ON THE APPROXIMATION BY LINEAR OPERATORS OF THE CLASS S_m

BY

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1. Let $C(X)$ be the space of all real-valued functions which are defined and continuous on $X = [a, b]$ and $B(X)$ be the Banach space of real-valued functions which are bounded on X. Let $\mathfrak{M}$ be a fixed set of linear operators $C(X) \to B(X)$ and H a linear subspace of $C(X)$.

Definition 1. *The closure of the subspace H relative to the class $\mathfrak{M}$ is a linear subspace $H(\mathfrak{M})$ of $C(X)$ defined in the following way: $f \in H(\mathfrak{M})$ if for each sequence of operators $(L_n)_{n=1}^{\infty} \subset \mathfrak{M}$, the equality*

$$\lim_{n\to\infty} \| h - L_n h \| = 0, \text{ for every } h \in H, \tag{1}$$

implies

$$\lim_{n\to\infty} \| f - L_n f \| = 0. \tag{2}$$

The theory of approximation by sequences of linear operators includes the treatment of the following three problems:

(A) Give necessary and sufficient conditions imposed to the subspace H, such that

$$H(\mathfrak{M}) = C(X). \tag{3}$$

(B) If $C(X) \setminus H(\mathfrak{M}) \neq \emptyset$ to describe the closure $H(\mathfrak{M})$, that is to obtain necessary and sufficient conditions which must be verified by an element $f \in C(X)$ such that $f \in H(\mathfrak{M})$.

(C) If the subspace H and the class $\mathfrak{M}$ for which (3) is fulfiled, are known, then to study the approximation properties of some concrete sequences of operators from $\mathfrak{M}$.

In (1) and (2) we consider the „sup"-norm. The formulation of the problems (A) and (B) are similar with the topics of Weierstrass-Stone's problem (see [8]—[9]). When $\mathfrak{M}$ is the set of linear positive operators, the problem (A) was solved by M. L. Brodski [3] and by Ju. A. Saskin [7]. Likewise for this case the problem (B) was solved by V. A. Baskakov [1] and independently by H. Bauer [2].

In [5], as a generalization of linear positive operators, P. P. Korovkin has introduced a new class of linear operators, the class S_m. For these operators the problem (A) is solved by R. M. Minkova and Ju. A. Saskin [6]. Our purpose is to give a solution for the problem (B), when $\mathfrak{M}$ is the class S_m of operators.

2. If $g \in C(X)$ vanishes at the point $x_0 \in X$ and g changes the sign, then we say that x_0 is a simple root. If $g(x_0) = 0$ and there is an open neighbourhood V of x_0 such that the sign of g is constant on $V \setminus \{x_0\}$, then we adopt the term of double root for x_0. If $g(a) = 0$ then a is a simple root and also the same convention for the second end-point b of X.

Let g be a function from $C(X)$ which has, on the interval X, at most m roots (taking into account the multiplicity). Likewise by ε_{x_0}, $x_0 \in X$, we denote the functional of evaluation which is defined as

$$\varepsilon_{x_0}(f) = f(x_0), \quad f \in C(X).$$

Definition 2. *A functional $F \in C(X)^*$ belongs to the class $S(m, g)$ if for each $f \in C(X)$ such that* sign $f(x) =$ sign $g(x)$, $x \in X$, *we have*

$$F(f) \geq 0;$$

A linear operator $L: C(X) \to B(X)$ belongs to the class S_m, iff for each $x_0 \in X$ there is a function $g_{x_0} \in C(X)$ such that

$$L_{x_0} \in S(m, g_{x_0}), \text{ where } L_{x_0}(f) = \varepsilon_{x_0}(Lf), \; f \in C(X).$$

When we do not use effectively the function g, instead of $S(m, g)$ we write $S(m)$. It is clear that S_0 is the set of linear positive operators and moreover

$$S_0 \subset S_1 \subset \ldots \subset S_m \subset \ldots .$$

If $x_0 \in X$ then $M^m_{x_0}(H)$ denotes the set of all linear functionals $F \in C(X)^*$ with the properties

1. $F \in S(m)$
2. $F(h) = \varepsilon_{x_0}(h)$ for every $h \in H$.

Definition 3. *A function $f \in C(X)$ is called (H_m)-harmonic at the point $x_0 \in X$, iff*

$$F_{x_0}(f) = \varepsilon_{x_0}(f) \text{ for every } F_{x_0} \in M^m_{x_0}(H).$$

However, this notion has a real meaning only in the case when the subspace H separates the points of X and contains the constant functions; in the case $m = 0$ the (H_0)-harmonic functions were introduced by H. Bauer [2].

Suppose now that $\mathfrak{M}_m$ is the set of all sequences of linear operators $L_n: C(X) \to B(X)$, $n = 1, 2, \ldots$, such that

(i) $\|L_n\| \leq M < +\infty$, $n = 1, 2, \ldots$,

(ii) $L_n \in S_m$, $n = 1, 2, \ldots$,

and in the same time we shall consider that H separates the points of X and $1 \in H$.

Lemma 1. (P. P. Korovkin [5]). *If $(F_n)_{n=1}^{\infty} \subset S(m)$ is a sequence weakly convergent to F, then $F \in S(m)$.*

Lemma 2. *Let $(F_n)_{n=1}^{\infty} \subset S(m)$, $\|F_n\| \leq M < +\infty$, $n = 1, 2, \ldots$, and $f_0 \in C(X)$. If f_0 is (H_m)-harmonic at the point $x_0 \in X$, then*

$$\lim_{n \to \infty} F_n(h) = \varepsilon_{x_0}(h) \text{ for every } h \in H,$$

implies

$$\lim_{n \to \infty} F_n(f_0) = \varepsilon_{x_0}(f_0).$$

Proof. Let (F_{n_k}), $k = 1, 2, \ldots$, be a subsequence weakly convergent to F. Because

$$\lim_{k \to \infty} F_{n_k}(h) = \varepsilon_{x_0}(h), \; h \in H,$$

we have

$$F(h) = \varepsilon_{x_0}(h), \quad h \in H.$$

If $F(f_0) \neq \varepsilon_{x_0}(f_0)$ then, by using lemma 1, we conclude that there is a functional $F \in M^m_{x_0}(H)$ which does not equal the functional of evaluation at f_0, and that is a contradiction.

Theorem. *The closure $H(\mathfrak{M}_m)$ coincides with the set of all (H_m)-harmonic functions on X.*

Proof. Let us suppose that $f_0 \in H(\mathfrak{M}_m)$ and that there is a point $x_0 \in X$ at which the function f_0 is not (H_m)-harmonic. Then there exists a functional $\varphi_{x_0} \in M^m_{x_0}(H)$ such that

$$\varphi_{x_0}(f_0) \neq \varepsilon_{x_0}(f_0).$$

We define the operators $L_n: C(X) \to B(X)$, $n = 1, 2, \ldots$, in the following way

$$(L_n f)(x) = \begin{cases} f(x) & \text{for } x \in X \setminus \{x_0\} \\ \varphi_{x_0}(f) & \text{for } x = x_0 \end{cases} \quad n = 1, 2, \ldots$$

If L_x is defined by $L_x(f) = \varepsilon_x(L_n f)$, $f \in C(X)$, then $L_{x_0} \in S(m)$ and $L_x \in S(0) \subset S(m)$ for $x \in X \setminus \{x_0\}$. Therefore for every $x \in X$, $L_x \in S(m)$, which is the same with the fact that $L_n \in S_m$, $n = 1, 2, \dots$. Because

$$\lim_{n \to \infty} \| h - L_n h \| = 0, \ h \in H,$$

we must have

$$\lim_{n \to \infty} \| f_0 - L_n f_0 \| = 0. \tag{4}$$

On the other hand the relation (4) is not valid taking into account that for each $n = 1, 2, \dots$, we have

$$(L_n f_0)(x_0) = \varphi_{x_0}(f_0) \neq f_0(x_0).$$

If there is a sequence $(L_n)_{n=1}^{\infty} \subset \mathfrak{M}_m$ for which

$$\lim_{n \to \infty} \| h - L_n h \| = 0, \ h \in H,$$

and

$$\lim_{n \to \infty} \| f_0 - L_n f_0 \| \neq 0,$$

that is, f_0 is a (H_m)-harmonic function which belongs to $C(X) \setminus H(\mathfrak{M}_m)$, then there is $\varepsilon > 0$, a sequence of natural numbers (n_k), $k = 1, 2, \dots$, $\lim_{k \to \infty} n_k = \infty$, and a sequence $(x_k)_{k=1}^{\infty} \subset X$, $\lim_{k \to \infty} x_k = x_0$, such that

$$| \varepsilon_{x_k}(f_0) - \varepsilon_{x_k}(L_{n_k} f_0) | \geq \varepsilon, \quad k = 1, 2, \dots . \tag{5}$$

Let $F_k(f) = \varepsilon_{x_k}(L_{n_k} f)$, $f \in C(X)$, $k = 1, 2, \dots$; according to the definition of the class $\mathfrak{M}_m$ and by lemma 2 it follows that

$$\lim_{k \to \infty} F_k(f_0) = \varepsilon_{x_0}(f_0),$$

which is in contradiction with (5).

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ASUPRA APROXIMĂRII PRIN OPERATORI LINIARI DÍN CLASA S_m

Rezumat

Fie $\mathfrak{M}_m$ mulţimea şirurilor uniform mărginite de operatori liniari $L_n : C(X) \to B(X)$ astfel încît $L_n \in S_m$, $n = 1, 2, \dots$, unde S_m este clasa de operatori introdusă de către P. P. Korovkin în lucrarea [5], ca o generalizare a mulţimii operatorilor pozitivi. În acest articol se rezolvă următoarea problemă: fixînd un subspaţiu liniar H al lui $C(X)$, să se găsească o condiţie necesară şi suficientă pe care să o verifice o funcţie $f_0 \in C(X)$ astfel încît pentru orice şir de operatori $L_n \in \mathfrak{M}_m$, $n = 1, 2, \dots$, din

$$\lim_{n \to \infty} \| h - L_n h \| = 0, \quad h \in H,$$

să rezulte

$$\lim_{n \to \infty} \| f_0 - L_n f_0 \| = 0.$$