MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

MATHEMATICA, Tome 23 (46), N° 2, pp. 225—230

THE ORDER OF STARLIKENESS OF CERTAIN INTEGRAL OPERATORS

by

PETRU T. MOCANU, DUMITRU RIPEANU and IOAN ŞERB

1. Introduction. Let α be a real number, $\alpha < 1$, and denote by $S^*(\alpha)$ the class of functions $f(z) = z + a_2z^2 + \ldots$, which are regular in the unit disc $U = \{|z| < 1\}$ and satisfy

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in U.$$

If $\alpha \in [0, 1)$, then $S^*(\alpha)$ is the well-known class of starlike functions of order α and $S^*(\alpha) \subset S^*(0) = S^*$, the class of starlike functions. We also are interested in the case $\alpha < 0$, and continue to call „starlike of order α" any function in $S^*(\alpha)$, although it is not necessarely starlike, not even univalent.

Let β and γ be real numbers and suppose $\beta > 0$, $\beta + \gamma > 0$ and $-\frac{\gamma}{\beta} \leqslant \alpha < 1$. From a more general result on Briot—Bouquet differential subordination [1, Theorem 1] it is easy to deduce that the integral operator $I_{\beta,\gamma}$ defined by $g = I_{\beta,\gamma}(f)$, where

$$g(z) = \left(\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(w) w^{\gamma-1} dw\right)^{1/\beta}, \quad z \in U,\ f \in S^*(\alpha) \tag{1}$$

maps $S^*(\alpha)$ into $S^*(\alpha)$, i.e. $I_{\beta,\gamma} \colon S^*(\alpha) \to S^*(\alpha)$.

For a given $\alpha \in \left[-\frac{\gamma}{\beta}, 1\right)$ we define the *order of starlikeness* of the class $I_{\beta,\gamma}(S^*(\alpha))$ by the largest number $\delta = \delta(\alpha; \beta, \gamma)$ such that $I_{\beta,\gamma}(S^*(\alpha)) \subset S^*(\delta)$.

In this paper we find $\delta(\alpha; \beta, \gamma)$, for appropriate choices of α, β and γ, by using the sharp subordination result recentely obtained in [1] and [3], as well as the useful lemma due to J. FENG and D. R. WILKEN [2]. Our general result includes some particular ones obtained by several authors [2, 4, 5, 6, 7, 8, 9].

2. Preliminaries. Let f and g be regular in U. We say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

We will make use of the following lemmas to prove our main result.

Lemma A. *Let $\beta > 0$, $\beta + \gamma > 0$ and $-\gamma/\beta \leqslant \alpha < 1$. Then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad q(0) = 1,$$

has a univalent solution in U, given by

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta} \tag{2}$$

where

$$Q(z) = \int_0^1 \left(\frac{1 - z}{1 - tz}\right)^{2\beta(1-\alpha)} t^{\beta+\gamma-1}\, dt, \quad z \in U, \tag{3}$$

and $q(z) \prec [1 + (1 - 2\alpha)z]/(1 - z)$.

If $p(z) = 1 + p_1 z + \ldots$ is regular in U and satisfies the differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \tag{4}$$

then $p(z) \prec q(z)$ and this subordination is sharp.

More general forms of this lemma may found in [1, Corollary 2.1] and [3, Corollary 1.1].

Lemma B. *Let $d\mu(t)$ be a positive measure on $[0, 1]$ and let $Q(z, t)$ be a complex-valued function defined on $U \times [0, 1]$, such that $Q(z, \cdot)$ is integrable on $[0, 1]$ for each $z \in U$. Suppose that $\operatorname{Re} Q(z, t) > 0$ in U, $Q(-r, t)$ is real and*

$$\operatorname{Re} \frac{1}{Q(z, t)} \geqslant \frac{1}{Q(-r, t)}$$

for $|z| \leqslant r < 1$ and $t \in [0, 1]$.

If

$$Q(z) = \int_0^1 Q(z, t) d\mu(t)$$

then

$$\operatorname{Re} \frac{1}{Q(z)} \geqslant \frac{1}{Q(-r)}, \quad \text{for } |z| \leqslant r.$$

This lemma is due to J. FENG and D. R. WILKEN [2, Lemma 2].

3. Main result.

THEOREM. *Let $\beta > 0$, $\beta + \gamma > 0$ and consider the integral operator $I_{\beta,\gamma}$ defined by (1). If $\alpha \in [-\gamma/\beta, 1)$ then the order of starlikeness of the class $I_{\beta,\gamma}(S^*(\alpha))$ is given by*

$$\delta(\alpha; \beta, \gamma) = \inf_{|z|<1} \operatorname{Re} q(z). \tag{5}$$

Moreover if $\alpha \in [\alpha_0, 1)$, where

$$\alpha_0 = \max\left\{\frac{\beta - \gamma - 1}{2\beta}, -\frac{\gamma}{\beta}\right\} \tag{6}$$

and $g = I_{\beta,\gamma}(f)$, for $f \in S^*(\alpha)$, then

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geqslant q(-r) = \frac{1}{\beta}\left[\frac{\beta + \gamma}{F(1, 2\beta(1 - \alpha), \beta + \gamma + 1; r/(1 + r))} - \gamma\right], \tag{7}$$

for $|z| \leqslant r < 1$ and

$$\delta(\alpha; \beta, \gamma) = q(-1) = \frac{1}{\beta}\left[\frac{\beta + \gamma}{F(1, 2\beta(1 - \alpha), \beta + \gamma + 1; 1/2)} - \gamma\right], \tag{8}$$

where q is given by (2) and $F(a, b, c; z)$ is the hypergeometric function. The extremal function is given by $g = I_{\beta,\gamma}(k)$, where $k(z) = z(1 - z)^{2(\alpha-1)}$.

Proof. If we let $p(z) = \dfrac{zg'(z)}{g(z)}$ from (1) we obtain

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{zf'(z)}{f(z)}.$$

Since $f \in S^*(\alpha)$ is equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z},$$

we deduce that $p(z)$ satisfies the differential subordination (4) and henc, by Lemma A, $p(z) \prec q(z)$ which implies (5).

Next we shall use the following well-known formulas

$$\int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a}\, dt = \frac{\Gamma(b)\Gamma(c - b)}{\Gamma(c)} F(a, b, c; z), \quad c > b > 0, \tag{9}$$

and

$$F(a, b, c; z) = (1 - z)^{-a} F\left(a, c - b, c; \frac{z}{z - 1}\right), \tag{10}$$

which hold for all $z \in \mathbf{C} \setminus (1, \infty)$.

Suppose $\alpha \in (\alpha_0, 1)$ where α_0 is given by (6) and denote $a = 2\beta(1 - \alpha)$, $b = \beta + \gamma$ and $c = \beta + \gamma + 1 = b + 1$. Since $c > b > 0$, from (3), by using (9) and (10), we deduce

$$Q(z) = (1 - z)^a \frac{\Gamma(b)\Gamma(c - b)}{\Gamma(c)} F(a, b, c; z) = \frac{1}{\beta + \gamma} F\left(a, c - b, c; \frac{z}{z - 1}\right) =$$

$$= \frac{1}{\beta + \gamma} F\left(1, a, c; \frac{z}{z - 1}\right), \quad z \in U.$$

Since $\alpha > \alpha_0$ implies $c > a$, by using again (9), we get

$$Q(z) = \int_0^1 \frac{1 - z}{1 - (1 - t)z}\, d\mu(t), \quad z \in U,$$

where

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}\, dt.$$

If we let $Q(z, t) = (1-z)/(1-(1-t)z)$, then $\operatorname{Re} Q(z, t) > 0$, $Q(-r, t)$ is real and

$$\operatorname{Re}\frac{1}{Q(z, t)} = \operatorname{Re}\frac{1-(1-t)z}{1-z} \geqslant \frac{1+(1-t)r}{1+r} = \frac{1}{Q(-r, t)},$$

for $|z| \leqslant r < 1$, $0 \leqslant t \leqslant 1$. By Lemma B we have

$$\operatorname{Re}\frac{1}{Q(z)} \geqslant \frac{1}{Q(-r)},$$

which implies (7). If we take $g = I_{\beta,\gamma}(k)$, then a simple computation yields $p = q$, which shows that (7) is sharp. In the case $\alpha = \alpha_0$, we obtain (7), by letting $\alpha \to \alpha_0^+$. From (7) we obtain (8), by letting $r \to 1^-$. This completes the proof of our theorem.

4. Particular cases. 1) Let $\beta > 0$ and $\beta + \gamma \geqslant 1$. In this case $\alpha_0 = (\beta - \gamma - 1)/2\beta$ and if we take $\alpha = \alpha_0$, i.e. $c = a$, by using $F(1, a, a; z) = (1-z)^{-1}$, we easily get $q(z) = (\beta + \gamma z)/\beta(1-z)$ and

$$\delta\left(\frac{\beta-\gamma-1}{2\beta};\ \beta, \gamma\right) = \frac{\beta-\gamma}{2\beta}$$

which shows that, if $\beta \geqslant \gamma \geqslant 1 - \beta$, the integral operator $I_{\beta,\gamma}$ maps each starlike function of order $(\beta - \gamma - 1)/2\beta$ into a starlike (univalent) function. This result is sharp if $\beta = \gamma \geqslant 1/2$. For instance, if $\beta = \gamma = 1$, we deduce that the Libera integral operator maps starlike functions of order $-1/2$ into starlike functions.

2) Let $\beta > 0$ and $\beta + \gamma \leqslant 1$. In this case $\alpha_0 = -\gamma/\beta$ and we have

$$\delta(-\gamma/\beta;\ \beta, \gamma) = \frac{1}{\beta\sqrt{\pi}}\frac{\Gamma(\beta+\gamma+1/2)}{\Gamma(\beta+\gamma)} - \gamma/\beta.$$

If in this last formula we take $\gamma = 0$ and $\beta = 1/\alpha \leqslant 1$, we get

$$\delta(0;\ 1/\alpha, 0) = \frac{\Gamma(1/\alpha + 1/2)}{\sqrt{\pi}\,\Gamma(1/\alpha + 1)},$$

which is the order of starlikeness of α-convex functions, for $\alpha \geqslant 1$, [4]. We note that $\delta(0;\ 1/\alpha, 0) = 0 \neq q(-1)$, for $\alpha \in [0, 1)$.

3) Let $\beta > 0$ and $\beta + \gamma = 1$. In this case $\alpha_0 = 1 - \frac{1}{\beta}$ and for $\alpha \in [1 - 1/\beta, 1)$ we have

$$\delta(\alpha;\ \beta, 1-\beta) = \begin{cases} \dfrac{1}{\beta}\left[\dfrac{1-2\beta(1-\alpha)}{2-2^{2\beta(1-\alpha)}} + \beta - 1\right], & \alpha \neq 1 - 1/2\beta \\ \dfrac{1}{\beta}\left[\dfrac{1}{2\ln 2} + \beta - 1\right], & \alpha = 1 - 1/2\beta. \end{cases}$$

If in this last formula we take $\beta = 1$, we obtain the order of starlikeness of convex functions of order α, for $\alpha \in [0, 1)$ [2], [9]. If we take $\beta = 1/2$, then, for $\alpha \in [-1, 1)$, we have

$$\delta(\alpha;\ 1/2, 1/2) = \begin{cases} \alpha/(1-2^{-\alpha}) - 1, & \alpha \neq 0 \\ 1/\ln 2 - 1 & \alpha = 0. \end{cases}$$

4) Let $\beta > 0$ and $\beta + \gamma = 2$. In this case $\alpha_0 = 1 - 3/2\beta$, and for $\alpha \in [1 - 3/2\beta, 1)$ we obtain

$$\delta(\alpha;\ \beta, 2-\beta) = \begin{cases} \dfrac{1}{\beta}\left[\dfrac{(1-2\beta(1-\alpha))(1-\beta(1-\alpha))}{2^{2\beta(1-\alpha)-1} - 2\beta(1-\alpha)} + \beta - 2\right], & \alpha \neq 1 - \dfrac{1}{2\beta},\ \alpha \neq 1 - 1/\beta \\ \dfrac{1}{\beta}\left[\dfrac{1}{2(1-\ln 2)} + \beta - 2\right], & \alpha = 1 - 1/2\beta \\ \dfrac{1}{\beta}\left[\dfrac{1}{2(2\ln 2 - 1)} + \beta - 2\right], & \alpha = 1 - 1/\beta, \end{cases}$$

If we take $\beta = 1$, we obtain the following result.

Corollary, *If $\alpha \in [-1/2, 1)$ then the order of starlikeness of the class $L(S^*(\alpha))$, where L is the Libera integral operator defined by*

$$L(f)(z) = \frac{2}{z}\int_0^z f(w)\, dw$$

is given by

$$\delta(\alpha;\ 1, 1) = \begin{cases} \dfrac{\alpha(2\alpha-1)}{2(2^{-2\alpha} + \alpha - 1)} - 1, & \alpha \neq 1/2,\ \alpha \neq 0 \\ \dfrac{1}{2(1-\ln 2)} - 1, & \alpha = 1/2 \\ \dfrac{1}{2(2\ln 2 - 1)} - 1, & \alpha = 0. \end{cases}$$

We note that $\delta(-1/2;\ 1, 1) = 0$, but $\delta(-1;\ 1, 1) = -1 \neq q(-1) = -1/4$.

Remarks. 1) If A_β is the operator defind by $A_\beta(f)(z) = z(f(z)/z)^\beta$, then we have $I_{\beta,\gamma} = A_{1/\beta} \circ I_{1,\beta+\gamma-1} \circ A_\beta$. Since A_β maps bijectively $S^*(\alpha)$ onto $S^*(\alpha\beta + 1 - \beta)$, we easily deduce the following relationship

$$\beta\delta(\alpha;\ \beta, \gamma) = \delta(\alpha\beta + 1 - \beta;\ 1, \beta + \gamma - 1) + \beta - 1,$$

which shows that for the problem involved in this paper it is sufficient to consider integral operators of the form $I_{1,\gamma}$, with $\gamma + 1 > 0$.

2) For some particular cases, in [5], [6] and [7] elementary proofs of the fact that $\min_{|z|=1} \operatorname{Re} q(z) = q(-1)$ were given. We note that in all those cases it was proved that $z = -1$ is the only point of minimum.

REFERENCES

[1] Eenigenburg, P., Miller, S. S., Mocanu, P. T., Reade, M. O., *On a Briot-Bouquet differential subordination* (submitted).

[2] Feng, J., Wilken, D. R., *A remark on convex and starlike functions*, J. London Math. Soc., (2), 21 (1980), 287–290.

[3] Miller, S. S., Mocanu, P. T., *Univalent solutions of Briot-Bouquet differential equations* (in preparation).

[4] Miller, S. S., Mocanu, P. T., Reade, M. O., *The order of starlikeness of alpha-convex functions*, Mathematica (Cluj), 20 (43), 1 (1978), 25–30.

[5] Mocanu, P. T., Reade, M. O., Ripeanu, D., *The order of starlikeness of a Libera integral operator*, Mathematica (Cluj), 19 (42), 1 (1977), 67–73.

[6] Mocanu, P. T., Ripeanu, D., *The order of starlikeness of the Libera transform of the class of starlike functions of order 1/2*, Mathematica (Cluj) (to appear).

[7] Mocanu, P. T., Ripeanu, D., Şerb, I., *On the order of starlikeness of convex functions of order α* (submitted).

[8] Ruscheweyh, S., Singh, V., *On a Briot-Bouquet equation related to univalent functions*, Rev. Roum. Math. Pures et Appl. 24 (1979), 285–290.

[9] Zmorovich, V. A., Korobkova, I. K., *On the order of starlikeness of convex functions of order α*, Dopov. Akad. Nauk Ukr. SSR, ser. A, 7 (1977), 584–587 (Russian).

Received 10. I. 1981

Department of Mathematics
Babeş-Bolyai University
3400 *Cluj-Napoca, România*