MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION

MATHEMATICA, Tome 23 (46), № 1, pp. 43–48

THE NONCONVEX MINIMIZATION PRINCIPLE IN ORDERED REGULAR BANACH SPACES

by

A. B. NÉMETH

0. Arising from the geometry of Banach spaces [1], the Bishop-Phelps-Ekeland variational principle have become in the last several years a useful method in solving a vide class of optimization problems for functionals [2]. In looking for new techniques for optimization in ordered vector spaces we shall give here two extensions of this principle. Our aim is to give real generalizations in the sense that replacing the adress space by the real line, to obtain as consequence the full statement of the principal result in [2]. To this end we shall consider as adress spaces ordered Banach spaces that are regular. We will also strive after to be in keeping with the terminology and the notation of the last mentioned monography.

1. Terminology and results. Let Y be a normed space over the reals, ordered by a closed (proper) cone K. The space Y (and equally, its positive cone K) is said to be *regular* if every monotone order bounded sequence in it is fundamental. If Y is a Banach space (abbreviated: B-space), then its regularity implies that K is a *normal* cone, i.e., a cone having the property that there exists a positive real number η such that $\|v + u\| \geqslant \eta$, whenever $u, v \in K$ and $\|u\| = \|v\| = 1$ (see [3], Theorem 1.6), and the normality of K implies that Y can be endowed with an equivalent *monotone norm*, that is, with a norm having the property that from $0 \leqslant u \leqslant v$ it follows that $\|u\| \leqslant \|v\|$, (see [5], Proposition II. 15). Hence without loss of the generality *we can consider in all what follows that the norm in the space Y is monotone.*

Let V be a complete metric space. The operator F from V to the regular B-space Y will be called *lower monotone semicontinuous* at x_0 in V, if $\lim\limits_{n\to\infty} x_n = x_0$ and $F(x_{n+1}) \leqslant F(x_n)$ for each n imply $\lim\limits_{n\to\infty} F(x_n) \leqslant F(x_0)$.

Suppose that X is a normed space. The operator P from X to Y is called *sublinear* if it is positive homogeneous and subadditive with respect to the order relation in Y.

2

The operator Q from the normed space X to Y will be said to be *positive δ-norm definite* (where δ is a positive real number), if $Q(X) \subset K$ and $\|Q(x)\| \geqslant \delta\|x\|$ for any x in X.

PROPOSITION 1. *Let Y be a B-space ordered by a closed regular cone K. Suppose that F is a lower monotone semicontinuous operator from the complete metric space (V, d) to Y. If for some $z \in V$ the set*

$$F(V) \cap (F(z) - K)$$

has a lower bound, then for each positive real number ε and each k in $K \setminus \{0\}$ there exists an element u in V such that

(i) $$F(u) \leqslant F(z) \text{ and}$$

(ii) $$(F(u) - \varepsilon k - K) \cap F(V) = \emptyset.$$

For every u satisfying (ii) *there exists v in V such that*

(iii) $$d(v, u) \leqslant 1,$$

(iv) $$F(v) \leqslant F(u) - \varepsilon k d(u, v) \text{ and}$$

(v) $$F(v) - F(w) - \varepsilon k d(v, w) \not\leqq K \text{ for any } w \neq v.$$

For V having a richer structure this proposition can be strengthened for instance as follows:

PROPOSITION 2. *Let Y be a B-space ordered by the closed regular cone K. Suppose that F is a lower monotone semicontinuous operator from the closed subset V of the B-space X to Y. Let P be a continuous sublinear, positive δ-norm definite operator from X to Y. If for some $z \in V$ the set*

$$F(V) \cap (F(z) - K)$$

has a lower bound, then for each positive real number ε there exists an element u in V such that

(i) $$F(u) \leqslant F(z) \text{ and}$$

(ii) $$(F(u) - \varepsilon H - K) \cap F(V) = \emptyset,$$

where H is defined by $H = \{y \in K : \|y\| = 1\}$.

For every u satisfying (ii) there exists a v in V such that

(iii) $$\|u - v\| \leqslant \delta^{-1},$$

(iv) $$F(v) \leqslant F(u) - \varepsilon P(u - v) \text{ and}$$

(v) $$F(v) - F(w) - \varepsilon P(v - w) \not\leqq K \text{ for any } w \in V \setminus \{v\}.$$

We observe that the assertion (ii) in Proposition 1 can be changed in a stronger one, similar to (ii) in the second proposition.

We shall begin with the proof of the last proposition and then shall give an outline of the proof of the first one.

2. The proof of Proposition 2. (i) *and* (ii). We have proved in [4], that if N is a subset in the regular B-space Y which has a lower bound, then for each a in N there exists an element b in N such that

(1) $$b \leqslant a \text{ and}$$

(2) $$(b - \varepsilon H - K) \cap N = \emptyset.$$

(See the proof of Proposition 4 in the above cited note). Putting now $N = F(V) \cap (F(z) - K)$ and $a = F(z)$, it follows the existence of an element u in V such to $F(u)$ play the role of b in the above assertions. Then (1) yields (i) and, after a straightforward verification, (2) yields (ii).

(v) We define an order relation $\prec$ on the set $F(V)$ by putting $F(p) \prec F(q)$ if

(3) $$F(p) \leqslant F(q) - \varepsilon P(q - p).$$

From the positivity, the sublinearity and the δ-norm definitness of P it follows that the relation $\prec$ is reflexive, antisymmetric and transitive.

Consider now the family of decreasing $\prec$-chains having the first element $F(u)$. With respect to the set theoretic inclusion, each totally ordered subfamily of it has a maximal element: it is simply the union of the members of this family. Hence it exists a maximal decreasing $\prec$-chain Z starting with $F(u)$. We shall show that Z contains a minimal element with respect to the relation $\prec$.

From the definition of the relation $\prec$ it follows that Z is also a decreasing $\leqslant$-chain. If we consider it as a decreasing net it is bounded and hence must be convergent by the regularity of Y. Indeed, if not, then there would exist a positive δ such that for any y_n in Z there would exist y_{n+1} in Z such that $y_{n+1} \leqslant y_n$ and $\|y_n - y_{n+1}\| \geqslant \delta$. The sequence (y_n) is bounded because $Z \subset F(V) \cup (F(z) - K)$ and the last set has a lower bound by hypothesis. Hence (y_n) must be fundamental, and we get a contradiction with the above inequality. Denote by y the limite of the decreasing net Z.

We define now the sequence (x_n) in V having the property that $x_0 = u$, $F(x_n) \in Z$ and for each $n \geqslant 1$,

(4) $$\|F(x_n) - y\| < 2^{-n}.$$

We have for an arbitrary natural number m

(5) $$F(x_{n+m}) \leqslant F(x_n) - \varepsilon P(x_n - x_{n+m})$$

and since (4), the δ-norm positive definitness of P and the monotonity of the norm in Y, it follows that

$$\delta\varepsilon\|x_n - x_{n+m}\| \leqslant \varepsilon\|P(x_n - x_{n+m})\| \leqslant \|F(x_n) - F(x_{n+m})\| < 2^{-n+1}.$$

Hence (x_n) is fundamental in V. Let v be its limit. From the lower monotone semicontinuity of F we have

$$F(v) \leqslant \lim_{n\to\infty} F(x_n) = y.$$

Passing to limit for $m \to \infty$ in (5) and using the continuity of P we get

$$F(v) \leqslant y \leqslant F(x_n) - \varepsilon P(x_n - v), \tag{6}$$

that is, $F(v) \prec F(x_n)$ for each n. Since $F(x_n)$ tends to y, it follows that for each x with the property that $F(x) \in Z$ there exists an n such that $F(x_n) \prec F(x)$. But then $F(v) \prec F(x)$ for any $F(x)$ in Z and hence $F(v) \in Z$, the $\prec$-chain Z being maximal. The maximality of Z implies also that it doesn't exist $w \in V \setminus \{v\}$ with the property that $F(w) \prec F(v)$, and this assertion is quite (v).

The point (iv) follows from (6) if we put $n = 0$.

(iii). Assume that $\|u - v\| \geqslant \delta^{-1}$. Then we have $\|P(u - v)\| \geqslant 1$ and hence $P(u - v) \geqslant \|P(u - v)\|^{-1} P(u - v) \in H$. It follows that

$$F(u) - \varepsilon P(u - v) - K \subset F(u) - \varepsilon H - K. \tag{7}$$

According (iv) we have

$$F(v) \in F(u) - \varepsilon P(u - v) - K,$$

and this together (7) contradict (ii). Q.E.D.

3. An outline of the proof of Proposition 1. The points (i) and (ii) can be obtained by using Proposition 2. To prove (v) we introduce the order relation $\prec$ in the set $F(V)$ by putting $F(p) \prec F(q)$ if

$$F(p) \leqslant F(q) - \varepsilon k d(p, q)$$

and then proceed as in the proof of (v) of the preceding paragraph. The same technique gives also the proof of (iv), and finally, (iii) can be verified also by contradiction. Q.E.D.

4. Some consequences and comments. If we put $Y = R$ and consider in R the usual ordering, we obtain as a direct cosequence of Proposition 1 the Theorem 1.1 in [2]. The condition to $F(V) \cap (F(z) - K)$ be minorized reduces to the existence of a lower bound for $F(V)$ and then (ii) can be obviously realized for $k = 1$ and for some u in V.

We observe that the conditions (ii) in our propositions are related to the existence of some „near to minimum points" of the lower bounded sets in ordered vector spaces. To prove the existence of points of this kind was the first thing that we had to done. It appears that the existence of such points in fact characterizes in some usual terms the considered ordered vector spaces [4].

Consider now an example of a sublinear operator defined on the B-space X with values in the regular B-space Y. Put

$$P(x) = P(x) = \|x\|k, \tag{8}$$

where k is an element of $K \setminus \{0\}$. If k is of norm δ, then P is a δ — norm positive definite continuous sublinear operator. For this P the Proposition 2 becomes Proposition 1 for the special case of V being a closed subset of a B-space.

Suppose that the positive cone K in the regular B-space Y has a nonempty interior. (A wide class of regular B-spaces having this property can be defined according the results in [6].) Assume that P is a δ-norm positive definite sublinear operator from the B-space X to Y with the property that $P(x) \in \operatorname{int} K$ whenever $x \neq 0$. If we consider $k \in \operatorname{int} K$ and $\|k\| = \delta$, then P defined by (8) furnishes a sublinear operator having this property.

In order to give an application, assume for the sake of simplicity, that the operator F in Proposition 2 is defined on all the space X. We define a directional derivative $F'(v\,;z)$ of F at the point v and the direction z in X by putting

$$F'(v\,;z) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda}\,[F(v + \lambda z) - F(v)], \tag{9}$$

if this limit exists. Because the ordered space Y is regular, it is complete by chains and because K has a nonempty interior, the order convergence and the topological convergence coincide in it [7]. Hence if F is a convex operator, $F'(v\,;z)$ exists for each v and z in X.

We have listed the above examples to illustrate the consistence of the conditions in the following

Corollary. *Let Y be a B-space ordered by a closed regular cone K having a nonempty interior. Consider the B-space X and the δ-norm-positive definite continuous sublinear operator P from X to Y having the property that $P(x) \in \operatorname{int} K$ whenever $x \neq 0$. Suppose that $F : X \to Y$ is a continuos operator having the property that $F(X)$ is bounded from below, and having directional derivatives in any point and any directions. Then for an arbitrary positive number ε there exists v in X such that*

$$-F'(v\,;z) - \varepsilon P(-z) \notin K$$

for any $z \neq 0$ in X.

Proof. According Proposition 2, there exists a v in X such that

$$F(v) - F(w) - (\varepsilon/2)\,P\,(v - w) \notin K \tag{10}$$

for any $w \neq v$. We shall prove that this v satisfies the condition in the corollary. Assume the contrary. Then there exists $z \neq 0$ such that

$$F'(v\,;z) + \varepsilon P(-z) \in -K. \tag{11}$$

Let $\lambda > 0$ and put $w = v + \lambda z$. By (10) we have 6

$$F(v) - F(v + \lambda z) - (\varepsilon/2)P(-\lambda z) \not\leq K$$

and hence

$$(1/\lambda)(F(v) - F(v + \lambda z)) - (\varepsilon/2)P(-z) \not\leq K.$$

Adding this relation to (11) we obtain

$$F'(v\,; zy) + (1/\lambda)(F(v) - F(v + \lambda z)) + (\varepsilon/2)P(-z) \not\leq K.$$

Now, $(\varepsilon/2)P(-z)$ is in int K and because the sum of the first two terms in this relation tends to 0 when $\lambda \downarrow 0$, we get a contradiction. Q.E.D.

REFERENCES

[1] Bishop, E., Phelps, R. R., *A proof that all Banach spaces are subreflexive*, Bull Amer. Mat. Soc. **67**, (1961), 97–98.

[2] Ekeland, I., *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (New Series) **1**, (1979), 443–474.

[3] Krasnosel'skii, M. A., *Positive Solutions of the Operatorial Equations*, (Russain) Fizmatgiz, Moskow, 1962.

[4] Németh, A. B., *Near to minimality in ordered vector spaces*, this journal **23** (46) № 2 (1981).

[5] Peressini, A. L., *Ordered Topological Vector Spaces*, Harper & Row, New York – Evanston – London, 1967.

[6] Vulih, B. Z., Danilenko, I. F., *On a method of the ordering of a normed space* (Russain), Vestnik Leningr. Gos. Univ. **19**, (1970), 18–22.

[7] Vulih, B. Z., Korsakova, O. S., *On spaces in wich the norm convergence coincides with the order convergence* (Russain) Matem. Zam. **13**, (1973) 259–268.

Received, 24. XII. 1979