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LINEAR OPERATORS THAT TRANSFORM A NORMAL CONE IN COMPLETELY REGULAR CONES

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The Fredholm resolvents of a wide class of operators, which are sublinear with respect to the ordering induced by the wedge W in the normed space Y, have the property that transform W into completely regular cones [6]. These resolvents approximate indefinitely the identity map in the topology of uniform convergence on norm bounded sets. This advantage is associated with the drawback that composing them with convex mappings with values in Y, the resulting operators fail to be convex with respect to the ordering induced by the transformed cone.

The linear operator A on Y has the property that composed with any W-convex operator yields a mapping which is $A(W)$-convex. The complete regularity of $A(W)$ remains of a crucial interest for applications. But it appears that when W isn't regular, the linear operator with this property cannot approximate indefinitely the identity map (Corollary 3). However, some important operators (see the example in 12) have good properties from this point of view. Hence we devote the present note to investigation of the linear operators A with the cone range $A(W)$ being a completely regular cone.

If the linear and positive operator A maps the closed normal cone C with nonempty interior, contained in the Banach space (B-space), Y, into a completely regular cone, then any abstract Hammerstein operator AF, where F is C-convex and continuous, is subdifferentiable at any interior point of the domain of F (see Proposition 19).

1. Operators with completely regular cone ranges. Let Y be a normed space over the reals and let C be a *cone* in Y, i.e., a subset having the properties $C + C \subset C$, $tC \subset C$ for any positive real number, t, and $C \cap (-C) = \{0\}$. The cone C *induces* a reflexive, transitive and antisymmetric order relation $\leqslant$ on Y if we put $u \leqslant v$ whenever $u - u \in C$. This order relation relates to the linear structure of Y by the properties: $u \leqslant v$ implies $u + w \leqslant v + w$ for any w in Y and $tu \leqslant tv$ for any positive real number t. Since in the sequel we have to do with different cones, we shall call C-ordering the ordering induced by C. Similarly, we shall use terms as C-order bound, C-monotone etc.

The cone C is said to be *normal* if there exists a positive number b such that $\|u\| \leqslant b\|v\|$, whenever $0 \leqslant u \leqslant v$.

The cone C is called *completely regular (regular)* if any C-monotone norm bounded (C-order bounded) sequence in Y is fundamental. Any regular cone which is complete is normal, and any completely regular cone is normal and hence regular ([3], theorems 1.6 and 1.7).

If A is a linear operator on Y, then the *cone range* $A(C)$ of A is obviously a cone. If $A(C) \subset C$, then A is called *positive*. For a wide class of cones the positivity of a linear operator implies its continuity. We shall in the present note ignore this aspect and shall explicitly require in all what follows the continuity of the considered linear and positive operators.

The linear operator A is said to be *of completely regular (regular) type*, if its cone range $A(C)$ is a completely regular (regular) cone. Since any completely regular cone is also regular, any linear operator of completely regular type is also of regular type.

We shall frequently use in the sequel Lemma 4 in [6] and hence we shall state it here in a slightly modified from.

1. Lemma. *The cone C in Y is completely regular (regular) if it contains no sequence (y_i) having the property that $\|y_i\| \geqslant d$ for any i and some positive d and for which the set $\left\{\sum_{i=1}^{n} y_i : n \in \mathbf{N}\right\}$ is norm bounded (C-order bounded).*

2. Proposition. *If C isn't a regular cone of the normed space Y, then no regular type linear and continuous operator (and hence no completely regular type linear and continuous operator) can have continuous left hand side inverse.*

Proof. The linear operator A has continuous left hand side inverse if and only if there exists a positive b such that

$$\|Ay\| \geqslant b\|y\| \tag{1}$$

for any y in Y (see e.g. V. 4.4. in [4]).

If C isn't regular, it contains by Lemma 1 a sequence (y_i) with the property that $\|y_i\| \geqslant d$ for any i and some positive d, for which the set $\left\{\sum_{i=1}^{n} y_i : n \in N\right\}$ is order bounded. Let y be a C-upper bound for this set. Then Ay will be an $A(C)$-upper bound for the set $\left\{\sum_{i=1}^{n} Ay_i : n \in \mathbf{N}\right\}$. According (1) and the property of (y_i) it holds

$$\|Ay_i\| \geqslant b\|y_i\| \geqslant bd > 0$$

for any i. Applying once again Lemma 1 we conclude that the cone $A(C)$ cannot be regular. Q.E.D.

We shall denote with $\mathfrak{L}(Y)$ the vector space of all linear and bounded operators acting in Y, endowed with the norm topology.

3. Carollary. *Let C be a cone in the B-space Y that isn't regular. Then the open unit sphere in $\mathfrak{L}(Y)$ with the centre at the identity map I ean contain no operator of regular (and henece no operator of completely regular) type.*

Proof. Any operator in the above open sphere has continuous inverse by a theorem of Banach (see e.g. V. 4.5 in [2]). Q.E.D.

4. *Remark.* In [6] it was shown that the identity map can be indefinitely approximated in the topology of the uniform convergence on norm bounded sets by the Fredholm resolvents of some sublinear operators. These resolvents transform the cone C whose closure isn't a subspace in some completely regular subcones of its. Restricted to the linear and continuous operators the considered topology is quite the norm topology. Intuitively the above cited result

means that a cone contains subcones „arbitrarily close to it" which are completely regular. Although, by Corollary 3, the transformation of a cone that isn't regular into a such subcone cannot be realised by a linear operator.

5. PROPOSITION. *The property of a cone in a normed space to be completely regular is preserved by any linear and bounded operator with continuous left hand side inverse.*

Proof. Let C be a completely regular cone and assume that $A(C)$ isn't completely regular for some linear and bounded A with continuous left hand side inverse. We have for any y the relation (1) for some positive b. Invoking Lemma 1, there exists a sequence (y_i) in C with $\|Ay_i\| \geqslant d$ for some positive d and any i, such that the set $\left\{\sum_{i=1}^{n} Ay_i : n \in \mathbf{N}\right\}$ is norm bounded. We have for any i the relation $\|y_i\| \geqslant d/\|A\| > 0$, while from (1), $\left\|\sum_{i=1}^{n} Ay_i\right\| \geqslant b\left\|\sum_{i=1}^{n} y_i\right\|$. Accordingly the set $\left\{\sum_{i=1}^{n} y_i : n \in \mathbf{N}\right\}$ is norm bounded and we have get via Lemma 1 a contradiction with the hypothesis that C is a completely regular cone. Q.E.D.

6. *Remark.* Obviously, any linear operator of finite range preserves the complete regularity of a cone. The operators constructed in 12 and 14 furnish other examples having this property. However, there exist linear and compact operators that transform some completely regular cones onto cones without this property (see the example in 17).

Let $\mathfrak{B}$ and $\mathfrak{C}$ be subsets in $\mathfrak{L}(Y)$. We shall say that $\mathfrak{C}$ *is modular over* $\mathfrak{B}$ if for any $n \in \mathbf{N}$, any B_i in $\mathfrak{B}$ and any C_i in $\mathfrak{C}$, the operator $\sum_{i=1}^{n} B_i C_i$ is in C. If $\mathfrak{B}$ contains the identity map, then it suffices to restrict n in the above definition to be $\geqslant 2$. It is straightforward to show that if $\mathfrak{B}$ contains all the positive multiples of the identity map (respectively, its multiples with scalars in $[0, 1]$), then if $\mathfrak{C}$ is modular over $\mathfrak{B}$, it is a convex cone (respectively, a convex set). If $\mathfrak{C}$ is modular over itself we shall say that it is *automodular.*

We shall need in our next reasonings the

7. LEMMA. *The sum of a finite number of completely regular subcones of a normal cone is a completely regular cone.*

Proof. Let C_1 and C_2 be completely regular subcones of the normal cone C and assume that $C_1 + C_2$ isn't completely regular. By Lemma 1 there exist the sequence (y_i^j) in C_j, $j = 1, 2$ and a positive number d so to have $\|y_i^1 + y_i^2\| \geqslant d$ for any i, while the set

$$\left\{\sum_{i=1}^{n} (y_i^1 + y_i^2) : n \in \mathbf{N}\right\} \tag{2}$$

is norm bounded. Passing to a subsequence we can assume without loss of the generality that

$$\|y_{i_k}^1\| \geqslant d/2, \quad k \in \mathbf{N}. \tag{3}$$

On the other hand if $\leqslant$ denotes the C-ordering in Y, we have

$$0 \leqslant \sum_{k=1}^{m} y_{i_k} \leqslant \sum_{k=1}^{m} (y^1_{i_k} + y^2_{i_k}) \leqslant \sum_{i=1}^{i_m} (y^1_i + y^2_i),$$

wherefrom, using the normality of the cone C and the norm boundendess of (2) it results that the set

$$\left\{\sum_{k=1}^{m} y^1_{i_k} : k \in \mathbf{N}\right\}$$

is norm bounded. But this, together with (3) contradicts the complete regularity of C_1. Q.E.D.

8. PROPOSITION. *Let C be a normal cone in the normed space Y. Let B denote the subset of C-positive operators in $\mathfrak{L}(Y)$ that transform any completely regular subcone in C in completely regular cone. Then B is an automodular convex cone in $\mathfrak{L}(Y)$.*

Proof. We have obviously $B_1B_2 \in \mathfrak{B}$ whenever B_1 and B_2 are in $\mathfrak{B}$. Further, by the inclusion

$$(B_1 + B_2)(C) \subset B_1(C) + B_2(C),$$

it follows that the cone in the left is completely regular being the subcone of the cone in the right, which is completely regular by Lemma 7. That is, $B_1 + B_2$ is in $\mathfrak{B}$. From Proposition 5 we have that I and any positive multiple of its are in $\mathfrak{B}$ and hence we are done. Q.E.D.

9. PROPOSITION. *Let C be a normal cone in the normed space Y. Let $\mathfrak{C}$ denote the set in $\mathfrak{L}(Y)$ of the operators that transform C in completely regular subcones of its. Then $\mathfrak{C}$ is an automodular convex cone in $\mathfrak{L}(Y)$, which is modular over $\mathfrak{B}$, where $\mathfrak{B}$ is the set in $\mathfrak{L}(Y)$ of C-positive operators transforming the completely regular subcones of C in completely regular ones.*

Proof. Since $\mathfrak{C}$ is contained in $\mathfrak{B}$ it suffices to prove that it is modular over $\mathfrak{B}$. For any B' in $\mathfrak{B}$ and any C' in $\mathfrak{C}$ the composed operator $B'C'$ is obviously in $\mathfrak{C}$. Because I is contained in $\mathfrak{B}$ we have only to prove that $B_1C_1 + B_2C_2$ is in $\mathfrak{C}$ whenever B_1 and B_2 are in $\mathfrak{B}$ and C_1 and C_2 are in $\mathfrak{C}$. But this follows directly from Lemma 7. Q.E.D.

10. *Remark.* From 14 it follows that the cone $\mathfrak{C}$ in general is not closed in the norm topology of the space $\mathfrak{L}(Y)$.

Let A and B be in $\mathfrak{L}(Y)$. We shall put $A \leqslant B$ if $B - A$ is a C-positive operator.

11. PROPOSITION. *Let C be a normal cone in Y. Let $\mathfrak{C}$ denote the set of C-positive operators in $\mathfrak{L}(Y)$ which transform C in completel)y regular cones. If for some A and B in $\mathfrak{L}(Y)$ there exist the positive scalars α and β such that*

$$\alpha B \leqslant A \leqslant \beta B, \tag{4}$$

then A is in $\mathfrak{C}$ if and only if B is in $\mathfrak{C}$.

Proof. The relation (4) defines in fact an equivalence relation. Hence it is sufficient to show that $B \in \mathfrak{C}$ implies $A \in \mathfrak{C}$. According the normality of C, for any sequence (z_i) in C the norms $\|Bz_i\|$ and $\|Az_i\|$, $i \in \mathbf{N}$ are in the same time bounded from above and respectively, lower bounded by a positive number. Using now Lemma 1 in the way we have done it in the preceeding proofs, we get the required implication. Q.E.D.

2. Examples. The operators of finite range transform a closed cone in completely regular ones. The image by a compact operator of a closed cone is a compactly generated cone and hence the compact operators can be suspected to improve esentially the properties of a cone. Are they of completely regular or of regular type? Unfurtunately they don't. The aim of this paragraph is to show that the property of an operator to be of completely regular as well as of regular type is far to be characterizable with a property like compactness. There are linear and continuous operators of rather general form which are of completely regular type, while some compact operators don't have this property. In the same time we complete the results in the preceeding paragraph.

Let $C[0, 1]$ denote the space of continuous real valued functions defined on $[0, 1]$ endowed with the uniform norm and ordered by the cone C of non-negative functions. This cone is closed and normal.

12. *The linear operators in $C[0, 1]$ with the representing kernels bounded rom above and from below by positive multiples of a measure function, which represents a linear and positive functional, are of completely regular type.*

Let A be a linear and positive operator in $C[0, 1]$ and assume that the representing kernel K of its (see e.g. VI. 9.46 in [1]) satisfies the following conditions:

(i) There exist a normalized function g of bounded variation on $[0, 1]$ and a positive real β, such that

$$0 \leqslant K(s, dt) \leqslant \beta g(dt)$$

for any s and t in $[0, 1]$;

(ii) There exist an s_0 in $[0, 1]$ and a positive scalar α such that

$$\alpha g(dt) \leqslant K(s_0, dt)$$

for any t in $[0, 1]$.

For any y in the cone C it holds by (i)

$$\|Ay\| = \sup_s \int_0^1 y(t)\, K(s, dt) \leqslant \beta \int_0^1 y(t)\, g(dt).$$

If $\|Ay\| \geqslant d$, then this relation together with (ii) yields

$$\alpha d/\beta \leqslant \int_0^1 y(t) K(s_0, dt) = (Ay)(s_0). \tag{5}$$

If (z_i) is a sequence in $A(C)$ with the property that $\|z_i\| \geqslant d$ for some positive d and any i, then we have by (5) the inequality

$$z_i(s_0) \geqslant \alpha d/\beta$$

and hence

$$\left\|\sum_{i=1}^{n} z_i\right\| \geqslant n\alpha d/\beta,$$

that is, the set $\left\{\sum_{i=1}^{n} z_i : n \in \mathbf{N}\right\}$ cannot be norm bounded. Thus by Lemma 1, $A(C)$ is a completely regular cone.

We have in particular, that *if the representing kernel K of the positive and compact operator A satisfies the condition*

$$K(s_0, t) \geqslant \alpha > 0$$

for any t in $[0, 1]$ *and some s_0 in this interval, then A is of completely regular type.*

Indeed, we have then

$$K(s_0, t)dt \geqslant \alpha dt,$$

and

$$K(s, t)dt \leqslant \beta dt$$

for any s and t since K is continuous and hence bounded.

13. *Example of a positive integral operator with continuous kernel acting in* $C[0, 1]$ *that isn't of completely regular type.*

Consider the increasing sequence (a_n) of distinct real numbers in $(0, 1/2)$. Let we construct the functions k_n by putting for any $n \in \mathbf{N}$

$$k_n(s, t) = \max\{0, (a_n - a_{n+1})^2 - (t - a_n - a_{n+1})^2 - (s - a_n - a_{n+1})^2\},$$
$$(s, t) \in [0, 1] \times [0, 1]$$

They have the properties

(i) k_n vanishes outside the square $[2a_n, 2a_{n+1}] \times [2a_n, 2a_{n+1}]$;

(ii) $0 \leqslant k_n(s, t) \leqslant \max\{0, (a_n - a_{n+1})^2 - (t - a_n - a_{n+1})^2\} = k_n(a_n + a_{n+1}, t)$;

(iii) $\max k_n(s, t) = (a_n - a_{n+1})^2$.

According (ii) and (iii) the function

$$K(s, t) = \sum_{n=1}^{\infty} k_n(s, t)$$

is non-negative and continuous on $[0, 1] \times [0, 1]$.

We shall show that the integral operator A defined by the relation

$$(Ay)(s) = \int_0^1 K(s, t)y(t)\, dt$$

isn't of completely regulat type with respect to the cone C of the non-negative functions in $C[0, 1]$. To this end, we consider the sequence (y_n) in C defined by

$$y_n(t) = \max\{0, c_n((a_n - a_{n+1})^2 - (t - a_n - a_{n+1})^2)\}, \quad n \in \mathbf{N},$$

where

$$c_n = \left(\int_{2a_n}^{2a_{n+1}} ((a_n - a_{n+1})^2 - (t - a_n - a_{n+1})^2)^2 \, dt \right)^{-1}. \tag{6}$$

Then we have the properties

(a) The function $z_n(s) = \int_0^1 k_n(s, t) y_n(t) \, dt$ vanish outside the interval $[2a_n, 2a_{n+1}]$;

(b) $\|z_n\| = 1$;

(c) $\int_0^1 k_m(s, t) y_n(t) \, dt = 0$ for any s, whenever $m \neq n$.

From (a), (c) and the definitions it follows that

$$z_n(s) = (Ay_n)(s).$$

The properties (a) and (b) imply

$$\left\| \sum_{i=1}^{n} z_i \right\| = 1$$

for any n in $\mathbf{N}$, wherefrom via Lemma 1 we conclude that $A(C)$ isn't a completely regular cone.

14. *The integral operator A constructed in* 13 *can be indefinitely approximated in the norm topology by positive integral operators of completely regular type.*

We refer for the notations to the preceeding point. Consider the function

$$K_m(s, t) = \sum_{n=1}^{m} k_n(s, t)$$

and let the operator A_m be defined by the relation

$$(A_m y)(s) = \int_0^1 K_m(s, t) y(t) \, dt.$$

From the properties (i) and (iii) of k_n we have for any y in C

$$0 \leqslant (Ay)(s) - (A_m y)(s) = \int_0^1 (K(s, t) - K_m(s, t)) y(t) \, dt =$$

$$= \int_0^1 \left(\sum_{n=m+1}^{\infty} k_n(s, t) \right) y(t)dt \leqslant \max_{n \geqslant m+1} (a_n - a_{n+1})^2 \int_0^1 y(t)dt \leqslant \|y\| \max (a_n - a_{n+1})^2.$$

and hence

$$\|A - A_m\| \leqslant \max_{n \geqslant m+1} (a_n - a_{n+1})^2$$

wherefrom A_m converges in the norm to A when $m \in \infty$.

We have to check that A_m is for any m of completely regular type. We observe first that for any y in C the function $A_m y$ attains its local maxima at the points $a_1 + a_2, \ldots, a_m + a_{m+1}$. Indeed, suppose that s is in the interval $[2a_j, 2a_{j+1}]$ $(j = 1, \ldots, m)$. Then by the property (i) of k_j,

$$(A_m y)(s) = \int_0^1 K_m(s, t)y(t)dt = \int_0^1 k_j(s, t)\, y(t)\, dt, \quad (s \in [2a_j, 2a_{j+1}]).$$

Now, by the property (ii) of k_j,

$$\int_0^1 k_j(s, t)y(t)\, dt \leqslant \int_0^1 k_j(a_j + a_{j+1}, t)\, y(t)\, dt = (A\ y)\,(a_j + a_{j+1}),$$

that is, for s in $[2a_j, 2a_{j+1}]$,

$$(A_m y)(s) \leqslant (A_m y)(a_j + a_{j+1}). \tag{7}$$

Consider now an arbitrary sequence (z_i) in $A_m(C)$ with the property that $\|z_i\| \geqslant d$ for some positive d and for any i. We have

$$z_i = A_m y_i, \ i \in \mathbf{N},$$

for some y_i in C. According to the property (i) of k_j it follows that $z_i(s) = 0$ for s in $[0, 1] \setminus [2a_1, 2a_{m+1}]$. By the relation (7) we have that the maximum of z must be attained on some point $a_j + a_{j+1}, j = 1, \ldots, m$. That is, since $\|z_i\| \geqslant d$, there exists at least a $j (1 \leqslant j \leqslant m)$ so to have

$$z_i(a_j + a_{j+1}) \geqslant d.$$

Because j can have a finite number of values, it follows that there exists an index h $(1 \leqslant h \leqslant m)$ and a subserquence (uz_{i_l}) of (z_i) such that

$$z_{i_l}(a_h + a_{h+1}) \geqslant d$$

for any l in $\mathbf{N}$. This means that

$$\sum_{l=1}^{r} z_{i_l}(a_h + a_{h+1}) \geqslant rd,$$

and hence the set

$$\left\{ \sum_{j=1}^{n} z_j : n \in \mathbf{N} \right\}$$

cannot be norm bounded. From Lemma 1 we have then that $A_n(C)$ is a completely regular cone.

15. *The operator A constructed in* 13 *is of regular type.* We have to show in accordance with Lemma 1, that if (z_i) is a sequence in $A(C)$ with the property that there exists a positive d such that $\|z_i\| \geqslant d$ for any i, then the set

$$\left\{\sum_{i=1}^{n} z_i : n \in \mathbf{N}\right\} \tag{8}$$

cannot be $A(C)$-order bounded (by any element in $A(C)$).

Let a be the limit of the sequence (a_i). Then $K(2a, t) = 0$ for any t in $[0, 1]$. Hence $z(2a) = 0$ for any z in $A(C)$. Assume that z is an element in $A(C)$ which is a C-order bound for the set (8). This means that

$$\sum_{i=1}^{n} z_i(s) \leqslant z(s), \qquad s \in [0, 1],\ n \in \mathbf{N}.$$

Since z is continuous, $z(a_j + a_{j+1}) \to z(2a) = 0$. Assume that $h \in \mathbf{N}$ has the property that $z(a_j + a_{j+1}) < d/2$ for any $j \geqslant h$. Since $z_i(s) \leqslant z(s)$ for any i and since $\|z_i\| \geqslant d$, it follows that the maximum of any element z_i must be attained at a point $s < a_h + a_{h+1}$. According the reasonings in the point 14, an s with this property must be one of the points $a_j + a_{j+1}$ for $j \leqslant h$. Hence we get a contradiction as in the above point with the norm boundendess of the set (8) which follows from the C-order boundendess of it. Now, if the set (8) would be $A(C)$-order bounded by some element in $A(C)$, then it would be also C-order bounded by the same element. But this contradicts, as we have seen above, the hypothesis that $\|z_i\| \geqslant d$ for any i. Thus $A(C)$ must be regular.

16. *Example of a positive integral operator in* $C[0, 1]$ *with continuous kernel, which isn't of regular type.*

We shall use the constructions in the example 13, restricting the terms fo the sequence (a_i) to satisfy $1/4 < a_i < 1/2$, $i \in \mathbf{N}$. Let be $a_0 = 1/4$ and put

$$k_0(s, t) = \max\{0, (a_0 - a_1)^2 - (t - a_0 - a_1)^2\}$$

Consider also the function

$$y_0(t) = \max\{0, c_0((a_0 - a_1)^2 - (t - a_0 - a_1)^2)\},$$

where c_0 is given by (6) with $n = 0$.

The function

$$K^1(s, t) = \sum_{n=0}^{\infty} k_n(s, t)$$

is continuous and non-negative and have the property that

$$\int_0^1 K^1(s, t)\, y_0(t)\, dt = 1$$

for any s in $[0, 1]$.

The elements

$$z_n(s) = \int_0^1 K^1(s, t)\, y_n(t)\, dt, \quad n \in \mathbf{N}$$

are of the norm 1 and have the property that $\left\|\sum_{n=1}^{m} z_n\right\| = 1$ for any m in $\mathbf{N}$. Let us denote by $e(s)$ the function identically 1 on $[0, 1]$, and let consider the difference

$$u_m(s) = e(s) - \sum_{n=1}^{m} z_n(s), \quad m \in \mathbf{N}.$$

This is for any m a non-negative function of norm $\leqslant 1$.

Consider the sequence (b_i), where $b_i = a_i - 1/4$, $i \in \mathbf{N}$, and put

$$h_n(s, t) = u_n(s) \max \{b_n - b_{n+1})^2 - (t - b_n - b_{n+1})^2\}, \quad n \in \mathbf{N}.$$

h_n is a non-negative continuous function vanishing outside the strip $[0, 1] \times [2b_n, 2b_{n+1}]$, satisfying the inequality $h_n(s, t) \leqslant (b_n - b_{n+1})^2$. Hence

$$K^2(s, t) = \sum_{n=1}^{\infty} h_n(s, t)$$

is a continuous non-negative function. Let

$$v_n(t) = \max \{0, c_n((b_n - b_{n+1})^2 - (t - b_n - b_{n+1})^2)\}$$

with c_n given by (6). We shall show that the compact operator A defined by

$$(Ay)(s) = \int_0^1 K(s, t)\, y(t)\, dt,$$

where $K = K^1 + K^2$ isn't of regular type.

We observe first that e and the sequence (z_n) are in $A(C)$. Further, we have

$$\int_0^1 K(s, t)\, v_n(t)\, dt = \int_{2b_n}^{2b_{n+1}} h_n(s, t)\, v_n(t)\, dt =$$

$$= c_n u_n(s) \int_{2b_n}^{2b_{n+1}} ((b_n - b_{n+1})^2 - (t - b_n - b_{n+1})^2)^2\, dt = u_n(s)$$

by the definition of the sequence (b_n) and of the numbers c_n, $n \in \mathbf{N}$.

The obtained relation shows that u_n is in $A(C)$ and that the set

$$\left\{\sum_{n=1}^{m} z_n : m \in \mathbf{N}\right\}$$

is $A(C)$-order bounded by the element e of $A(C)$. But $\|z_n\| = 1$ for any n in $\mathbf{N}$, and invoking Lemma 1 again we conclude that A isn't of regular type.

17. *Example of a linear, positive and compact operator in c that transforms a completely regular cone in a cone that isn't completely regular.*

Denote by c the space of convergent sequences of real numbers, endowed with the usual norm. Let C be the cone of the sequences in c with non-negative terms. The subcome C_1 in C of the nondecreasing sequences is completely regular. Indeed, if y is in C_1, $y = (y^i)$, $y^i \in \mathbf{R}$, then $\|y\| = \lim y^i$. Accordingly, for y_1 and y_2 in C_1 we have $\|y_1 + y_2\| = \|y_1\| + \|y_2\|$ and hence there cannot exist any sequence (y_i) of elements in C_1 such that $\|y_i\| \geqslant d$ for some positive d and any i, for which $\left\{\sum_{i=1}^{n} y_i : n \in \mathbf{N}\right\}$ is a norm bounded set. That is, C_1 is completely regular by Lemma 1.

Let us consider the infinite matrix of real numbers denoted by A,

$$A = (a_{ij})_{i,j=1,2,\ldots}, \quad a_{ij} = 2^{-i}\,\delta_i^j$$

with δ_i^j standing for the Kronecker symbol. If we define Ay for some y in c as to be the multiplication of A by the (column) vector y, then A can be interpreted as a linear operator in c. It is straightforward to see that A is compact.

Define the sequence (y_i) of the elements in the completely regular cone C_1 by putting

$$y_n = (\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, 2^n, 2^n, \ldots)$$

Then

$$Ay_n = z_n = (z_n^1, z_n^2, \ldots, z_n^m, \ldots),$$

where $z_n^m = 2^{-m} y_n^m$, that is,

$$z_n = (\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, 2^0, 2^{-1}, 2^{-2}, \ldots,).$$

We have $\|Ay_n\| = \|z_n\| = 1$ and

$$\|Ay_1 + Ay_2 + \ldots + Ay_n\| < 2$$

for any n in $\mathbf{N}$. That is, $A(C_1)$ isn't completely regular by Lemma 1.

3. The subdifferentiability of some Hammerstein type operators. A totally ordered subset of the ordered vector space Y is said to be a *chain*. The space Y is said to be *chain complete* if any chain that is bounded from below (from above) has an infimum (a supremum) in Y. If Y is a regular space ordered by a closed cone, then the limit of any monotonically decreasing (increasing) sequence is also the infimum (supremum) of this sequence (see II.3.2 in [8]). Hence isn't difficult to show (see for example the reasoning in the proof of Proposition 2 in [7]), that a space with this property is chain complete. Thus for it we have the conditions used in [5] in order to prove the existence of the subgradients for convex mappings. Ths operator F from the vector space X to the ordered vector space $(Y, \leqslant)$ is said to be *convex* if

$$F(tx_1 + (1-t)x_2) \leqslant tF(x_1) + (1-t)F(x_2)$$

for any x_1 and x_2 in X and any t in $[0, 1]$. The linear operator A from X to Y is said to be a subgradient of F at x if

$$F(x+u) - F(x) \geqslant Au$$

for any u in X.

Suppose that X and Y are B-spaces and that Y is ordered by a closed, normal cone with nonempty interior. Then if F is a continuous convex operator from X to Y, then from the existence of a subgradient of F, it follows its continuity. There exist examples (see e.g. [4]) showing that even for rather nice convex operators there are points in the domain of them at which no subgradient exists. We shall use in this paragraph the results we have estabilished in order to give some sufficient conditions for the existence of subgradients. First of all we prove the following preparatory result:

18. LEMMA. *The closure of any completely regular cone is completely regular too.*

Proof. Assume that C is completely regular and $\bar{C}$ isn't. Then there exist a $d > 0$ and a sequence (y_i) in $\bar{C}$ with the property that $\|y_i\| \geqslant d$ for any i, so to $\left\{\sum_{i=1}^{n} y_i : n \in \mathbf{N}\right\}$ be a norm bounded set (Lemma 1). Suppose that $\left\|\sum_{i=1}^{n} y_i\right\| \leqslant \alpha$ for any n. Let z_i be elements in C which satisfy the conditions $\|z_i\| \geqslant d/2$ and $\|z_i - y_i\| < 2^{-i}$ for any i. Then

$$\left|\left\|\sum_{i=1}^{n} y_i\right\| - \left\|\sum_{i=1}^{n} z_i\right\|\right| \leqslant \sum_{i=1}^{n} \|y_i - z_i\| < 1$$

and hence

$$\left\|\sum_{i=1}^{n} z_i\right\| \leqslant \left|\left\|\sum_{i=1}^{n} z_i\right\| - \left\|\sum_{i=1}^{n} y_i\right\|\right| + \left\|\sum_{i=1}^{n} y_i\right\| < 1 + \alpha$$

for any n. That is, the set $\left\{\sum_{i=1}^{n} z_i : n \in \mathbf{N}\right\}$ is norm bounded. Thus we have get a contradiction via Lemma 1 with the hypothesis that C is a completely regular cone. Q.E.D.

19. PROPOSITION. *Let Y be a B-space ordered by a closed normal cone C with nonempty interior and let F be a continuous convex mapping from the B-space X to Y. If A is a positive operator in Y of completely regular type, then the abstract Hammerstein operator AF has continuous subgradients at any point of X.*

Proof. From Lemma 18, $\overline{A(C)}$ will be a closed completely regular cone. The operator AF will be convex with respect to the $\overline{A(C)}$-ordering in Y and hence it will have $\overline{A(C)}$-subgradients in any point of X. Since $\overline{A(C)} \subset C$, these subgradients will be C-subgradients too. Hence they will be continuous operators by our comments at the beginning of this paragraph. Q.E.D.

20. COROLLARY. *Let the space $C[0, 1]$ be ordered by the cone of non-negative functions and let F be a continuous convex operator acting in it. Consider the Hammerstein operator defined by*

$$G(x)(s) = \int_0^1 F(x(t))\, K(s, dt),$$

where the kernel K satisfies the conditions in 12. *Then G has continuous subgradients in each point of $C[0, 1]$.*

Proof. The positive cone in $C[0, 1]$ is closed, normal and has nonempty interior. The linear operator defined by

$$(Ay)(s) = \int_0^1 y(t) K(s, dt)$$

is of completely regular type by 12. Now, $G = AF$ and hence we are in the conditions of Proposition 19. Q.E.D.

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REFERENCES

1. Dunford, N., J. T. Schwartz, *Linear Operators I*, Interscience, New York — London, 1958.
2. Kantorovič, L. V., G. P. Akilov, *Functional Analysis*, (Russian), Nauka, Moskow, 1977.
3. Krasnosel'skii, M. A., *Positive Solutions of the Operational Equations* (Russian), Fizmatgiz, Moskow, 1962.
4. Levin, V. L., *Subdifferentials of convex mappings and composed functions*, Sibirsk. Mat. Ž., 13 (1972), 1295—1303.
5. Németh, A. B., *Some differential properties of the convex mappings*, Mathematica (Cluj), 22 (45) (1980), 107—114.
6. Németh, A. B., *Nonlinear operators that transform a wedge*, Studia Univ. Babeş-Bolyai Math., XXV, 4 (1980), 55—69.
7. Németh A. B., *The nonconvex minimization principle in ordered regular Banach spaces*, Mathematica (Cluj) 23 (46) (1981), 43—48.
8. Peressini, A. L., *Ordered Topological Vector Spaces*, Harper & Row, New York — Evanston — London, 1967.

OPERATORI LINIARI CE TRANSFORMĂ UN CON NORMAL ÎN CONURI COMPLET REGULARE

(Rezumat)

În lucrare sînt studiaţi operatorii liniari şi continui care transformă un con normal în conuri complet regulare. Se dau condiţii suficiente pentru ca un operator liniar şi continuu din spaţiul funcţiilor continue definite pe un interval compact de pe axa reală, să aibă această proprietate. Se construiesc operatori liniari şi compacţi definiţi în acest spaţiu, care nu transformă conul funcţiilor pozitive într-un con regular.