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ON THE LIPSCHITZ PROPERTIES OF CONTINUOUS CONVEX FUNCTIONS

by

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Let $(X, \mathfrak{P})$ be a real locally convex Hausdorff space, where $\mathfrak{P}$ denotes a directed family of semi-norms generating the topology of X. Let C be a subset of X and M a subset of C. A function $f: C \to \mathbf{R}$ is called *Lipschitz* on M if there exist a semi-norm $p \in \mathfrak{P}$ and a number $L > 0$ such that

$$|f(x) - f(y)| \leqslant Lp(x - y), \tag{1}$$

for all $x, y \in M$. The function f is called *locally Lipschitz* on C if every $x \in C$ has a neighborhood V such that f is Lipschitz on $V \cap C$. A classical theorem (see, e.g. [3]) asserts that every convex function defined on an open interval of the real axis is Lipschitz on every compact interval contained in its domain of definition. This result was extended to convex functions defined on open convex subsets of the Euclidean n-dimensional space (see [7]), and then to continuous convex functions defined on an open convex subset of a normed linear space (see [2], [4], [5], [6]), respectively of a locally convex Hausdorff space ([1]).

The aim of this Note is to give a new and shorter proof of the result from [1], by the methods of convex analysis.

Let $(X, \mathfrak{P})$ be as above, and let $f: X \to \mathbf{R} \cup \{\infty\}$ be a function on X. A *subgradient* of f at a point $x \in X$ (such that $f(x) < \infty$) is a continuous linear functional Φ on X such that

$$\Phi(y - x) \leqslant f(y) - f(x), \tag{2}$$

for all $y \in X$. The set (possibly empty) of all subgradients of f at x is denoted by $\partial f(x)$ and is called the *subdifferential* of f at x. When we are dealing with a convex function f, defined on a convex subset C of X, we consider f automatically defined on all X, by $f(x) = \infty$ for $x \in X \setminus C$. It is well known (see [2] or [4]) that if C is a convex open subset of X and $f: C \to \mathbf{R}$ is a continuous convex function, then $\partial f(x) \neq \emptyset$ for all $x \in C$.

THEOREM. *Let $(X, \mathfrak{P})$ be a locally convex Hausdorff space, C a nonvoid convex open subset of X and $f: C \to \mathbf{R}$ a continuous convex function. Then*

(a) *the function f is locally Lipschitz on C,*

and

(b) *the function f is Lipschitz on every compact subset of C.*

Proof. For $x \in X$, $p \in \mathfrak{P}$ and $r>0$ put $B(x, p, r) = \{y \in X : p(y - x) \leqslant r\}$. Let now f be a continuous convex function defined on the open convex subset C of X and let $x_0 \in C$. By the continuity of f there exist the numbers m, M, r_0, $m \leqslant M$, $r_0 > 0$ and a semi-norm $p \in \mathfrak{P}$ such that $B(x_0, p, r_0) \subset C$ and

$$m \leqslant f(x) \leqslant M, \tag{3}$$

for all $x \in B(x_0, p, r_0)$. Let $0 < r < r_0$ and let $V = B(x_0, p, r)$. For each $x \in V$ there exists a continuous linear functional Φ_x on X, such that

$$\Phi_x(y - x) \leqslant f(y) - f(x), \tag{4}$$

for all $y \in X$. Since $B(x, p, r_0 - r) \subset B(x_0, p, r_0)$ for all $x \in V$, the inequalities (3) and (4) yield

$$\Phi_x(y - x) \leqslant M - m, \tag{5}$$

for all $y \in B(x, p, r_0 - r)$. But, as it is well known (and easy to check), the inequality (5) implies

$$|\Phi_x(z)| \leqslant (r_0 - r)^{-1}(M - m)p(z), \tag{6}$$

for all $z \in X$.

Now, if $x, y \in V$ then by (4) and (6)

$$f(x) - f(y) \leqslant \Phi\,(x - y) \leqslant (r_0 - r)^{-1}(M - m)p(x - y).$$

Interchanging the roles of x and y, it follows

$$|f(x) - f(y)| \leqslant (r_0 - r)^{-1}(M - m)p(x - y), \tag{7}$$

for all $x, y \in V$, which proves that f is Lipschitz on V. Therefore f is locally Lipschitz on C.

Suppose now, K is a compact subset of C. Since f is continuous on C, there exist, for every $x \in K$, a semi-norm $p_x \in \mathfrak{P}$, and the positive numbers r_x, M_x such that $B(x, p_x, r_x) \subset C$ and

$$|f(y)| \leqslant M_x, \tag{8}$$

for all $y \in B(x, p_x, r_x)$. The set K being compact we can select a finite subcovering $\{B(x_i, p_{x_i}, r_{x_i}/2) : i = 1, \ldots, n\}$ of the covering $\{B(x, p_x, r_x/2) : x \in K\}$ of K. Put $p_i = p_{x_i}$, $M_i = M_{x_i}$ and $B_i = B(x_i, p_i, r_i/2)$ for $i = 1, \ldots, n$. Let $M = \max\{M_i : i = 1, \ldots, n\}$, $r = \min\{r_i/2 : i = 1, \ldots, n\}$ and let $p \in \mathfrak{P}$ be such that $p \geqslant p_i$, for $i = 1, \ldots, n$ (the family $\mathfrak{P}$ is directed). For every $x \in \bigcup_{i=1}^{n} B_i$ let Φ_x be a subgradient of f at x. Let now, $x, y \in K$ and let $j \in \{1, \ldots, n\}$ be such that $x \in B_j$. Then $B(x, p, r) \subset B_j$, and since $|f(z)| \leqslant M$ for all $z \in \bigcup_{i=1}^{n} B_i$, reasoning as above, one concludes

that $|\Phi_x(z)| \leqslant 2r^{-1}M\,p(z)$, for all $z \in X$ and $|f(x) - f(y)| \leqslant 2r^{-1}M\,p(x-y)$, which shows that f is Lipschitz on K.

Remarks 1. In the inequality (7) we have obtained the same delimitation for the Lipschitz constant as in [2], p. 13.

2. It is sufficient in the Theorem to suppose the function f bounded from above on a neighborhood of some point in C, because this condition implies the continuity of f ([2]).

REFERENCES

[1] Cobzaş, S., and Muntean, I., *Continuous and locally Lipschitz convex functions*, Mathematica (Cluj), **18**, 41–51. (1976)

[2] Ekeland, I., and Temam, R., *Analyse convexe et problèmes variationelles*, Dunod, Paris (1974).

[3] Hardy, G., Littlewood J. L., and Plóya, G., *Inequalities*, Cambridge Univ. Press, Cambridge (1934).

[4] Holmes, R. B., *A course on optimization and best approximation*, Lectures Notes in Math. v. 257, Springer V., Berlin-Heidelberg-New York (1972).

[5] Roberts, A. W., and Varberg, D. E., *Another proof that convex functions are locally Lipschitz*, Amer. Math. Monthly, **81**, 1014–1016 (1974).

[6] Roberts, A. W., and Varberg, D. E., *Convex functions*, Academic Press, New York (1973).

[7] Wayne State University, Mathematics Department Coffee Room, *Every convex function is locally Lipschitz*, Amer. Math. Monthly, **79**, 1121–1124 (1972).

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