MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION

MATHEMATICA, Tome 23 (46), N° 2, 1981, pp. 239—243

NEAR TO MINIMALITY IN ORDERED VECTOR SPACES

by

A. B. NÉMETH

0. In [2] we had to consider two near to minimality notions in ordered vector spaces.

The first of them defines points in a subset of the ordered vector space with the property that translating them downward in a given direction (characterized by an element of the positive cone) with a given distance, the obtained points be minimum points for the set.

The second notion defines points of a subset with the property that moving them downward in any direction characterized by the elements of the positive cone, the obtained points be minimal for the set. We have here again to translate points „with a given distance". While in the above case this distance can be characterized by the element of the cone characterizing the deplacement's direction, here we need a notion of distance in general. This and the applications in [2] justify to consider the simplest possible case: the case when we have to do with ordered normed vector spaces, and to characterize the length of the deplacement by its norm.

The present note aims to show that the ordered vector spaces with the property that each subset of them which is bounded from below have near to minimum points in one of the above sense, can be characterized by some usual notions of the ordered vector space theory. In the first case they are the semi-Archimedian ordered vector spaces and in the second case they are the regular normed spaces (see 1. of this note for definitions).

1. Notations and terminology. Let X be a vector space over the reals ordered by the (proper) cone K. The cone K will be called the *positive cone* of X. The expressions „the ordered vector space X has the property p" and „the cone K has the property p" will be used in the sequel as having the same meaning.

The ordered vector space X (and equally, its positive cone K) will be said to be *Archimedian* if $x \leqslant 0$ whenever $\alpha x \leqslant a$ for some a in K and all the reals $\alpha > 0$. X is called *almost Archimedian* if $-\beta a \leqslant x \leqslant \beta a$ for some a in K and all real numbers $\beta > 0$ implies $x = 0$. (See [3] p. 4.)

The ordered vector space X will be called *semi-Archimedian* if the relation $0 \leqslant \alpha x \leqslant a$ for some a in K and all $\alpha > 0$ implies $x = 0$.

An Archimedian ordered vector space is almost Archimedian, and an almost Archimedian one is semi-Archimedian.

We give simple examples illustrating the consistency of the above notions. Let us consider $X = R^2$ and the cones:

(1) $$K = \{x = (x^1, x^2) \in \mathbf{R}^2 : |x^1| < x^2\} \cup \{(0, 0)\}),$$

(2) $$K = \{x = (x^1, x^2) \in \mathbf{R}^2 : x^2 > 0\} \cup \{(0, 0)\},$$

(3) $$K = \{x = (x^1, x^2) \in \mathbf{R}^2 : x^2 > 0 \text{ or } x^2 = 0 \text{ and } x^1 \geqslant 0\}.$$

A straightforward verification shows that the cone (1) is almost Archimedian but it isn't Archimedian, while the cone (2) is semi-Archimedian but isn't almost Archimedian. The cone (3) is an example of a cone which isn't semi-Archimedian.

According the propositions II.1.29 and II.1.31 in [3], if there exists a Hausdorff locally convex topology on X such that the closure of K is a cone, then X is almost Archimedian. This fact shows that the condition to be almost Archimedian or semi-Archimedian is quite mild for an ordered vector space.

Assume now that X is also a normed space. The cone K of the positive elements in X is called *regular* if every monotone order bounded sequence in X is fundamental. (See [1] p. 34.)

2. The positive direction near to minimum property. Consider the ordered vector space X with the positive cone K. Let ε be a positive real number and let k be an element in $K \setminus \{0\}$. The point in the subset M of X is said to be an *εk-near to minimum (εk-NM) point* of M if

$$(x - \varepsilon k - K) \cap M = \emptyset.$$

The ordered vector space X is said to have the *positive direction near to minimum (PDNM) property* if each subset M of X which is bounded from below has εk-NM points, for any $\varepsilon > 0$ and any k in $K \setminus \{0\}$.

PROPOSITION 1. *The ordered vector space X has the PDNM property if and only if each subset of it that is bounded from below, has k-NM points.*

Proof. Let ε be a positive number. Then

$$(x - \varepsilon k - K) \cap M = \varepsilon((\varepsilon^{-1}x - k - K) \cap (\varepsilon^{-1}M)).$$

We observe that M is bounded from below if and only if $\varepsilon^{-1}M$ is. This together with the above relation give the proof. Q.E.D.

PROPOSITION 2. *The ordered vector space X has the PDNM property if and only if is semi-Archimedian.*

Proof. According Proposition 1 it suffices to show that each subset M in X that is bounded from below has k-NM points, for any k in $K \setminus \{0\}$, if and only if X is semi-Archimedian.

Let us assume that X is semi-Archimedian and it contains a subset M which is bounded from below by c, but hasn't any k-NM points, that is,

$$(x - k - K) \cap M \neq \emptyset$$

for any x in M. Let x_0 be arbitrarily choosen in M. We construct the sequence $(x_i)_{i=0}^{\infty}$ by choosing

$$x_i \in (x_{i-1} - k - K) \cap M$$

arbitrarily ($i = 1, 2, \ldots$). This means that

$$x_i \leqslant x_{i-1} - k$$

for any i. Adding these relations from 1 to n, we obtain

$$x_n \leqslant x_0 - nk,$$

and since $c \leqslant x_n$ for any n, it follows that $0 \leqslant nk \leqslant x_0 - c$. But then

$$0 \leqslant \alpha k \leqslant x_0 - c$$

for any positive α and we get a contradiction with the hypothesis that X is semi-Archimedian.

Assume now that X has the *PDNM* property but it isn't semi-Archimedian, that is, there exist x and y in K, $x \neq 0$, such that

(4) $$0 \leqslant \alpha x \leqslant y$$

for any $\alpha > 0$.

Define the set M by

$$M = \{\lambda y - \mu x : \lambda, \mu > 0\}.$$

Then $M \subset K$ since from the relation (4) we have $\lambda\left(y - \frac{\mu}{\lambda} x\right) \geqslant 0$ for any positive λ and μ. Hence M is bounded from below by 0. According our assumption there exists a z in M such that

(5) $$(z - x - K) \cap M = \emptyset.$$

Let be $z = \lambda y - \mu x$ with given $\lambda, \mu > 0$. We have $z - x \in z - x - K$ and $z - x = \lambda y - (\mu + 1)x \in M$. These relations contradict (5). Q.E.D.

3. The cone direction near to minimum property. Let X be a normed space ordered by the cone K. Let us introduce the notation

$$H = \{x : x \in K, \|x\| = 1\}.$$

The element x of the subset M of X will be said an *εH near to minimum (εH-NM) point of M*, where ε is a positive real number, if

$$(x - \varepsilon H - K) \cap M = \emptyset.$$

The space X will be said to have the *cone direction near to minimum (CDNM) property* if each subset in it that is bounded from below has εH-NM points for any positive ε. By a similar way with that in the proof of Proposition 1 we can verify

PROPOSITION 3. *The space X has the CDNM property if and only if each subset in it that is bounded from below has H-NM points.*

PROPOSITION 4. *The space X has the CDNM property if and only if is regular.*

We need in the proof the following

Lemma. *The space X is regular if and only if it has the following property: for any sequence x_i in K, $i = 1, 2, \ldots$ with the property that there exists an $\varepsilon > 0$ such that $||x_i|| \geq \varepsilon$ for each* i, *the set of sums* $\sum_{i=1}^{n} x_i$, $n = 1, 2, \ldots$ *cannot have any upper bound.*

Proof. Assume that $(y_i)_{i=1}^{\infty}$ is an increasing order bounded sequence that isn't fundamental. Then there exists a positive number ε such that, passing if necessary to a subsequence of it (and using the same notation for this), it holds

$$||y_{n+1} - y_n|| \geq \varepsilon,\ n = 1, 2, \ldots .$$

Put $x_n = y_{n+1} - y_n$. Then we have

$$\sum_{i=1}^{n} x_i = y_{n+1} - y_1 \leq a - y_1,$$

where a denotes an upper bound of $(y_i)_{i=1}^{\infty}$. This relation shows that there exist $\varepsilon > 0$ and the sequence $(x_i)_{i=1}^{\infty}$, x_i in K and $||x_i|| \geq \varepsilon$ for any i, such that the sums $\sum_{i=1}^{n} x_i$, $n = 1, 2, \ldots$ are bounded from above.

Assume now that X is regular and the property in Lemma doesn't hold, that is, there exist the number $\varepsilon > 0$ and the sequence $(x_i)_{i=1}^{\infty}$, $x_i \in K$ and $||x_i|| \geq \varepsilon$ for each i, which has the property that the set of sums $\sum_{i=1}^{n} x_i$, $n = 1, 2, \ldots$ has an upper bound. Then $(y_n)_{n=1}^{\infty}$, where $y_n = \sum_{i=1}^{n} x_i$, is an increasing sequence bounded from above. By the regularity of X this sequence must be fundamental. But $||y_{n+1} - y_n|| = ||x_{n+1}|| \geq \varepsilon$, $n = 1, 2, \ldots$, contradiction that completes the proof. Q.E.D.

Remark. The condition in Lemma can be reformulated as follows: for any sequence x_i in K, $i = 1, 2, \ldots$, $||x_i|| = 1$ for any i, the set of sums $\sum_{i=1}^{n} x_i$, $n = 1, 2, \ldots$ cannot be bounded from above. The equivalence of these two forms can be shown by the method we used in the proof of Proposition 1.

The proof of Proposition 4. From Proposition 3 it follows that it suffices to prove that X is regular if and only if each subset in it that is bounded from below has *H-NM* points.

Assume that in X exists a subset M, bounded from below by c, without *H-NM* points, i.e., with the property that

$$(x - H - K) \cap M \neq \emptyset$$

for every x in M. Let x_0 be choosen arbitrarily in M. We construct the sequence $(x_n)_{n=0}^{\infty}$ by choosing

$$x_n \in (x_{n-1} - H - K) \cap M,\ n = 1, 2, \ldots .$$

We have then

$$x_n \leq x_{n-1} - h_n,$$

where $h_n \in H$, $n = 1, 2, \ldots$. Adding from 1 to m the above relations, we obtain

$$\sum_{n=1}^{m} h_n \leq x_0 - x_m \leq x_0 - c.$$

Because $h_m \in K$, $||h_n|| = 1$ for each n and the above relation holds for any m, we get using Lemma that X cannot be regular.

Suppose now that each subset of X that is bounded from below has *H-NM* points but X isn't regular. That is, using now the remark after the lemma, there exist x_i in K, $||x_i|| = 1$ for each $i = 1, 2, \ldots$ such that the set of sums $\sum_{i=1}^{n} x_i$, $n = 1, 2, \ldots$ has an upper bound c. This means that the set $M = \left\{-\sum_{i=1}^{n} x_i : n = 1, 2, \ldots\right\}$ is bounded from below by $-c$. Let x be an arbitrarily choosen element in M. Then $x = -\sum_{i=1}^{n} x_i$ for some n. We have $x - x_{n+1} = -\sum_{i=1}^{n+1} x_i \in M$, and $x - x_{n+1} \in x - H \subset x - H - K$. That is,

$$(x - H - K) \cap M \neq \emptyset$$

for every x in M. This contradiction completes the proof. Q.E.D.

REFERENCES

[1] Krasnosel'skii, M. A., *Positive Solutions of the Operational Equations* (russian), Fizmatgiz, Moskow, 1962.

[2] Németh, A. B., *The nonconvex minimization principle in ordered regular Banach spaces*, to appear in this journal.

[3] Peressini, A. L., *Ordered Topological Vector Spaces*, Harper Row, New York — Evanston — London, 1967.

Received, 18. XII. 1979

Universitatea „Babeş-Bolyai"
Facultatea de matematică
3400 CLUJ–NAPOCA
Romania