STUDIA UNIV. BABEŞ–BOLYAI, MATHEMATICA, XXVIII, 1983

THE HEAT TRANSFER IN A VISCOUS UNSTEADY FLOW THROUGH CIRCULAR DUCTS

DOINA BRĂDEANU

1. **Formulation of the thermal problem.** a) *The problem equations.* Let us considered a viscous incompressible fluid, with thermal conductivity λ, in unsteady flow inside a circular cylinder (tube) of radius R and very long lenght L. It is assumed that at the initial moment $t = 0$ the fluid is at rest and is subjected to a pressure gradient $(p_0 - p_L)/L$ where p_0 is the pressure in the cross section $z = 0$ and $p_L < p_0$ is the pressure of the fluid in circular section $z = L$ (here, Oz is the axis of the duct). It is assumed also that $\tilde{T}^{(0)}$ is the temperature of the fluid at the moment $t = 0$ and that $\tilde{T}_w$ is the temperature of the duct. Suppose that the flow, which is produced in these conditions, is unsteady, axisymmetrical (straight lines). The thermal conductivity of the fluid is not neglected. In the domain of the flow, let us now introduce the cylindrical coordinates (r, z, φ) where r is the radial coordinate and φ is the polar angle (fig. 1). Then, the velocity and temperature field in the fluid is represented by the scalar functions $v_z(r, t)$ and $\tilde{T}(r, t)$ where t is time.

The momentum and energy equations (Poiseuille flow) are deduced from the equations of the Navier-Stokes type in the form [1], [3]

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \tag{1}$$

$$\rho c_v \frac{\partial \tilde{T}}{\partial t} = \mu\left(\frac{\partial v_z}{\partial r}\right)^2 + \frac{\lambda}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}}{\partial r}\right) \tag{2}$$

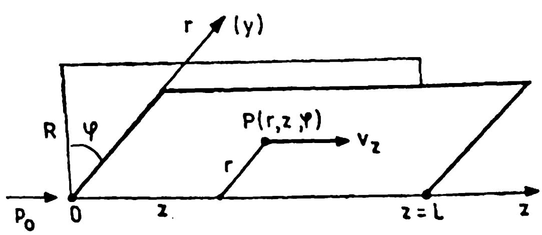

Fig. 1.

where the notations are in the usual form; the term $\mu(\partial v_z/\partial r)^2$ represents the dissipation function (a measure of the heat produced by the dissipation of the mechanical energy by friction) μ and λ are the constant coefficients of the viscosity and thermal conductivity, ρ — the density and c_v — the specific heat.

We can now make the following transformations for the independent variables and functions

$$y = \frac{r}{R}, \quad \tau = \frac{\mu t}{\rho R^2}, \quad U = \frac{v_z}{\left(-\frac{\partial p}{\partial z}\right)\frac{R^2}{4\mu}}, \quad \theta = \frac{\widetilde{T} - \widetilde{T}^{(0)}}{\widetilde{T}_w - \widetilde{T}^{(0)}} \tag{3}$$

and we introduce the notations (σ — Prandth number)

$$m = \frac{R^4(\partial p/\partial z)^2}{16\,\mu^2 c_p(\widetilde{T}_w - \widetilde{T}^{(0)})}, \quad \sigma \equiv \frac{\mu c_p}{\lambda}$$

Then, the equations (1)—(2) take the form

$$\frac{\partial U}{\partial \tau} = 4 + \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial U}{\partial y}\right), \quad (y, \tau) \in \Omega \tag{4}$$

$$\frac{\partial \theta}{\partial \tau} = m\left(\frac{\partial U}{\partial y}\right)^2 + \frac{1}{\sigma}\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial \theta}{\partial y}\right), \quad (y, \tau) \in \Omega \tag{5}$$

$$U(y, 0) = 0, \quad U(0, \tau) = \text{finite}\left(\frac{\partial \theta}{\partial y}(0, \tau) = 0\right), \quad U(1, \tau) = 0 \tag{6}$$

$$(0 < y \leqslant 1;\ \tau > 0)$$

$$\theta(y, 0) = 0, \quad \theta(0, \tau) = \text{finite}\left(\frac{\partial \theta}{\partial y}(0, \tau) = 0\right), \quad \theta(1, \tau) = 1 \tag{7}$$

$$(0 < y < 1;\ \tau > 0); \qquad \Omega = (0, 1) \times (0, \tau_1)$$

For this problem we seek the solutions $U, \theta \in C^{2,1}(\Omega)$ of which derivatives $\partial/\partial y = 0$ for $y = 0$ — by this condition the discontinuity in these equations is eliminated — and which, of course, must verify the boundary condition and the symmetry condition (6)—(7).

Let us consider the set of the functions

$$S = \{U, \theta\ U, \theta \in C^{2,1}(\Omega),\ U \text{ and } \theta \text{ verifies } (6) - (7)\}$$

In set S, [5], [4], the motion equation has the exact solution

$$U(y, \tau) = 1 - y^2 - 8\sum_{n=1}^{\infty}\frac{J_0(\alpha_n y)}{\alpha_n^3 J_1(\alpha_n)}e^{-\alpha_n^2 \tau}, \tag{8}$$

where J_0 is the Bessel function of the zero order and the first kind ($J_0(\alpha) = 0$):

$$J_p(\alpha_n y) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(k+p)!}\left(\frac{\alpha_n y}{2}\right)^{2k+p}, \qquad p = 0, 1, 2, \ldots \tag{9}$$

2. **The exact solution of the energy equation.** The energy equation, (5), is linear and nonhomogeneous under a nonhomogeneous boundary condition. For solving this problem we seek the solution in the form $\theta = \theta_0 + \theta_1$, where θ_0 verifies the homogeneous equation subjected to nonhomogeneous conditions (given in this problem) and θ_1 verifies the nonhomogeneous equation under

homogeneous conditions. Let $A(\theta)$ be the linear operator from the energy equation; thus, for the verification of the equation we have

$$A(\theta) = A(\theta_0) + A(\theta_1) = g(y, \tau)\,;\; g(y, \tau) = m\left(\frac{\partial U}{\partial y}\right)^2$$

$$\theta(y, 0) = \theta_0(y, 0) + \theta_1(y, 0) = 0,\; \theta(1, \tau) = \theta_0(1, \tau) + \theta_1(1, \tau) = 1$$

a) *The homogeneous energy equation under nonhomogeneous conditions.* Let $\theta_0(y, \tau)$ be the solution of the homogeneous energy equation under nonhomogeneous boundary conditions and let the change of the functions be

$$V = \theta_0(y, \tau) - 1\,; \qquad (a \equiv 1/\sigma)$$

Then, we have the problem

$$\frac{\partial V}{\partial \tau} = a\left(\frac{\partial^2 V}{\partial y^2} + \frac{1}{y}\,\frac{\partial V}{\partial y}\right) \tag{10}$$

$$V(y, 0) = -1,\; V(1, \tau) = 0\,;\; (V(0, \tau) = \text{finite})$$

We use the separation method of the variables by choosing

$$V(y, \tau) = Y(y)T(\tau)$$

and we obtain the differential system

$$\frac{dT}{T} = -\,\alpha^2 a\, d\tau$$

$$\frac{d^2Y}{dy^2} + \frac{1}{y}\,\frac{dY}{dy} + \alpha^2 Y = 0$$

where α is an unknown constant value. The solutions of these equations are

$$T(\tau) = C_0 e^{-\alpha^2 a\tau},\; Y(y) = C_1 J_0(\alpha y) + C_2 N_0(\alpha y)$$

where C_0, C_1 and C_2 are arbitrary integration constants and J_0, N_0 are Bessel functions of zero order and 1[st] and 2[nd] kind (the equation of the function $Y(y)$ is just the Bessel equation of $n = 0$ order).

The general solution has the following form

$$V(y, \tau) = C_0 e^{-a\alpha^2\tau}\,[C_1 J_0(\alpha y) + C_2 N_0(\alpha y)] \tag{11}$$

It is known, [4], that the Neumann function $N_0(x) \to -\infty$ when $x \to 0$. Consequently, in order to have a finite solution, imposed by the physical problem (17), we take $C_2 = 0$.

The boundary condition on the wall of the duct $V(1, \tau) = 0$ provides the equation

$$J_0(\alpha) = 0$$

Let $\alpha_1, \alpha_2, \ldots$ be the positive roots of this equation which at the same time are the eigenvalues of Bessel's ordinary differential operator. To thesse eigenvalues the eigenfunctions system corresponds $J_0(\alpha_n y)$, $n = 1, 2, 3, \ldots$

Consequently, the problem (10) has the particular solutions of the form (B_n are arbitrary constants)

$$V_n(y, \tau) = B_n e^{-a\alpha_n^2\tau} J_0(\alpha_n y), \ n = 1, 2, 3, \dots$$

The linear problem (10) has the solution

$$V(y, \tau) = \sum_{n=1}^{\infty} B_n e^{-a\alpha_n^2\tau} J_0(\alpha_n y) \tag{11'}$$

in which the coefficients B_n must be determined.

Let us impose the initial condition $V(y, 0) = -1$, and then we obtain

$$-1 = \sum_{n=1}^{\infty} B_n J_0(\alpha_n y)$$

that is, the Fourier-Bessel series of the function $f(y) = -1$. In order to determine the B_n constants we use the orthogonality property on [0, 1] with the weight y of Bessel's function. We obtain

$$-\int_0^1 J_0(\alpha_m y)\, y\, dy = \sum_{n=1}^{\infty} B_n \int_0^1 J_0(\alpha_m y)\, J_0(\alpha_n y)\, y\, dy =$$

$$= B_m \int_0^1 [J_0(\alpha_m y)]^2 y\, dy = \frac{1}{2} B_m [J_0'(\alpha_m)]^2 \tag{12}$$

if we use the orthogonality property and the known formula from the theory of the Bessel functions

$$\int_0^R r[J_0(\alpha_m r)]\, dr = \frac{R^2}{2} [J_0'(\alpha_m R)]^2 \tag{13}$$

(α_m — eigenvalues, $J_0(\alpha_m r)$ — eigenfunctions)

We now calculate the integral from the left side of the equality (13). If we introduce the Bessel function J_0, in accordance with (9) and then integrate, we obtain

$$\int_0^1 J_0(\alpha_m y)\, y\, dy = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{\alpha_m}{2}\right)^{2k+1} \frac{1}{\alpha_m} = \frac{1}{\alpha_m} J_1(\alpha_m)$$

It is also known that $J_0'(x) = -J_1(x)$, [4]. Then, from (12) we obtain the values

$$B_m = -\frac{2}{\alpha_m J_1(\alpha_m)}$$

One can now write, by means of (11'), the solution $V(y, \tau)$.

The solution of the homogeneous energy equation under nonhomogeneous boundary conditions is

$$\theta_0(y, \tau) = 1 - 2\sum_{n=1}^{\infty} e^{-\frac{1}{a}\alpha_n^2 \tau} \frac{J_0(\alpha_n y)}{\alpha_n J_1(\alpha_n)} \tag{14}$$

(α_n — are the positive roots of the equation $J_0(\alpha) = 0$; see (22))

b) *The solution of the nonhomogeneous energy equation subjected to homogeneous boundary conditions.* Let α_n, $n = 1, 2, \ldots$ be the eigenvalues of the operator from the homogeneous energy equation. The values α_n are the roots of the equation $J(\alpha) = 0$. Let $J_0(\alpha_1 y), J_0(\alpha_2 y), \ldots, J_0(\alpha_n y), \ldots$ be the eigenfunctions corresponding with the eigenvalues α_n. The functions θ_1 and g are developed in a series of $J_0(\alpha_n y)$ eigenfunctions (generalized Fourier series), introduced by the homogeneous equation, which forms an orthogonal system of functions (a system of linear independent functions; a complete system).

Let us introduce the Fourier-Bessel expansions by setting

$$\theta_1(y, \tau) = \sum_{n=1}^{\infty} c_n(\tau) J_0(\alpha_n y) \tag{15}$$

$$g(y, \tau) = \sum_{n=1}^{\infty} d_n(\tau) J_0(\alpha_d y) \tag{16}$$

$$(\theta_1(y, 0) = 0, \qquad \theta_1(1, \tau) = 0) \tag{16'}$$

where the Fourier coefficients $c_n(\tau)$ and $d_n(\tau)$ are the unknown functions which are to be determined.

The generalized Fourier expansions (15)—(16) are substituted in energy equation (5), which, reduces to the identity

$$\sum_n c_n' J_0(\alpha_n y) \equiv \sum_n d_n J_0(\alpha_n y) + a\left(\sum_n c_n \frac{d^2 I_0}{dy^2} + \frac{1}{y}\sum_n c_n \frac{dJ_0}{dy}\right)$$

with

$$\frac{d^2 J_0(\alpha_n y)}{dy^2} + \frac{1}{y}\frac{dJ_0(\alpha_n y)}{dy} + \alpha_n^2 J_0(\alpha_n y) = 0$$

This identity is further reduced to

$$\sum_n \left(\frac{dc_n}{d\tau} - d_n + a\, c_n \alpha_n^2\right) J_0(\alpha_n y) \equiv 0$$

From here the resulting nonhomogeneous ordinary equations are

$$\frac{dc_n(\tau)}{d\tau} + a\,\alpha_n^2 c_n(\tau) = d_n(\tau), \; n = 1, 2, 3, \ldots \tag{17}$$

$$c_n(0) = 0, \; n = 1, 2, 3, \ldots$$

whose solutions are

$$c_n(\tau) = e^{-a\alpha_n^2\tau}\int_0^\tau e^{a\alpha_n^2 t} d_n(t)\,dt$$

The solution of the nonhomogeneous energy equation (under homogeneous conditions) can be written in the following form

$$\theta_1(y, \tau) = \sum_{n=1}^{\infty}\int_0^\tau e^{-a\alpha_n^2(\tau-t)} d_n(t)\, J_0(\alpha_n y)\,dt \tag{18}$$

Now, we must calculate the Fourier coefficients $d_n(t)$. In order to do this, we set

$$y\,g(y, \tau)J_0(\alpha_m y) = \sum_{n=1}^{\infty} d_n(\tau)\, J_0(\alpha_n y)\, J_0(\alpha_m y)y$$

From here, by integrating on $[0, 1]$, we find

$$\int_0^1 g(y, \tau)\, J_0(\alpha_m y)\, y\,dy = d_m(\tau)\int_0^1 [J_0(\alpha_m y)]^2\, y\,dy =$$

$$= \frac{1}{2}\, d_m(\tau)\, [J_0'(\alpha_m)]^2 = \frac{1}{2}\, d_m(\tau)\, [J_1(\alpha_m)]^2$$

if we take into account the orthogonal Bessel functions and formula (13). We obtain the formulas

$$d_n(\tau) = \frac{2}{[J_1(\alpha_n)]^2}\int_0^1 g(s, \tau)\, J_0(\alpha_n s)\, s\,ds, \qquad n = 1, 2, 3, \ldots$$

The solution of the problem, from (18), is

$$\theta_1(y, \tau) = \int_0^\tau\int_0^1 G(y, s, \tau - t)\, g(s, t)\,ds\,dt \tag{19}$$

if we introduce Green's function:

$$G(y, s, \tau - t) = \sum_{n=1}^{\infty}\frac{2s}{[J_1(\alpha_n)]^2}\, J_0(\alpha_n y)\, J_0(\alpha_n s)\, e^{-a\alpha_n^2(\tau-t)} \tag{20}$$

c) *The solution of the energy equation.* The solution of the energy equation (5) under initial and boundary conditions (7) is

$$\theta(y, \tau) = \theta_0(y, \tau) + \theta_1(y, \tau) = 1 - 2\sum_{n=1}^{\infty} e^{-\frac{1}{\tau}\alpha_n^2\tau}\frac{J_0(\alpha_n y)}{\alpha_n J_1(\alpha_n)} + \int_0^\tau\int_0^1 G(y, s, \tau - t)\, g(s, t)\,ds\,dt. \tag{21}$$

where:

— The Green function $G(y, s, \tau - t)$ for the Bessel operator has the expression (20).

— The function $g(y, t)$ has the expression

$$g(y, t) = m\left(\frac{\partial U}{\partial y}\right)^2$$

with m given in (3) and U given in (8)

— The coefficients α_n are the positive roots (found in Tables) of the equation

$$J_0(\alpha) = 0$$

$$(\alpha_1 = 2{,}4048;\ \alpha_2 = 5{,}5201;\ \alpha_3 = 8{,}6537;\ \alpha_4 = 11{,}7915;\ \alpha_5 = 14{,}9309;\ \ldots) \qquad (22)$$

— J_0 and J_1 are the Bessel functions of the first kind and of the zero and first order, given in (9).

(Received October 12, 1981)

REFERENCES

1. L. G. Loiţianski, *Mehanika jidkosti i gaza*, Moskva, 1950.
2. H. Schlichting, *Teoria progranicinogo sloia*, Moskva, 1956.
3. P. Brădeanu, *Mecanica fluidelor*, Ed. tehnică, Bucureşti, 1973.
4. N. N. Lebedev, *Funcţii speciale şi aplicaţiile lor*, Ed. tehnică, Bucureşti, 1973.
5. O. Iordache, O. Smigelschi, *Ecuaţiile fenomenelor de transfer de masă şi căldură*, Ed. tehnică, Bucureşti, 1981.

TRANSFERUL DE CĂLDURĂ ÎN MIŞCAREA NESTAŢIONARĂ A UNUI FLUID VÎSCOS PRIN CONDUCTE CILINDRICE CIRCULARE

(Rezumat)

Se presupune că mişcarea nestaţionară în conductă este axialsimetrică pe traiectorii rectilinii şi că în momentul iniţial (t = 0) fluidul este în repaos şi este supus la un gradient de presiune. Conductibilitatea termică a fluidului şi disipaţia nu sînt neglijabile dar termenul de convecţie este eliminat din ecuaţia energiei. Ecuaţia neomogenă a energiei are condiţii la limită neomogene şi este rezolvată cu ajutorul funcţiilor Bessel şi a funcţiei Green.