STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXIV, 1, 1989

THE ERROR ANALYSIS IN INTERPOLATION BY COMPLEX SPLINE FUNCTIONS

C. IANCU*

Received: January 20, 1989

REZUMAT. — Studiul erorii în interpolarea prin funcţii spline complexe. Se evaluează norma funcţiei $F = \sigma - f$, unde f este o funcţie analitică în domeniul Ω deschis din $\mathbf{R}^2$, cînd cunoaştem valorile $f_k = f(z_k)$, $k = 0, 1, \ldots, n+1$, pe nodurile z_k ale unei diviziuni date pe o curbă închisă Γ, rectificabilă Jordan, conţinută în Ω, iar σ este o funcţie spline cubică complexă de interpolare a lui f. În plus f şi σ îndeplinesc condiţia lui Hölder.

1. **Introduction.** Let Γ be a closed curve, Jordan rectifiable, that belong to the open domain Ω from $\mathbf{R}^2$. One considers the partition

$$\Delta_\Gamma : \{P_0, P_1, \ldots, P_n, P_{n+1};\ P = P_{n+1}\} \tag{1}$$

that divides the curve Γ in the arcs Γ_k from P_{k-1} to P_k, $k = 1, \ldots, n+1$.

On denotes

$$h_k = z_k - z_{k-1},\quad k = 1, \ldots, n+1 \tag{2}$$

where $z_k = x_k + iy_k$; $x_k, y_k \in \mathbf{R}$ is the affix of

$$P_k,\ k = 0, 1, \ldots, n+1\ \ (z_0 = z_{n+1})$$

Knowing the values $f_k = f(z_k)$, $k = 0, 1, \ldots, n+1$, of a given function f, that is analytic in Ω, in the paper [13] it was constructed a complex cubic spline σ, of the form

$$\sigma(z) = \frac{M_k - M_{k-1}}{6h_k}(z - z_{k-1})^3 + \frac{M_{k-1}}{2}(z - z_{k-1})^2 + m_{k-1}(z - z_{k-1}) + f_{k-1},\ z \in \Gamma_k,$$
$$k = 1, 2, \ldots, n+1, \tag{3}$$

where $m_j = \sigma'(z_j)$ and $M_j = \sigma''(z_j)$, $j = 0, 1, \ldots, n+1$, while h_j, $j = 1, 2, \ldots, n+1$ is given by the formula (2), that interpolates the function f.

From the conditions

$$\begin{cases} \sigma(z_k) = f_k,\ k = 0, 1, \ldots, n+1,\ (f_{n+1} = f_0) \\ \sigma'(z_k) = m_k,\ k = 0, 1, 2, \ldots, n+1, \end{cases} \tag{4}$$

* *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

that can be written in the form

$$\begin{cases} M_k' = 6\dfrac{f_k - f_{k-1}}{h_k^2} - 6\dfrac{m_{k-1}}{h_k} - 2M_{k-1} \\ m_k = 3\dfrac{f_k - f_{k-1}}{h_k} - 2m_{k-1} - \dfrac{M_{k-1}}{2}h_k, \quad k = 1, 2, \ldots, n+1 \end{cases} \tag{5}$$

with $m_0 = a$, $M_0 = b$; $a, b \in \mathbf{R}$, it follows that the spline function (3) exists and it is unique [13].

In the paper [13] it is also evaluated the value

$$|\sigma(z) - f(z)| \text{ for } z = \frac{z_{k-1} + z_k}{2}, \quad z \in \Gamma_k, \quad k = 1, \ldots, n+1$$

when f satisfies the Hölder's condition on Γ and it is obtained

$$|\sigma(z) - f(z)| \leqslant \frac{|h_k|}{8}\left[A\,|h_k|^{\mu-1}\left(1 + \frac{8}{2^\mu}\right) + \frac{1}{2}|M_{k-1}| \cdot |h_k| + 3\,|m_{k-1}|\right], \tag{6}$$

where A is the Hölder's constant, $\mu \in (0, 1)$ is the Hölder's exponent and M_{k-1}, m_{k-1}, $k = 2, \ldots, n+1$, are obtained by (5) with initial data M_0, m_0.

2. In the present paper one estimates the norm of the error function F, $F = \sigma - f$, in the conditions that $\sigma, f \in H^\mu(\Gamma)$ — the Banach space of the functions φ which satisfies the condition

$$|\varphi(t'') - \varphi(t')| \leqslant A\,|t'' - t'|^\mu, \quad \forall t', t'' \in \Gamma, \tag{7}$$

with the norm [15]

$$\|\varphi\|_{H^\mu} = \|\varphi\|_\infty + M_\mu(\varphi) \tag{8}$$

where

$$M_\mu(\varphi) = \sup_{t', t'' \in \Gamma} \frac{|\varphi(t') - \varphi(t'')|}{|t' - t''|^\mu}. \tag{9}$$

Now, if we consider the parameter s (the length of the arc), $s = s(t)$, $t \in \Gamma$, we have

$$s_k = s(z_k),\ k = 0, 1, \ldots, n+1.$$

Next, one denotes by

$$\lambda_k = s_k - s_{k-1}, \tag{10}$$

the measure of the arc Γ_k, $k = 1, \ldots, n+1$.

Taking into account the formulas (2) and (10), we have

$$|h_k| \leqslant \lambda_k,\ k = 1, 2, \ldots, n+1 \tag{11}$$

First, we estimate the value

$$\Delta_k F = |F(t') - F(t'')|, \quad \forall t', t'' \in \Gamma_k \ (t' \neq t''), \tag{12}$$

where

$$F(t') = \sigma(t') - f(t'), \qquad F(t'') = \sigma(t'') - f(t'')$$

We have

$$F(t') - F(t'') = \sigma(t') - f(t') - \sigma(t'') + f(t'') = f(t'') - f(t') + \frac{M_k - M_{k-1}}{6h_k}\left[(t' - z_{k-1})^3 - (t'' - z_{k-1})^3\right] + \frac{M_{k-1}}{2}\left[(t' - z_{k-1})^2 - (t'' - z_{k-1})^2\right] + m_{k-1}\left[(t' - z_{k-1}) - (t'' - z_{k-1})\right] + f_{k-1} - f_{k-1}, \qquad \forall t', t'' \in \Gamma_k.$$

So, one obtains

$$\Delta_k F \leqslant |f(t'') - f(t')| + \frac{1}{6}\left|\frac{M_k - M_{k-1}}{h_k}\right| \cdot |(t' - z_{k-1})^3 - (t'' - z_{k-1})^3| + \frac{1}{2}|M_{k-1}| \cdot |(t' - z_{k-1})^2 - (t'' - z_{k-1})^2| + |m_{k-1}| \cdot |t' - z_{k-1} - t'' + z_{k-1}| = |f(t'') - f(t')| + \frac{1}{6}\left|\frac{M_k - M_{k-1}}{h_k}\right| \cdot |[(t' - z_{k-1}) - (t'' - z_{k-1})] \cdot [(t' - z_{k-1})^2 + (t' - z_{k-1})(t'' - z_{k-1}) + (t'' - z_{k-1})^2]| + \frac{1}{2}|M_{k-1}| \cdot |[(t' - z_{k-1}) - (t'' - z_{k-1})] \cdot [(t' - z_{k-1}) + (t'' - z_{k-1})]| + |m_{k-1}| \cdot |t' - t''|,$$

which, for $f \in H^{\mu}(\Gamma)$ can be written in the form:

$$\Delta_k F \leqslant A|t' - t''|^{\mu} + \frac{1}{6}\left|\frac{M_k - M_{k-1}}{h_k}\right| \cdot |t' - t''| \cdot |(t' - z_{k-1})^2 + (t' - z_{k-1})(t'' - z_{k-1}) + (t'' - z_{k-1})^2| + \frac{1}{2}|M_{k-1}| \cdot |t' - t''| \cdot |(t' - z_{k-1}) + (t'' - z_{k-1})| + |m_{k-1}| \cdot |t' - t''|. \tag{14}$$

If $t', t'' \in \Gamma_k$ then we have the following three cases:

$$\begin{aligned} &\text{a)} \quad z_{k-1} \prec t' \prec t'' \prec z_0 \prec z_k \\ &\text{b)} \quad z_{k-1} \prec t' \prec z_0 \prec t'' \prec z_k \\ &\text{c)} \quad z_{k-1} \prec t' \prec z_0 \prec t'' \prec z_k, \end{aligned} \tag{15}$$

where $x \prec y$ means that x precede y on Γ and $z_0 = (z_{k-1} + z_k)/2$.

Taking into account (10), (11) and (15, a), it follows that

$$\begin{aligned} &|t' - z_{k-1}| \leqslant \frac{|z_k - z_{k-1}|}{2} \leqslant \frac{1}{2}\lambda_k \\ &|t'' - z_{k-1}| \leqslant \frac{1}{2}\lambda_k \\ &|t'' - t'| \leqslant \frac{1}{2}\lambda_k. \end{aligned} \tag{16}$$

Using these inequalities, from (14) one obtains

$$\Delta_k F \leqslant \frac{1}{2}\lambda_k\left\{A\left(\frac{1}{2}\lambda_k\right)^{\mu-1}+\frac{1}{6}\left|\frac{M_k-M_{k-1}}{h_k}\right|\cdot\left|\frac{\lambda_k^2}{4}+\frac{\lambda_k^2}{4}+\frac{\lambda_k^2}{4}\right|+\right.$$
$$\left.+\frac{1}{2}|M_{k-1}|\cdot\left|\frac{1}{2}\lambda_k+\frac{1}{2}\lambda_k\right|+|m_{k-1}|\right\}= \tag{17}$$
$$=\frac{1}{2}\lambda_k\left\{A\left(\frac{1}{2}\lambda_k\right)^{\mu-1}+\frac{\lambda_k^2}{8}\left|\frac{M_k-M_{k-1}}{h_k}\right|+\frac{\lambda_k}{2}|M_{k-1}|+|m_{k-1}|\right\}.$$

In the same way, for the case (15,b) one obtains the inequalities

$$|t'-z_{k-1}|\leqslant\frac{\lambda_k}{2}$$
$$|t''-z_{k-1}|\leqslant\frac{\lambda_k}{h} \tag{18}$$
$$|t''-t'|\leqslant\lambda_k$$

that implies

$$\Delta_k F\leqslant A(\lambda_k)^{\mu}+\frac{1}{6}\left|\frac{M_k-M_{k-1}}{h_k}\right|\lambda_k\cdot\left(\frac{\lambda_k^2}{4}+\frac{\lambda_k^2}{4}+\lambda_k^2\right)+$$
$$+\frac{1}{2}|M_{k-1}|\cdot\lambda_k\left(\frac{\lambda_k}{2}+\lambda_k\right)+|m_{k-1}|\cdot\lambda_k= \tag{19}$$
$$=\lambda_k\left\{A(\lambda_k)^{\mu-1}+\frac{7}{24}\lambda_k^2\left|\frac{M_k-M_{k-1}}{h_k}\right|+\frac{3}{4}\lambda_k|M_{k-1}|+|m_{k-1}|\right\}$$

respectively for the case (15,c), the inequalities

$$|t'-z_{k-1}|\leqslant\lambda_k$$
$$|t''-z_{k-1}|\leqslant\lambda_k \tag{20}$$
$$|t''-t'|\leqslant\frac{1}{2}\lambda_k$$

and

$$\Delta_k F\leqslant A\left(\frac{1}{2}\lambda_k\right)^{\mu}+\frac{1}{6}\left|\frac{M_k-M_{k-1}}{h_k}\right|\cdot\frac{1}{2}\lambda_k(\lambda_k^2+\lambda_k^2+\lambda_k^2)+$$
$$+\frac{1}{2}|M_{k-1}|\cdot\frac{1}{2}\lambda_k(\lambda_k+\lambda_k)+|m_{k-1}|\cdot\frac{1}{2}\lambda_k= \tag{21}$$
$$=\frac{1}{2}\lambda_k\left\{A\left(\frac{1}{2}\lambda_k\right)^{\mu-1}+\frac{\lambda_k^2}{2}\left|\frac{M_k-M_{k-1}}{h_k}\right|+\lambda_k|M_{k-1}|+|m_{k-1}|\right\}.$$

So, it is proved the following:

PROPOSITION 1. *Let Γ be a closed and Jordan rectifiable curve in the open domain Ω from $\mathbb{R}^2$, Δ_Γ a partition on Γ defined by* (1), $f\in H^{\mu}(\Gamma)$ *and* $f_k=f(z_k)$, $k=0, 1, \ldots, n+1$.

Then

$$\Delta_k F \leqslant \frac{1}{2}\lambda\left\{A\left(\frac{\lambda}{2}\right)^{\mu_1} + \frac{\lambda^2}{8}\left|\frac{M_k - M_{k-1}}{h_k}\right| + \frac{\lambda}{2}\,|M_{k-1}| + |m_{k-1}|\right\} = L_{k,1} \tag{22}$$

in the case (15;a)

$$\Delta_k F \leqslant \lambda\left\{A(\lambda)^{\mu-1} + \frac{7}{24}\lambda^2\left|\frac{M_k - M_{k-1}}{h_k}\right| + \frac{3}{4}\lambda\,|M_{k-1}| + |m_{k-1}|\right\} = L_{k,2} \tag{23}$$

in the case (15;b)

$$\Delta_k F \leqslant \frac{1}{2}\lambda\left\{A\left(\frac{\lambda}{2}\right)^{\mu-1} + \frac{1}{2}\lambda^2\left|\frac{M_k - M_{k-1}}{h_k}\right| + \lambda\,|M_{k-1}| + |m_{k-1}|\right\} = L_{k,3} \tag{24}$$

in the case (15;c)

where $\lambda = \max\ \{\lambda_k \mid k = 1, 2, \ldots, n+1\}$.

Using these rezults we can estimate the norm $\|F\|_{H^\mu}$. By (8) and (9) we have

$$\|F\|_{H^\mu} = \|F\|_\infty + M_\mu(F) \tag{25}$$

where

$$\|F\|_\infty = \sup_{t\in\Gamma_k} |F(t)|$$

and

$$M_\mu(F) = \sup_{\substack{t',t''\in\Gamma_k\\ t'\neq t''}} \frac{|F(t') - F(t'')|}{|t' - t''|^\mu}$$

If one denotes by

$$L_k = \max\left\{\frac{L_{k,1}}{|t' - t''|^\mu}, \frac{L_{k,2}}{|t' - t''|^\mu}, \frac{L_{k,3}}{|t' - t''|^\mu}\right\} \tag{26}$$

then it follows:

PROPOSITION 2. *In the conditions of the Proposition* 1, *if* $\sigma \in H^\mu(\Gamma)$ *then*

$$\|F\|_{H^\mu} \leqslant \|F\|_\infty + L_k,\ \ \forall\Gamma_k,\ k = 1, 2, \ldots, n+1.$$

BIBLIOGRAPHIE

1. Ahlberg, J. H., Nilson, E. N., *Ortogonality Properties of Spline Functions*, J. of Math. Anal. and Appl. 11 (1965), p. 321–337.
2. Ahlberg, J. H., Nilson, E. N., Walsh, J. L., *Complex Cubic Splines*, Transactions of the American Mathematical Society, Volume 129, Nr. 1 (1967), p. 391–413.
3. Atteia, M., *Analyse Numerique et Approximation. Fonctions Spline dans le Champ Complexe*, C.R. Acad. Sc. Paris, t. 273 (11 oct. 1971), p. 678–681, Serie A.
4. Atteia, M., Fage, C., Gaches, J., *Etude et convergence des fonctions spline complexes*, R.A.I.R.O. Analyse numérique, vol. 18, Nr. 3 (1984), p. 219–236.
5. Atkinson, K., *The Numerical Evaluation of the Cauchy Transform on Simple Closed Curves*, SIAM J. Numer Anal., 9 (1972), 284–299.

6. Chaterjee, A., Dikshit, H. P., *Complex Cubic Spline Interpolation*, Acta Mathematica Academiae Scientiarum Hungaricae, Tomus 36 (3–4) (1980), p. 243–249.
7. Chen Hanlin, *Complex Spline Functions*, *Scientia Sinica*, Vol. XXIV, Nr. 2 (1981), p. 160–169.
8. Chien-Ke Lu, *Error Analysis for Interpolating Complex Cubic Splines With Deficiency* 2, Journal of Approximation Theory 36 (1982), 183–196.
9. Chien-Ke Lu, *The Approximation of Cauchy-Type Integrals by Some Kinds of Interpolatory Splines*, Journal of Approximation Theory 36 (1982), 197–212.
10. Elliot, D., Paget, D. F., *An Algorithm for the Numerical Evaluation of Certain Cauchy Principal Value integrals*, Numer. Math. 19 (1972), 373–385.
11. Elliot, D., Paget, D. F., *On the Convergence of a Quadrature Rule for Evaluating Certain Cauchy Principal Value Integrals*, Numer. Math. 23 (1975), 311–319.
12. Hunter, D. B., *Some Gauss Type Formulae for the Evaluation of Cauchy Principal Values of Integrals*, Numer. Math. 19 (1972), 419–424.
13. Iancu, C., *Sur une Fomction Spline Cubique Complexe*, Seminar on Numerical and Statistical Calculus, Preprint Nr. 9 (1987), 73–84.
14. Muskhelishvilli, N. I., *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen, 1953.
15. Muskhelishvilli, N. I., *Singular Integral Equations*, Noordhoff, Groningen, 1953.
16. Hamburg, P., Mocanu, P., Negoescu, N., *Analiză matematică* (*Funcţii complexe*), Ed. Did. şi Ped., Bucureşti, 1982.
17. Stancu, D. D., *Consideraţii asupra interpolării polinomiale a funcţiilor de mai multe variabile*, Buletinul Universităţii „Babeş–Bolyai", Seria Ştiinţelor naturii, I, 1957, fasc. 1–2, 43–82.
18. Stancu, D. D., *Asupra aproximării funcţiilor de două variabile prin polinoame de tip Bernstein, Citeva evaluări asimptotice*, Studii şi cercetări de matematică (Cluj), Tomul XI, 1, 1960, 171–176. p.
19. Stancu, D. D., *Analiză numerică*, vol. I (litografiat), Cluj-Napoca, 1977.

RECENZII

J. R. Giles, **Convex Analysis with Application in Differentiation of Convex Functions,** Research Notes in Mathematics, Nº 58, Pitman, Boston — London — Melbourne, 1982.

The aim of this book is to study the differentiability properties of convex functions with a special emphasis on their connections with other areas of functional analysis as — geometry of Banach spaces, integration in Banach spaces, Radon-Nikodym property as well as their significance in applications to optimization theory and fixed point theorems. The main themes of the book are: the duality between convex sets and convex functions and the relations between spaces with Radon-Nikodym property and the differentiability properties of convex functions defined on these spaces. The book is selfcontained, the only prerequisities being a knowledge of the basic concepts of functional analysis and topology.

The first two chapters have an introductory character presenting the background material needed in the rest of the book. Chapter 1, Convexity in linear spaces, is concerned with topics as convex sets, convex functions, separation theorems (algebraic theory), while the second chapter, Convexity in linear topological spaces, presents the fundamentals of locally convex space theory.

The core of the book is Chapter 3, The differentiation of convex functions, which is concerned with topics as: lower continuity and lower semi-continuity of convex functions, Gâteaux differentiability in linear and normed spaces (including Kenderov's theorem on weak Asplund spaces) and Fréchet differentiability. The famous Bishop-Phelps theorem on subreflexivity of Banach spaces is presented in detail, in some varied formulations with many and consistent applications. The chapter closes with a section on the exposed structure of convex sets containing Strasewicz-Klee theorem, weak*-Asplund spaces and other topics. The last chapter of the book, Chapter 4, Two convexity problems in the geometry of Banach spaces, is concerned with Mazur's intersection property and convexity of Chebyshev sets, showing that the developed theory applies to prove how topological properties, associated with a norm, characterize convexity of certain sets in the space.

Many exercises, some of them being referred to in the proofs, and oustanding research problems are included in the places where they arise logically in the development of the theory.

The result is a fine book on convex analysis and its applications, which is higly recommended to all interested in this field of research.

S. COBZAŞ

V. Barbu, G. Da Prato, **Hamilton—Jacobi equations in Hilbert spaces,** Research Notes in Mathematics vol. 86, Pitman, Boston — London — Melbourne, 1983, 172 p.

The study of Hamilton—Jacobi equations in infinite dimensional spaces, particularly in Hilbert spaces, is motivated by the theory of deterministic distributed parameter systems and by the control of stochastic distributed systems. The aim of this book is to gather together, in the form of lecture notes, the results obtained in recent years by the authors on the existence and approximation of H—J equations in Hilbert spaces. The investigation is based on two interrelated methods: a constructive approximating method and the dynamic programming method.

In order to make the book self-contained, the authors survey in the first Chapter of the book, entitled Preliminaries, the basic results from convex analysis and from the theory of evolution equations in Banach spaces. Chapter 2, Existence in the class of convex functions, is concerned with initial value problem for the H—J equation $\Phi_t(t, x) + F(\Phi_x(t, x)) - (Ax, \Phi_x(t, x)) = g(t, x), t \in [0, T], x \in D(A), \Phi(0, x) = \Phi_0(x)$. The solution Φ is a function from $[0, T] \times H$ to H, the operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0 — semigroup of linear continuous operators on H and F, Φ_0, g are real-valued convex functions (with respect to x), defined on H and $[0, T] \times H$, respectively. The authors's approach to the problem consists in approximating the operator $\Phi \to F(\Phi_x)$ (which is an accreative operator in an appropriate function space) by $\varepsilon^{-1}(\Phi - (\Phi^* + \varepsilon F)^*$ (here $*$ stands for the Fenchel conjugate) and letting ε tend to zero in the approximating equation. It is shown that $\Phi \to - F(\Phi_x)$ is the infinitesimal generator of a semigroup of contractions on $C(H)$ (the space of all continuous functions bounded on the balls of H). A special attention is paid to the case