y', x + G(x)\rangle = \langle u('y), x\rangle + \langle y', G(x)\rangle \geq 0,$$

and because $x + Ax \in \ker (u(y') + y')$,

(4) $$\langle u(y'), x\rangle + \langle y', Ax\rangle = 0.$$

Subtracting the equality (4) term by term from the relation (3) we get a contradiction with (2). Thus, $G(x) - Ax \in K$, i.e., $A \in \partial G(0) = \partial F(z)$.

Assume in addition that $x_0 + y_0 \in \ker (u(y') + y')$ for any $y'$ in $\Gamma$. Then by the one-to-ones of $B$, $Bx_0 = x_0 + y_0$ and $Ax_0 = p_2Bx_0 = p_2(x_0 + y_0) = y_0$, and we obtain what the last part of the proposition states. Q.E.D.

*Remarks.* 1) It can be shown that for any convex mapping $F$ as above, any $Y$ with a lineally closed cone $K$ having an algebraic interior point can be determined a set $\Gamma$ in $K^*$ and a mapping $u : \Gamma \to X^*$ so as to have condition (i) in our proposition.

2) If $Y = \mathbf{R}^n$ and the cone $K$ of positive elements in $Y$ is miniedral (i.e., the ordering in $Y$ is a latticial one) then the existence of subgradients



for any convex operator can be derived from the above proposition. Moreover, we can find a method of an effective construction of the subgradients. We do not still prove by this method the existence of subgradients for the more general case of spaces $Y$ with the chain completeness property (see [1] and [6]).

3) We observe that if $Y = \mathbf{R}^1$ then the proposition is in fact the algebric variant of the so called epigraph method in the proving the existence of subgradients for a convex functional (see e.g. [2]).

**2. The operator theoretic approach of the problem of existence of the algebraic subgradients.** Let $Y$ be an ordered vector space with the positive cone K. The functional $f : Y \to \mathbf{R}$ is said to be *increasing* if for $y_1, y_2$ in $Y$ in the relation $y_1 \leq y_2$ it follows $f(y_1) \leq f(y_2)$. Let $F : U \to Y$ be a convex mapping from the convex subset $U$ of the vector space $X$ to the ordered vector space $Y$. If $f : Y \to \mathbf{R}$ is an increasing and convex functional, then $fF : U \to \mathbf{R}$ is a convex functional. If $z$ is a point in the algebraic interior of $U$, then $\partial fF(z) \neq \emptyset$. (This is a consequence of the classical result for continuous case adapted for the algebraic subgradients [2], and also from Proposition 1, Remarks 2 and 3.) In particular any $y'$ in $K^*$ is an increasing functional and therefore for $z$ as above it holds $\partial(y'F)(z) \neq \emptyset$.

Assume that K is dually supported by $\Gamma \subset K^*$.

2. Proposition. *For $\Gamma$, $F$, $U$ and $z$ as above $\partial F(z) \neq \emptyset$ if and only if there exists a mapping $\varphi : \Gamma \to X^*$ and for each $x$ in $U - z$ there exists $y$ in $Y$ with the properties:*

(i) $\varphi(y') \in \partial(y'F)(z)$;

(ii) $\langle\varphi(y'), x\rangle = \langle y', y\rangle$ *for any $y'$ in $\Gamma$.*

*There exists an $A$ in $\partial F(z)$ with the property $Ax_0 = y_0$ if and only if the conditions (i) and (ii) hold and*

(iii) $\langle\varphi(y'), x_0\rangle = \langle y', y_0\rangle$ *for any $y'$ in $\Gamma$.*

*Proof. The necessity.* Let $A \in \partial F(z)$, i.e.,

$$Ax \leq F(z + x) - F(z)$$

provided $x$ is in $U - z$. For any $y'$ in $K^*$ it holds then

$$\langle y', Ax\rangle \leq \langle y', F(z + x) - F(z)\rangle.$$

and hence

$$\langle A'y', x\rangle \leq (y'F)(z + x) - (y'F)(z),$$

where $A'$ is the adjoint of the operator $A$. This relation proves that $A'y'$ is in $\partial(y'F)(z)$. Set $\varphi := A'|\Gamma$. Let $x$ be an arbitrary element of $X$, put $y = Ax$ and assume that $y'$ is in $\Gamma$. Then

$$\langle\varphi(y'), x\rangle = \langle A'y', x\rangle = \langle y', Ax\rangle = \langle y', y\rangle$$

that is, the conditions (i) and (ii) hold for $\varphi$ defined in this way.

If $Ax_0 = y_0$ then we have obviously $\langle\varphi(y'), x_0\rangle = \langle y', y_0\rangle$ for any $y'$ in $\Gamma$.

*The sufficience.* By the condition (ii) it can be defined a mapping from $X$ to the subsets of $Y$. This mapping is in fact a point-to-point one. Indeed, assume that for a given $x$ there exist $y_1$, $y_2$ in $Y$ such that

$$\langle\varphi(y'),\ x\rangle = \langle y',\ y_1\rangle = \langle y',\ y_2\rangle$$

for any $y'$ in $\Gamma$, that is, $\langle y',\ y_1 - y_2\rangle = 0$ for any $y'$ in $\Gamma$. Because $K$ is dually supported by $\Gamma$ we have $y_1 = y_2$, i.e., the relation $\langle\varphi(y'),\ x\rangle = \langle y',\ y\rangle$ for any $y'$ in $\Gamma$, defines a mapping $A : X \to Y$, and this mapping is by the definition a linear one. So we have

$$\langle\varphi(y'),\ x\rangle = \langle y',\ Ax\rangle \text{ for any } y' \text{ in } \Gamma, \tag{5}$$

where $A$ is a linear operator from $X$ to $Y$. From the condition (i) for any $x$ in $U - z$ we have

$$\langle y',\ Ax\rangle = \langle\varphi(y'),\ x\rangle \leqslant (y'F)(z + x) - (y'F)(z),$$

hence

$$\langle y',\ Ax\rangle \leqslant \langle y',\ F(z + x) - F(z)\rangle \text{ for any } y' \text{ in } \Gamma.$$

$K$ being dually supported by $\Gamma$ this relation yields

$$F(z + x) - F(z) - Ax \in K.$$

that is, $A \in \partial F(z)$. From the condition (iii) and the relation (5) it follows also that $Ax_0 = y_0$. Q.E.D.

*Remarks.* 1) We can derive the proof of Proposition 2 from Proposition 1. Between the mappings $u$ and $\varphi$ arising in the above propositions which map $\Gamma$ to $X^*$, we have the relation $u = -\varphi$. From the proof of Proposition 2 derived in this way we can also conclude that $\ker\,(x' + y')$ is a supporting hyperplane of epi $G$ $(G := F(z + \cdot) - F(z))$ if and only if $x' \in \partial(y'F)(z)$.

2) We have already observed (Remark 2 after Proposition 1) that if $Y = \mathbf{R}^n$ and it is ordered by a miniedral cone, the subgradients of any convex mapping can be effectively constructed. We shall now show how this can be done applying Proposition 2. Because K is miniedral it is dually supported by $\Gamma = \{y'_1, \ldots, y'_n\}$, where $y'_i \in K^*$, $i = 1, \ldots, n$ are linearly independent functionals. (See e.g. [3]) Let $\{y_1, \ldots, y_n\}$ be a basis in $Y$ with the property

$$\langle y'_i,\ y_j\rangle = \delta^j_i,\ i, j = 1, \ldots, n,$$

where $\delta^j_i$ is the Krocker symbol. For every $y'_i$ we choose an element $\varphi(y'_i)$ in $\partial(y'_i F)(z)$. The mapping $\varphi : y'_i \to \varphi(y'_i)$ satisfies the conditions (i) and (ii) of Proposition 2. Indeed, (i) holds by the construction. Let $x$ in $X$ be arbitrary. Set

$$y := \sum_{i=1}^{n} c_i y_i,$$

where $c_i := \langle\varphi(y'_i),\ x\rangle$. It can be shown by a straightforward verification that it holds (ii), i.e., $\langle\varphi(y'_i),\ x\rangle = \langle y'_i,\ y\rangle$, $i = 1, \ldots, n$. (This prove once more that $\partial F(z) \neq \emptyset$).



Let $\{x_\nu : \nu \in J\}$ be a Hamel basis in $X$ and for every $x_\nu$ let $y_\nu$ be the element in $Y$ so that

$$\langle\varphi(y'_i),\ x_\nu\rangle = \langle y'_i,\ y_\nu\rangle,\ i = 1, \ldots, n.$$

Assume that $A$ is the linear extension to $X$ of the mapping $x_\nu \to y_\nu$, $\nu \in J$. Consider an arbitrary element $x$ in $X$. Then $x = \Sigma c_\nu x_\nu$ with a finite number of $c_\nu$'s different from 0 and

$$\langle\varphi(y'_i),\ x\rangle = \Sigma c_\nu\langle\varphi(y'_i),\ x_\nu\rangle = \Sigma c_\nu\langle y'_i,\ y_\nu\rangle = \langle y'_i,\ \Sigma c_\nu y_\nu\rangle = \langle y'_i,\ Ax\rangle.$$

Because $\varphi(y'_i) \in \partial(y'_i F)(z)$ we have on the other hand

$$\langle\varphi(y'_i),\ x\rangle \leqslant (y'_i F)(z + x) - (y'_i F)(z),\ i = 1, \ldots, n.$$

Hence

$$\langle y'_i,\ Ax\rangle \leqslant \langle y'_i,\ F(z + x) - F(z)\rangle$$

for any $x$ in $X$ and any $i = 1, \ldots, n$. Because $K$ is dually supported by $\{y'_1, \ldots, y'_n\}$, this means

$$Ax \leqslant F(z + x) - F(z),$$

i.e., $A \in \partial F(z)$.

*A final comment.* We have used the term „algebraic subgradient" in the title to point aut our approach: i.e., the consideration of the problem of existence of subgradients without any topological assumptions. In the case of the topological vector spaces and continuous convex mappings is usual to consider as subdifferential the set of continuous subgradients. The importance of algebraic subgradients even in this case derives from the fact that in rather general conditions the two subdifferentials coincide.

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