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To Professor CAIUS JACOB
for his seventieth anniversary

THE ORDER OF STARLIKENESS OF THE LIBERA TRANSFORM OF THE CLASS OF STARLIKE FUNCTIONS OF ORDER 1/2

by

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1. Introduction. A function f is said to be starlike of order α if it is regular in the unit disc $U = \{z : |z| < 1\}$, $f(0) = 0$, $f'(0) = 1$ and the inequality

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha$$

holds for $z \in U$. The class of starlike functions of order α shall be denoted by S^*_α. If $0 \leqslant \alpha < 1$, then $S^*_\alpha \subset S^*_0 \equiv S^*$, the class of starlike functions.

The order of starlikeness of a family $\mathcal{F}$ of starlike functions is defined by the largest number $\alpha = \alpha[\mathcal{F}]$ such that $\mathcal{F} \subset S^*_\alpha$.

In [2] R. LIBERA showed that if $f \in S^*$ then the function $F = L(f)$ defined by

$$F(z) = \int_0^z f(w)dw \tag{1}$$

is also in S^*, that is $L(S^*) \subset S^*$. The order of starlikeness of the class $L(S^*)$ was obtained in [3].

In this paper we find the order of starlikeness of the class $L(S^*_{1/2})$. We first show that a simple application of a recent result of P. EENIGENBURG S. MILLER, M. READE and the first author [1] reduces our problem to a computation involving a specific function. It is this elementary computation which enables us to obtain the order of starlikeness of $L(S^*_{1/2})$.

2. Preliminaries. Let $g(z)$ and $G(z)$ be regular in U. We say $g(z)$ is subordinate to $G(z)$, written $g(z) \prec G(z)$, if $G(z)$ is univalent, $g(0) = G(0)$ and $g(U) \subset G(U)$.

The following sharp result concerning a Briot-Bouquet differential subordination was obtained in [1]:

Lemma 1. *Let $p(z)$ be regular in U, $p(0) = 1$, and let it satisfy the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 - (1 - 2\delta)z}{1 + z} \equiv h(z),$$

with $\beta > 0$, $\operatorname{Re} \gamma \geqslant 0$ and $-\operatorname{Re} \gamma/\beta \leqslant \delta < 1$. Then the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \quad q(0) = 1,$$

has a univalent solution $q(z)$ and $q(z) \prec h(z)$. In addition $p(z) \prec q(z)$ and this subordination is sharp.

It is easy to check that q is given by

$$q(z) = \frac{zK'(z)}{K(z)} = \frac{\beta + \gamma}{\beta} \left[\frac{H(z)}{K(z}\right]^{\beta} - \frac{\gamma}{\beta}$$

where

$$K(z) = \left(\frac{\beta + \gamma}{z} \int_0^z H^{\beta}(w) w^{\gamma - 1}\, dw\right)^{1/\beta}$$

and

$$H(z) = z \exp \int_0^z \frac{h(w) - 1}{w}\, dw.$$

Lemma 2. *If $f \in S^*_{1/2}$ and $F = L(f)$ is defined by (1), then*

$$\frac{zF'(z)}{F(z)} \prec \frac{zK'(z)}{K(z)}$$

where

(2) $$K(z) = \frac{2}{z} \int_0^z \frac{w\, dw}{1 + w} = \frac{2}{z} [z - \ln\ (1 + z)], \ (\ln\ 1 = 0).$$

Proof. If we let $p(z) = zF'(z)/F(z)$, from (1) we get

$$p(z) + \frac{zp'(z)}{p(z) + 1} = \frac{zf'(z)}{f(z)} \prec \frac{1}{1 + z}$$

and the conclusion of Lemma 2 follows immediately from Lemma 1 if we take $\beta = \gamma = 1$ and $\delta = 1/2$.

3. Main result.

THEOREM *The order of starlikeness of the class $L(S^*_{1/2})$ where L is the Libera integral operator defined by (1) is given by*

$$\alpha[L(S^*_{1/2})] = \min_{|z|=1} \operatorname{Re}\ q(z) = q(1) = \frac{2 \ln 2 - 1}{2(1 - \ln 2)} = 0.629 \ldots$$

where

(3) $$q(z) = \frac{z^2}{(1 + z)[z - \ln\ (1 + z)]} - 1.$$

Proof. If we let $q(z) = zK'(z)/K(z)$, where $K(z)$ is given by (2), from Lemma 2 we obtain

$$\alpha[L(S^*_{1/2})] = \inf_{|z|<1} \operatorname{Re}\ q(z) = \min_{|z|=1} \operatorname{Re}\ q(z),$$

where $q(z)$ is given by (3). Let

$$R(t) = \operatorname{Re}\ q(e^{it}), \quad t \in (-\pi, \pi).$$

We shall prove that

$$\min_{-\pi<t<\pi} R(t) = R(0) = q(1)$$

by showing that

(5) $$R(t) > R(0), \text{ for all } t \in (-\pi, \pi), \quad t \neq 0.$$

Since $R(-t) = R(t)$, it is sufficient to suppose $t \in (0, \pi)$.

From (3) we get

$$R(t) = \frac{1}{N(t)}\left[2 - t \operatorname{tg} \frac{t}{2} (1 + 2 \cos t) + (1 - 2 \cos t) \ln 2(1 + \cos t)\right] - 1,$$

where

$$N(t) = (2 \sin t - t)^2 + [-2 \cos t + \ln 2(1 + \cos t)]^2.$$

The equation $2 \sin t - t = 0$ has a unique root t_0 in the interval $(0, \pi)$ and $t_0 > \pi/2$. Hence $N(t_0) = u^2(\sqrt{4 - t_0^2})$, where $u(s) = s + \ln (2 - s)$, $s \in (0, 2)$. Since $u(s) = 0$ has a unique root $s_0 > 3/2$ and $\sqrt{4 - t_0^2} < 3/2$, we deduce $N(t_0) > 0$. Therefore $N(t) > 0$ for all $t \in (0, \pi)$. It follows that

$$\operatorname{sign}\ [R(t) - R(0)] = \operatorname{sign}\ H(t),$$

where

(6) $$\begin{aligned} H(t) &= (2 - a)N(t)[R(t) - R(0)] = \\ &= -2a + t[4 \sin t - (2 - a) \operatorname{tg} \frac{t}{2} (1 + 2 \cos t)] - t^2 + \\ &+ (2 - a + 2a \cos t) \ln\ [2(1 + \cos t)] - \ln^2[2(1 + \cos t)], \end{aligned}$$

with $a = 2 \ln 2 = 1.386294 \ldots$

We shall write successively

$$(7)\qquad H_1(t) = \frac{H'(t)}{2\left(\operatorname{tg}\frac{t}{2} - a\sin t\right)} = \ln\,[2(1+\cos t)] +$$

$$+ \frac{-2a\sin t + t[a - 2(a-1)\cos t - 2a\cos^2 t]}{2(a - 1 + a\cos t)\sin t}$$

$$(8)\qquad H_2(t) = \frac{2(a-1+a\cos t)^2\sin^2 t}{2 - 4a + 3a^2 + 3a(a-1)\cos t}\,H'_1(t) =$$

$$= t + \frac{-2+3a-3a^2+a(4-3a)\cos t}{2-4a+3a^2+3a(a-1)\cos t}\sin t$$

$$H'_2(t) = -\frac{(1-\cos t)(a-1+a\cos t)[4-14a+21a^2-9a^3-3a(a-1)(3a-4)\cos t]}{[2-4a+3a^2+3a(a-1)\cos t]^2}$$

We have $2 - 4a + 3a^2 + 3a(a-1)\cos t \geqslant 2 - a > 0$ and

$$4 - 14\,a + 21a^2 - 9a^3 - 3a(a-1)(3a-4)\cos t \geqslant 2P_1(a),$$

with $P_1(s) = 2 - 13s + 21s^2 - 9s^3$. It is easy to show that $P_1(s) > 0$, for $s \in (1,\ 1.4)$, which implies $P_1(a) > 0$. We conclude that $H'_2(t)$ vanishes only at the point $t_1 = \pi - \arccos\,[(a-1)/a]$, $\pi/2 < t_1 < \pi$.

Since $(7 - \sqrt{17})/8 < 1 < a < (7 + \sqrt{17})/8 = 1.39\ldots$, we deduce

$$H_1(0) = \frac{2 - 7a + 4a^2}{2(2a-1)} < 0.$$

Therefore we have the following table

t	0		$\pi/2$		t_1		t_2		π
$H'_2(t)$		$-$	$-$	$-$	0	$+$	$+$	$+$	
$H_2(t)$	0		$\searrow$		$H_2(t_1) < 0$	$\nearrow$	0	$\nearrow$	$+\infty$
$H'_1(t)$	$-$		$-$	$-$	∞	$-$	0	$+$	$+\infty$
$H_1(t)$	$H_1(0)$	0 $\searrow$		$-\infty$	|	$+\infty$ $\searrow$	$H_1(t_2)$	$\nearrow$	$+\infty$

From Table 1 we deduce that $H_2(t)$ has a unique root t_2, in the interval $(0,\ \pi)$ and $t_1 < t_2$. We shall show that $H_1(t_2) > 0$. Since $H_2(t_2) = 0$, from (8) we get

$$(9)\qquad t_2 = -\frac{-2+3a-3a^2+a(4-3a)\cos t_2}{2-4a+3a^2+3a(a-1)\cos t_2}\sin t_2.$$

From (7) and (9) we deduce

$$(10)\qquad H_1(t_2) = H_3(\cos t_2)$$

with

$$(11)\qquad H_3(s) = \frac{a(2-3a) - 2(2-3a+3a^2)s + 2a(3a-4)s^2}{2[2-4a+3a^2+3a(a-1)s]} - \ln\,[2(1+s)].$$

Hence

$$H'_3(s) = \frac{P_2(s)}{2(1+s)\,[2-4a+3a^2+3a(a-1)s]^2}$$

with

$$P_2(s) = a(-4 + 14a - 21a^2 + 9a^3) + (a-1)(2-a)(2-3a)^2 s +$$
$$+ 2a(16 - 47a + 51a^2 - 18a^3)s^2 - 6a^2(a-1)(3a-4)s^3.$$

We have $P_2(-1) = 2(2-a)^2 < 0$ and $P_2(0) = -aP_3(a)$, with $P_3(s) = 4 - 14s + 21s^2 - 9s^3$. It is easy to show that $P_3(s) > 0$, for $s \in (1,1\cdot 4)$, which implies $P_2(0) < 0$. Since

$$P'_2(s) = (a-1)(2-a)(2-3a)^2 + 4a(16-47a+51a^2-18a^3)s -$$
$$-18a^2(a-1)(3a-4)s^2,$$

we deduce $P'_2(0) > 0$ and $P'_2(-1) = P_4(a)$, with $P_4(s) = -8 - 28s + 58s^2 - 39s^3 + 9s^4$. Further we have $P'_4(s) = -28 + 116s - 117s^2 + 36s^3$ and $P''_4(s) = 2(58 - 117s + 54s^2)$. We easily deduce that $P'_4(s)$ has a minimum in the interval $(1,1.4)$ at the point $s_0 = 1.398\ldots$, where $P''_4(s_0) = 0$, and $P'_4(s_0) = (250 - 129\,s_0)/18 > (250 - 129 \times 1.4)/18 > 0$. Hence $P'_4(s) > 0$, for $s \in (1,1.4)$. Since $P_4(1) = -8$ and $P_4(1.4) = -5.9616$, we deduce $P'_2(-1) = P_4(a) < 0$. Therefore we have the following table

s	-1		s_3		s_2		s_1		0
$P'_2(s)$	$P'_2(-1) < 0$		$-$		$-$		0	$+$	$P'_2(0) > 0$
$P_2(s)$	$P_2(-1) > 0$		$\searrow$		0	$\searrow$	$P_2(s_1)$	$\nearrow$	$P_2(0) < 0$
$H'_3(s)$	$+$	$+$	$+$		0		$-$		$-$
$H_3(s)$	$-\infty$	$\nearrow$	0	$\nearrow$	$H_3(s_2)$		$\searrow$		$H_3(0) > 0$

From (8) we deduce $H_2(3\pi/4) = H_4(a)$, with

$$H_4(s) = \frac{3\pi}{4} - \frac{1}{\sqrt{2}}\,\frac{2(2+\sqrt{2}) - (2-\sqrt{2})s + 3s^2}{2(2+\sqrt{2}) - (5+\sqrt{2})s + 3s^2}$$

Since

$$H'_4(s) = -\frac{3+2\sqrt{2}}{2}\,\frac{2(2+\sqrt{2}) - 3s^2}{[2(2+\sqrt{2}) - (5+\sqrt{2})s + 3s^2]^2} < 0,$$

for $s \in (1,\ 1.4)$, we have

$$H_4(a) > H_4(1.4) = \frac{1}{2}\left(\frac{3\pi}{2} - \frac{1155 + 1699\sqrt{2}}{789}\right) >$$

$$> \frac{1}{2}\left(3.14 \times \frac{3}{2} - \frac{1155 + 1700 \times 1.42}{789}\right) = .092\ldots > 0.$$

Hence $H_2(3\pi/4) > 0$ and by using Table 1 we deduce $t_2 < 3\pi/4$, which implies $\cos t_2 > -1/\sqrt{2}$. From (11) we get $H_3(-1/\sqrt{2}) = H_5(a)$, with

$$H_5(s) = \ln 2 - \ln (2 + \sqrt{2}) + \frac{2(1 + \sqrt{2}) + 3s - 3s^2}{2(2 + \sqrt{2}) - ((5 + \sqrt{2})s + 3s^2}.$$

We have

$$H_5'(s) = \frac{(2 + \sqrt{2})\, P_5(s)}{[2(2 + \sqrt{2}) - (5 + \sqrt{2})s + 3s^2]^2},$$

with $P_5(s) = 8 + 5\sqrt{2} - 6(2 + \sqrt{2})s + 3s^2$.

Since $P_5(1) < 0$ and $P_5(1.4) = -(73 + 85\sqrt{2})/25 < 0$, we deduce $P_5(s) < 0$, for $s \in (1, 1.4)$, which implies $H_5'(s) < 0$, for $s \in (1, 1.4)$. Hence

$$H_5(a) > H_5(1.4) = \ln 2 - \ln (2 + \sqrt{2}) + 2(-77 + 290\sqrt{2})/789 >$$

$$> -\frac{3}{5} + 2(-77 + 290 \times 1.41)/789 = .241\ldots > 0,$$

which shows that $H_3(-1/\sqrt{2}) > 0$. From Table 1 we deduce $s_3 < -1/\sqrt{2}$, which implies $\cos t_2 > s_3$. Using (10) and Table 2, we get $H_1(t_2) > 0$. Hence from (7) and Table 1 we deduce $H'(t) > 0$, for $t \in (0, \pi)$, which shows that $H(t)$ given by (6) is increasing for $t \in (0, \pi)$. Since $H(0) = 0$, we deduce $H(t) > 0$, for all $t \in (0, \pi)$, which implies (5). Since $\lim_{t \to \pi} R(t) = \infty$, from (5) we deduce that $\min_{|z|=1} \operatorname{Re} q(z)$ occurs if and only if $z = 1$. This completes the proof of the Theorem.

Remark. The order of starlikeness of the class $L(S_\alpha^*)$; for all $\alpha \in [-1/2, 1)$ was recently found in [4]. By the elementary method used in the present paper, as well as in [3] and [5], we have shown a little more, namely that $z = 1$ is the only point of the involved minimum.

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