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THE ORDER OF STARLIKENESS OF A LIBERA INTEGRAL OPERATOR

by

PETRU T. MOCANU, Cluj-Napoca

MAXWELL O. READE, Ann Arbor

DUMITRU RIPEANU, Cluj-Napoca

1. Introduction. A function f is said to be starlike of order α if f is regular in the unit disc $U = \{z : |z| < 1\}$, $f(0) = 0$, $f'(0) = 1$ and the inequality

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha$$

holds for $z \in U$. The class of starlike functions of order α shall be denoted by $S^*(\alpha)$. If $0 \leqslant \alpha < 1$ then $S^*(\alpha) \subset S^*(0) \equiv S^*$, the class of starlike functions (i.e. normalized univalent functions for which $f(U)$ is starlike with respect to the origin).

The order of starlikeness of a family $\mathcal{F}$ of starlike functions is defined by the largest number $\alpha = \alpha[\mathcal{F}]$ such that $\mathcal{F} \subset S^*(\alpha)$.

In [1], R. LIBERA showed that if $f \in S^*$ then the function $F = L(f)$ given by

$$F(z) = \frac{2}{z}\int_0^z f(t)dt \tag{1}$$

is also in S^*, that is $L(S^*) \subset S^*$.

In this paper we find the order of starlikeness of $L(S^*)$. We first show that a simple application of a recent result of ST. RUSCHEWEYH and V. SINGH [2] reduces our problem to a computation involving a specific function. It is this lengthy but elementary computation which enables us to obtain the order of starlikeness of $L(S^*)$.

2. Lemma. Let $\mathfrak{P}$ denote the class of regular functions $p(z)$ in U which satisfy $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in U$. One of our main tools is the following result of Ruscheweyh and Singh concerning an extremal problem in the class $\mathfrak{P}$ related to a differential equation of Briot-Bouquet type [2].

Lemma. *Let $a > 0$, $b \in C$ with $\operatorname{Re} b > 0$. For $p \in \mathfrak{P}$ let q be the regular solution of the differential equation*

$$zq'(z) + (q(z) - p(z))(aq(z) + b) = 0 \tag{2}$$

which satisfies the initial condition $q(0) = 1$. Then q is regular in U and

$$\min_{|z|=r} \operatorname{Re} q(z) \geqslant \min_{|z|=r} \operatorname{Re} Q(z) \tag{3}$$

for all $r < 1$, where Q is the solution of (2) which corresponds to

$$p(z) = (1 - z)/(1 + z).$$

3. Main result

THEOREM *The order of starlikeness of $L(S^*)$ where L is the Libera integral operator defined by* (1) *is given by*

$$\alpha[L(S^*)] = \min_{|z|=1} \operatorname{Re} \frac{zK'(z)}{K(z)} = \frac{3 - 4 \ln 2}{2(2 \ln 2 - 1)} = 0.29435 \ldots \tag{4}$$

where $K(z)$ is the function obtained from (1) *when $f(t)$ is the Koebe function $t/(1 + t)^2$, that is,*

$$K(z) = \frac{2}{z}\int_0^z \frac{t}{(1-t)^2}\,dt = \frac{2}{z}\left[\ln(1+z) - \frac{z}{1+z}\right], \quad (\ln 1 = 0) \tag{5}$$

Proof. For $f \in S^*$ let

$$p(z) = \frac{zf'(z)}{f(z)} \text{ and } q(z) = \frac{zF'(z)}{F(z)},$$

where F is given by (1). Then a simple calculation shows that q satisfies the differential equation

$$zq'(z) + (q(z) - p(z))(q(z) + 1) = 0, \; q(0) = 1,$$

which is of the form (2) with $a = b = 1$. It is easy to see that if $p(z) = (1 - z)/(1 + z)$ we have $q(z) = Q(z) = zK'(z)/K(z)$. Hence from (3) of the Lemma we get

$$\min_{p \in \mathfrak{P}} \min_{|z|=r} \operatorname{Re} q(z) = \min_{|z|=r} \operatorname{Re} \frac{zK'(z)}{K(z)}. \tag{6}$$

Since $f \in S^*$ if and only if $p \in \mathfrak{P}$, from (6) we obtain

$$\alpha[L(S^*)] = \inf_{|z|<1} \operatorname{Re} \frac{zK'(z)}{K(z)} = \min_{|z|=1} \operatorname{Re} \frac{zK'(z)}{K(z)}. \tag{7}$$

Now our problem is to find the above minimum. For $z = e^{i\theta}$, $\theta \in (-\pi, \pi)$ we let

$$R(\theta) = \operatorname{Re} \frac{zK'(z)}{K(z)},$$

where $K(z)$ is given by (5). We shall prove that

$$\min_{-\pi<\theta<\pi} R(\theta) = R(0) = \frac{3 - 4 \ln 2}{2(2 \ln 2 - 1)} \tag{8}$$

by showing that

$$R(\theta) > R(0), \text{ for all } \theta \in (-\pi, -\pi), \; \theta \neq 0. \tag{9}$$

Since $R(-\theta) = R(\theta)$, it is sufficient to suppose $\theta \in [0, \pi)$. From (5) we obtain

$$\frac{zK'(z)}{K(z)} = -1 + \frac{z^2}{(1+z)[-z + (1+z)\ln(1+z)]}, \quad (\ln 1 = 0).$$

Therefore

$$R(\theta) = -1 + S(\theta),$$

where

$$S(\theta) \equiv \frac{1}{N(\theta)}[-1 + \theta \sin\theta + \cos\theta \ln 2(1 + \cos\theta)],$$

with

$$N(\theta) \equiv 2 - 2\theta\sin\theta + \theta^2(1 + \cos\theta) -$$

$$-2(1 + \cos\theta)\ln 2(1 + \cos\theta) + (1 + \cos\theta)\ln^2 2(1 + \cos\theta).$$

We first remark that $N(\theta)$ can be written in the following form

$$N(\theta) = \frac{1}{2}[N_1^2(\theta) + N_2^2(\theta)],$$

where

$$N_1(\theta) \equiv 2\cos\theta + \theta\sin\theta - (1 + \cos\theta)\ln 2(1 + \cos\theta)$$

$$N_2(\theta) \equiv -2\sin\theta + \theta(1 + \cos\theta) + \sin\theta \ln 2(1 + \cos\theta).$$

We have $N(0) = 2(2 \ln 2 - 1) > 0$. If there exists $\theta_0 \in (0, \pi)$ such that $N_1(\theta_0) = N_2(\theta_0) = 0$ then it is easy to see that θ_0 must satisfy $\theta_0 - \operatorname{tg}(\theta_0/2) = 0$ and $\pi/2 < \theta_0 < \pi$. Since $4/(1 + \theta_0^2) < e$, a simple calculation shows that

$$N_2(\theta_0) = \left[\ln\frac{4}{1 + \theta_0^2} - 1\right]\sin\theta_0 < 0,$$

which contradicts $N_2(\theta_0) = 0$. Hence $N(\theta) > 0$ for all $\theta \in [0, \pi)$. It follows that

(10 $\quad \operatorname{sign}[R(\theta) - R(0)] = \operatorname{sign}[S(\theta) - S(0)] = \operatorname{sign} S_1(\theta),$

where

$$S(0) = \frac{1}{2(2\ln 2 - 1)}$$

$$S_1(\theta) \equiv 2(2\ln 2 - 1)\,N(\theta)\,[S(\theta) - S(0)] = -p + p\theta\sin\theta - \theta^2(1+\cos\theta) +$$
$$+ (2 + p\cos\theta)\ln 2(1+\cos\theta) - (1+\cos\theta)\ln^2 2(1+\cos\theta),$$

with

$$p = 4\ln 2 = 2.7725884\ \ldots$$

We shall write successively

$$S_2(\theta) \equiv \frac{S_1(\theta)}{1+\cos\theta} = p\,\frac{-1+\theta\sin\theta}{1+\cos\theta} - \theta^2 + \frac{2+p\cos\theta}{1+\cos\theta}\ln 2(1+\cos\theta) - \tag{11}$$
$$- \ln^2 2(1+\cos\theta)$$

$$S_3(\theta) \equiv \frac{1+\cos\theta}{(4-p+2\cos\theta)\,\mathrm{tg}\,(\theta/2)}\,S_2'(\theta) = \ln 2(1+\cos\theta) - \tag{12}$$
$$- \frac{2+\theta(2-p+2\cos\theta)\,\mathrm{ctg}\,(\theta/2)}{4-p+2\cos\theta}$$

$$S_4(\theta) \equiv \frac{(1-\cos\theta)(4-p+2\cos\theta)^2}{12-6p+p^2+4(3-p)\cos\theta}\,S_3'(\theta) = \tag{13}$$
$$= \theta - 2\,\frac{14-7p+p^2+(10-3p)\cos\theta}{12-6p+p^2+4(3-p)\cos\theta}\,\mathrm{tg}\,\frac{\theta}{2}\,.$$

We have

$$S_4'(\theta) = \frac{(1-\cos\theta)\,P_1(\cos\theta)}{(1+\cos\theta)\,[12-6p+p^2+4(3-p)\cos\theta]^2} \tag{14}$$

with

$$P_1(s) = -288 + 280\,p - 100\,p^2 + 16\,p^3 - p^4 +$$
$$+ (-432 + 368p - 104\,p^2 + 10\,p^3)s - 16(3-p)^2 s^2.$$

For $s \in [-1, 1]$ we have

$$\frac{1}{2}\,P_1'(s) = -216 + 184p - 52p^2 + 5p^3 - 16(3-p)^2 s \geqslant \frac{1}{2}\,P_1'(1) = P_2(p),$$

with

$$P_2(s) = -360 + 280s - 68s^2 + 5s.$$

It is easy to see that $P_2(s)$ is an increasing function in the interval $[2.77, 3]$ and $P_2(2.77) = 0.112\ldots. > 0$. Hence $P_2(p) > 0$ and $P_1'(s) > 0$ for all $s \in [-1, 1]$, which shows that $P_1(s)$ increases in the interval $[-1, 1]$ from $P_1(-1) = -p(p-2)^3 < 0$ to $P_1(1) = -P_3(p)$ with $P_3(s) = 864 - 744s + 220s^2 - 26s^3 + s^4$. It is easy to show that $P_3(s)$ decreses in the interval $[2.77, 3]$ and $P_3(2.77) = -2.5708\ldots < 0$. It follows that $P_3(p) < 0$ and $P_1(1) > 0$ (in fact, we have $P_1(1) = 2.6705\ldots$). Hence there exists a unique $s_1 \in (-1, 1)$such that $P_1(s_1) = 0$.

Let $\theta_1 = \arccos s_1$. From (14) we have $S_4'(\theta_1) = 0$. Since $12 - 6p + p^2 - 4(3-p)\cos\theta \geqslant 12 - 6p + p^2 - 4(3-p) = p(p-2) > 0$ for all $\theta \in [0, \pi]$, we deduce that $S_4'(\theta) > 0$ for $0 < \theta < \theta_1$ and $S_4'(\theta) < 0$ for $\theta_1 < \theta < \pi$. We conclude that $S_4(\theta)$ increases from 0 to $S_4(\theta_1) > 0$ in the interval $[0, \theta_1]$ and decreases from $S_4(\theta_1)$ to $-\infty$ in the interval $[\theta_1, \pi)$. Thus, there exists a unique $\theta_2 \in (\theta_1, \pi)$ such that $S_4(\theta_2) = 0$.

If we let

$$S_5(\theta) = 4 - p + 2\cos\theta,$$

then $S_5(\pi/2) = 4 - p > 0$ and $S_5(\pi) = 2 - p < 0$, so that $S_5(\theta)$ has a single root $\theta = \pi - \theta_3$ in the interval $[0, \pi)$ with $0 < \theta_3 = \arccos\frac{4-p}{2} < \frac{\pi}{2}$. From (13) we get $S_4(\pi - \theta_3) = S_6(p)$, where

$$S_6(s) = \pi - \arccos\frac{4-s}{2} - \sqrt{\frac{6-s}{s-2}}\,.$$

For $s \in [2, 3]$ we have

$$S_6'(s) = \frac{4-s}{(s-2)\sqrt{(s-2)(6-s)}} > 0,$$

so that

$$S_6(p) > S_6(2.7) = \pi - \arccos\frac{13}{20} - \sqrt{\frac{33}{7}} = 0.107\ldots > 0$$

(in fact, we have $S_6(p) = 0.337\ldots$).

From (12) we deduce

$$S_3(0) = \lim_{\theta \searrow 0} S_3(\theta) = \frac{-20 - 10p - p^2}{2(6-p)}\,.$$

Since $5 - \sqrt{5} < 2.77 < p < 5 + \sqrt{5}$, it follows that $S_3(0) > 0$ (in fact, we have $S_3(0) = 0.938\ldots$).

Since $S_4(\theta_2) = 0$, from (13) we get

$$\theta_2 = 2\,\frac{14-7p+p^2+(10-3p)\cos\theta_2}{12-6p+p^2+4(3-p)\cos\theta_2}\,\mathrm{tg}\,\frac{\theta_2}{2}$$

and substituting into (12) we deduce

$$S_3(\theta_2) = S_7(\cos\theta_2)$$

with

$$S_7(s) = \ln(1 + s) - \frac{(4 - p)[20 - 10p + p^2 + 4(5 - p)s]}{4[12 - 6p + p^2 + 4(3 - p)s]}.$$

We have

$$S_7'(s) = \frac{(2 - p)P_4(p) + 2P_5(p)s + 16(3 - p)^2 s^2}{(1 + s)[12 - 6p + p^2 + 4(3 - p)s]^2}$$

with

$$P_4(s) = 72 - 52s + 12s^2 - s^3$$
$$P_5(s) = 144 - 136s + 44s^2 - 5s^3$$

Since $P_4'(s) = -52 + 24s - 3s^2$ and $P_5'(s) = -136 + 88s - 15s^2$ are negative for all s real, we get $P_4(p) < P_4(2.7) = -0.603\ldots < 0$ and $P_5(p) < P_5(2.7) = -0.855\ldots < 0$ (in fact, we have $P_4(p) = -1.241\ldots$ and $P_5(p) = -1.401\ldots$)

Therefore $S_7'(s) > 0$, for $-1 < s \leqslant 0$. On the other hand we have $S_4(2\pi/3) = S_8(p - 2)$, with

$$S_8(s) = \frac{2\pi}{3} - \sqrt{3}\,\frac{4 - 2s + 2s^2}{2 + s^2}.$$

Since

$$S_8'(s) = 3\sqrt{3}\,\frac{2 - s^2}{(2 + s^2)^2} > 0$$

for $0 < s < 1$, we deduce $S_8(p - 2) < S_8(0.7) = \frac{2}{3}\left(\pi - \frac{432}{249}\sqrt{3}\right) = 0.091\ldots > 0$ (in fact, we have $S_8(p - 2) = 0.176\ldots$). Hence $S_4(2\pi/3) > 0$, which shows that $\theta_2 > 2\pi/3$. Since $\cos\theta_2 < \cos(2\pi/3) = -1/2$, we have

$$S_3(\theta_2) = S_7(\cos\theta_2) < S_7(-1/2) = -\frac{(p - 2)(2p - 5)}{6 - 4p + p^2} < 0.$$

We summarize the above partial results in the following diagram

θ	0		$\pi - \theta_3$		θ_2		π
$S_4(\theta)$	0	$+$	$+$	$+$	0	$-$	$-\infty$
$S_3'(\theta)$	$+$	$+$	$+$	$+$	0	$-$	$-$
$S_3(\theta)$	$S_3(0) > 0$	$\nearrow$	$+\infty \mid_{-\infty}$	$\nearrow$	$S_3(\theta_2) < 0$	$\searrow$	$-\infty$
$S_5(\theta)$	$+$	$+$	0	$-$	$-$	$-$	$-$
$S_2'(\theta)$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$S_2(\theta)$	0	$\nearrow$		$\nearrow$		$\nearrow$	$+\infty$

From this diagram and (11) we deduce $S_1(0) = 0$ and $S_1(\theta) > 0$ for all $\theta \in (0, \pi)$. Consequently, from (10) we get (9) and (8). From (8) and (7) we get (4). This completes the proof of the Theorem.

BIBLIOGRAPHY

[1] Libera, R. J., *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755–758.

[2] Ruscheweyh, St., V. Singh, *On a Briot-Bouquet equation related to univalent functions*, Rev. Roum. Math. Pures et Appl. (to appear).

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