AN EXTREMAL PROBLEM FOR THE TRANSFINITE DIAMETER OF A CONTINUUM

by

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In this paper we solve an extremal problem connected with the transfinite diameter of a continuum by using Schiffer's variational method [2] and also the same simple geometric arguments as described in the paper of Reich and Schiffer [1]. As the matter of fact, our problem is quite similar to those solved in [1].

Let

$$(1) \qquad \Phi(c_1, c_2, c_3) = |c_1 - c_2| + |c_2 - c_3| + |c_3 - c_1| ,$$

where c_1, c_2, c_3 are complex numbers. It is obvious that the function (1) , which represents the perimeter of the triangle (c_1, c_2, c_3) , is invariant under translations and rotations of the plane.

Let E be a continuum in the complex plane, and let c_1, c_2, c_3 be three arbitrary points belonging to E. Our problem is to find

$$(2) \qquad \sup_{c_1, c_2, c_3, E} \frac{\Phi(c_1, c_2, c_3)}{d(E)}$$

where $d(E)$ is the transfinite diameter of E.

The result is the following

THEOREM . *If* E *is a continuum in the plane and* c_1, c_2, c_3 *belong to* E , *then*

(3) $|c_1 - c_2| + |c_2 - c_3| + |c_3 - c_1| \leq 3^{3/2} 4^{1/3}\, d(E)$.

This inequality is sharp, equality being achieved if and only if E *is the union of three segments of equal length making angles of* $2\pi/3$ *with each other, having a common initial point, and* c_1, c_2, c_3 *as endpoints*.

PROOF. It is well-known that if c_1, c_2 belong to E , then $|c_1 - c_2| \leq 4d(E)$, [3]. Hence $\Phi(c_1, c_2, c_3)/\, d(E) \leq 12$, which shows that (2) exist and is assumed. Let E, c_1, c_2, c_3 be extremal for (2), and let D be the complementary domain of E which contains the point at infinity. Consider the conformal mapping

(4) $$w = f(z) = d(E)\left[z + a_o + \frac{a_1}{z} + \ldots \right]$$

of $1 < |z| < \infty$ onto D .

For $w_o \in D$, we consider the variation

(5) $$w^* = w + \frac{\lambda}{w - w_o}$$

where $|\lambda|$ is sufficently small. Denote by c_1^* , c_2^* , c_3^* , E^* the images of c_1, c_2, c_3, E by (5).

If we set $\Phi = \Phi(c_1, c_2, c_3)$ and $\Phi^* = \Phi(c_1^*, c_2^*, c_3^*)$, then we have the following variational formulas

(6) $$\log \Phi^* = \log \Phi - \mathrm{Re}\left\{ \frac{\lambda}{\Phi} A(c_1, c_2, c_3; w_o) \right\} + o(\lambda) ,$$

where

$$A(c_1, c_2, c_3; w_o) = \frac{|c_1 - c_2|}{(c_1 - w_o)(c_2 - w_o)} + \frac{|c_2 - c_3|}{(c_2 - w_o)(c_3 - w_o)} + \frac{|c_3 - c_1|}{(c_3 - w_o)(c_1 - w_o)}$$

and

(7) $$\log d(E^*) = \log d(E) - \operatorname{Re}\left\{\frac{\lambda}{z_0^2 f'(z_0)^2}\right\} + o(\lambda) ,$$

with $w_0 = f(z_0)$, [2].

Since $\Phi^*/d(E) \leq \Phi/d(E)$, we have

$$\log \Phi^* - \log d(E^*) \leq \log \Phi - \log d(E)$$

and by using (6) and (7) we obtain

$$\operatorname{Re}\left\{\frac{\lambda}{z_0^2 f'(z_0)^2} - \frac{\lambda}{\Phi} A(c_1,c_2,c_3;w_0)\right\} + o(\lambda) \leq 0$$

for all small enough values of $|\lambda|$. From this we conclude that if E, c_1, c_2, c_3 are extremal for (2), then the extremal function (4) satisfies the differential equation

(8) $$\frac{w - \Psi(c_1,c_2,c_3)}{(w-c_1)(w-c_2)(w-c_3)} = \frac{(dz)^2}{z^2(dw)^2}$$

where

$$\Psi(c_1,c_2,c_3) = \frac{1}{\Phi}\left[c_1|c_2-c_3| + c_2|c_3-c_1| + c_3|c_1-c_2|\right]$$

Since

$$\Psi(c_1+a, c_2+a, c_3+a) = \Psi(c_1,c_2,c_3) + a ,$$

and the extremal points are determined within an additive constant, we can suppose $\Psi(c_1,c_2,c_3) = 0$, i.e.

(9) $$c_1|c_2-c_3| + c_2|c_3-c_1| + c_3|c_1-c_2| = 0 .$$

The differential equation (8) becomes

(1o) $$\frac{w(dw)^2}{(w-c_1)(w-c_2)(w-c_3)} = \frac{(dz)^2}{z^2} , \quad |z| > 1 .$$

As in [1] it is easy to show that the extremum continuum E is the set of values omitted by the extremal function f , and the range D of f has no exterior points.

The extremal points c_k are distinct from each other and distinct from 0. Indeed, if $c_1=0$, then from (9) we deduce $c_2|c_3|=-c_3|c_2|$ and we have $\Phi \leq 2\max\{|c_2|,|c_3|\}$, and $\Phi/d(E)\leq 8$. If $c_1=c_2=a$, $c_3=b$, then $\Phi=2|b-a|\leq 8d(E)$. In each case the value of $\Phi/d(E)$ is not extremal. We remark that the extremal points c_1,c_2,c_3 can not be collinear, since in this case we also have $\Phi=2|c_1-c_2|\leq 8d(E)$, if we suppose that c_2 lies between c_1 and c_3.

Since Φ is invariant under rotations we can suppose $c_1>0$.

The extremum continuum E consists of the union of three analytic arcs γ_k, $k=1,2,3$, having 0 as the only common initial point and c_1,c_2,c_3 as endpoints. The three arcs γ_k meet 0 in equally spaced angles [1]. Using the same topological argument as in [1], we conclude that there exist numbers t_k, $0<t_k<1$, such that

$$(11)\qquad \operatorname{Im}\frac{c_{k+1}+c_{k+2}}{c_k}t_k=\operatorname{Im}\frac{c_{k+1}c_{k+2}}{c_k^2}, \quad k=1,2,3,$$

where we denote $c_4=c_1$, $c_5=c_2$.

From (9) we obtain

$$\operatorname{Im}\frac{c_{k+2}}{c_k}=-\frac{|c_{k+2}-c_k|}{|c_{k+1}-c_k|}\operatorname{Im}\frac{c_{k+1}}{c_k}$$

and

$$\operatorname{Re}\frac{c_{k+2}}{c_k}=-\frac{|c_{k+2}-c_{k+1}|}{|c_{k+1}-c_k|}-\frac{|c_{k+2}-c_k|}{|c_{k+1}-c_k|}\operatorname{Re}\frac{c_{k+1}}{c_k}.$$

Hence

$$(12)\qquad \operatorname{Im}\frac{c_{k+1}+c_{k+2}}{c_k}=\frac{|c_{k+1}-c_k|+|c_{k+2}-c_k|}{|c_{k+1}-c_k|}\operatorname{Im}\frac{c_{k+1}}{c_k}$$

$$(13)\qquad \operatorname{Im}\frac{c_{k+1}c_{k+2}}{c_k^2}=-\left[\frac{|c_{k+2}-c_{k+1}|}{|c_{k+1}-c_k|}+2\frac{|c_{k+2}-c_k|}{|c_{k+1}-c_k|}\operatorname{Re}\frac{c_{k+1}}{c_k}\right]\operatorname{Im}\frac{c_{k+1}}{c_k}$$

Using (9),(11),(12) and (13) we find that the extremal points c_1, c_2, c_3 satisfy the following conditions

(14) $c_1|c_2-c_1| + c_2|c_3-c_1| + c_3|c_1-c_2| = 0$

(15) $(|c_3-c_1| - |c_2-c_1|)\, t_1 = |c_3-c_2| + 2|c_3-c_1| \mathrm{Re}\frac{c_2}{c_1}$

(16) $(|c_1-c_2| - |c_3-c_2|)\, t_2 = |c_1-c_3| + 2|c_1-c_2| \mathrm{Re}\frac{c_3}{c_2}$

(17) $(|c_2-c_3| - |c_1-c_3|)\, t_3 = |c_2-c_1| + 2|c_2-c_3| \mathrm{Re}\frac{c_1}{c_3}$

where $c_1 > 0$ and $t_k \in (0,1)$.

We shall show that this conditions imply

(18) $|c_1 - c_2| = |c_2 - c_3| = |c_3 - c_1|$.

If we let

(19) $c_2 - c_1 = d = re^{it}$, $c_3 - c_1 = \delta = \varrho e^{i\tau}$

condition (14) becomes

(2o) $|c_3 - c_2| = |\delta - d| = -\left[r + \varrho + \frac{r\varrho}{c_1}(e^{it} + e^{i\tau})\right]$

From (15) we obtain

(21) $(r - \varrho)(1-t_1) = \frac{r\varrho}{c_1}\left[\cos t - \cos\tau - i(\sin t + \sin\tau)\right]$

If $r\varrho = 0$, then from (21) we deduce $r = \varrho = 0$ and from (19) we get the trivial solution $c_1 = c_2 = c_3$ which is not possible. Thus $r\varrho > 0$ and from (21) we obtain $\sin t + \sin\tau = 0$, which implies $\cos t = \pm\cos\tau$. Suppose $\cos\tau = -\cos t$. Then (2o) becomes $|\delta - d| = -(r + \varrho)$, that is, $r = \varrho = 0$ which is not possible. Therefore we have only the case $\cos t - \cos\tau = \sin t + \sin\tau = 0$ and from (21) we obtain $r = \varrho$. From (19) we deduce

(22) $c_2 = c_1 + re^{it}$, $c_3 = c_1 + re^{-it}$.

Employing (22) together with (16) and (17) ,we obtain

(23) $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$,

where $x = r/c_1$ and

$$(24)\begin{cases} a = -3 + 4\sin^2 t + (1 - 2|\sin t|)t_2 \\ b = 2\cos t\left[(-3 + (1 - 2|\sin t|)t_2\right] \\ c = -3 + (1 - 2|\sin t|)t_2 \\ a' = 1 + (1 - 2|\sin t|)t_3 \\ b' = 2\cos t\left[1 + 2|\sin t| + (1 - 2|\sin t|)t_3\right] \\ c' = 1 + 4|\sin t| + (1 - 2|\sin t|)t_3. \end{cases}$$

On the other hand,employing (22) together with (14) or (15) we get

(25) $x = -\dfrac{1 + |\sin t|}{\cos t}$.

If $\sin t = 0$,then $x = r/c_1 = -1$,which is not possible. If $\sin t > 0$,then from (23),(24) and (25) we obtain

$$(1-t_2)\sin^2 t\,(1 - 2\sin t) = 0$$

and

$$(1-t_3)\sin^2 t\,(1 - 2\sin t) = 0$$

hence $\sin t = 1/2$ and from (22) we get (18).In the case $\sin t < 0$ we obtain

$$(1-t_2)\sin^2 t\,(1 + 2\sin t) = 0$$

$$(1-t_3)\sin^2 t\,(1 + 2\sin t) = 0$$

hence $\sin t = -1/2$ and from (22) we also get (18).

We remark that (18) holds if we only suppose $t_k \neq 1$.

We conclude that the extremal points c_1, c_2, c_3 must satisfy (18),that is, (11) is of the form $0 \cdot t_k = 0$,for $k=1,2,3$. As in [1] this means that the arc γ_k coincides with the segment from 0 to c_k, for $k=1,2,3$. We have $c_2 = \omega c_1$, $c_3 = \omega^2 c_1$, where $\omega^3 = 1$ and the differential equation (1o) becomes

$$\frac{z^2 w}{w^3 - c_1^3}\left(\frac{dw}{dz}\right)^2 = 1 .$$

The extremal function will be

$$f(z) = d(E)\, z(1 + z^{-3})^{2/3} \quad ,$$

where $d(E) = 4^{-1/3}c_1$. Moreover the extremal value of Φ is $\Phi = 3\cdot 3^{1/2}c_1 = 3^{3/2}4^{1/3}d(E)$. This completes the proof of our Theorem.

COROLLARY. If the function

$$f(z) = z + a_0 + \frac{a_1}{z} + \ldots$$

is regular and univalent in $1 < |z| < \infty$ and E is the complement of its range, then the perimeter of any triangle with vertices in E is less or equal to $3^{3/2}4^{1/3}$. The equality holds if and only if the function f is

$$f(z) = a_0 + z(1 + e^{it}z^{-3})^{2/3}$$

In this case E is the union of three segments of equal length $L = 4^{1/3}$, making angles of $2\pi/3$ with each other, having a common intial point. The vertices of the triangle are the endpoints of the three segments.

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