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ON THE LEGENDRE TRANSFORM AND ITS APPLICATIONS

Arpad PAL* and Mira-Cristiana ANISIU**

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REZUMAT. - Transformarea lui Legendre şi aplicaţiile sale. Transformarea lui Legendre este folosită în mecanică la schimbări de variabile în sisteme de ecuaţii diferenţiale. În lucrare se prezintă unele proprietăţi ale transformării în R^n şi se indică aplicaţii în probleme de mecanică generală, mecanică cerească şi electricitate.

1. Introduction. The Legendre transform permits the change of dependent and independent variables. It is very useful in mechanics and thermodynamics. For example, let us consider the inner energy $E = E(S, V)$, which depends on the entropy S and the volume V. Then the total differential of E will be

$$dE = TdS + PdV,$$

with

$$T = E_S(S, V), \; P = E_V(S, V)$$

the absolute temperature and the pressure.

Now T and V will be the new independent variables, which means that

* "Babeş-Bolyai" University, Astronomical Observatory, 3400 Cluj-Napoca, Romania

** Romanian Academy, Mathematical Institute, P.O.Box 68, 3400 Cluj-Napoca, Romania

from $T = E_S(S, V)$ we have to obtain $S = S(T, V)$.

We can find a new function $F = F(T, V)$ given by $F = E - TS$ for which we have

$$dF = -SdT + PdV$$

and hence

$$S = -F_T(V, T),\ P = F_V(V, T).$$

So using the function F we can make the change of variables, of course imposing some conditions on the derivatives of E in order to obtain $S = S(T, V)$. The Legendre transform of E will then be the function $-F$.

The Legendre transform appears in [6], but it seems to have already been known to Euler. A natural generalization was given later by Fenchel [5]. The Fenchel transform has the property that it is defined for arbitrary functions. It is very useful not only in mechanics, but also in optimization. So, this old transform has its place in recent books on mechanics as Arnold [2] or Choquard [4], on differential equations as Amann [1], on convex analysis and optimization as Willem [8], on in more comprehensive ones as that of Zeidler [9]. We mention that this transform can be studied in the more general setting of Banach spaces or dual pairs, as in the books of Barbu and Precupanu [3] or Precupanu

[7], but for the applications given in this paper we consider only the $\boldsymbol{R}^n$ case.

In what follows we expose the definition and the main properties of the Fenchel transform for various classes of functions. Then we emphasize its key role in connecting the Lagrangian and the Hamiltonian setting of some outstanding problems of mechanics and electricity.

2. The conjugate of a function. This section contains general results on the conjugate of a function, as treated for example in [8], [3], [7] or [1].

Let the real function $F: \boldsymbol{R}^n \to]-\infty,\infty]$ be given so that the effective domain of $F, D(F) = \{u \in \boldsymbol{R}^n: F(u) < \infty\}$ is nonvoid.

The *conjugate* (or the *Fenchel transform*) of F is the function $F^*: \boldsymbol{R}^n \to]-\infty,\infty]$ given by

$$F^*(v) = \sup_{u \in D(F)} \{<v,u> - F(u)\}, \tag{1}$$

where $<v,u> = \sum_{k=1}^{n} v_k u_k$ is the inner product on $\boldsymbol{R}^n$.

From the definition we obtain at once the *Fenchel* (*Young*) *inequality*

$$F(u) + F^*(v) \geq <v,u>, \ \forall\, u,v \in \boldsymbol{R}^n. \tag{2}$$

It also follows easily that for two given functions F_i with $D(F_i) \neq \emptyset$, $i = 1, 2$ so that $F_1 \leq F_2$, we have the reversed inequality $F_1^* \geq F_2^*$.

The function F^* is always convex, so we shall remind some related definitions.

A set $C \subset \boldsymbol{R}^n$ is said *convex* if for every two points $x,y \in C$, the line segment

$$[x,y] = \{z \in \boldsymbol{R}^n: z = (1-a)x + ay,\ a \in [0,1]\}$$

lies completely in C.

A function $F: \boldsymbol{R}^n \to]-\infty,\infty]$ is called:

- *convex*, if for every $x,y \in C$ and $t \in]0,1[$,

$$F((1-t)x + ty) \leq (1-t)F(x) + tF(y);$$

- *strictly convex*, if $D(F) \neq \emptyset$ and

$$F((1-t)x + ty) < (1-t)F(x) + tF(y)$$

for every $x,y \in D(F)$, $x \neq y$, $t \in]0,1[$;

- *continuous*, if $u_k \to u$ implies $F(u_k) \to F(u)$;

- *inferior semi-continuous* (i.s.c.), if $u_k \to u$ implies $\underline{\lim}\, F(u_k) \geq F(u)$.

The *epigraph* of the function $F: \boldsymbol{R}^n \to]-\infty,\infty]$ is the set

$$\operatorname{epi} F = \{(u,t) \in \boldsymbol{R}^n: F(u) \leq t\}.$$

It is clear that F is convex if and only if epi F is convex.

If F is a convex function and the graph of F lies above the hyperp

$u \mapsto \langle v, u \rangle$, then $-F^*(v)$ represents the minimal distance from the graph of F to this hyperplane, in the vertical direction. If the hyperplane intersects the graph of F, then $F^*(v)$ represents the maximal distance in the vertical direction between the graph of F and the hyperplane, considering the points for which the graph of F lies under the hyperplane.

A function G: $\mathbf{R}^n \to \mathbf{R}$ is said to be *affine* if it has the form

$$G(u) = \langle v, u \rangle + a, \text{ where } v \in \mathbf{R}^n, a \in \mathbf{R}.$$

For an i.s.c. convex function, the following geometric description holds.

THEOREM 1. *A function* F: $\mathbf{R}^n \to]-\infty,\infty]$ *is i.s.c. and convex if and only if it is the pointwise supremum of the affine functions dominated by F.*

As a consequence of this theorem, F^* is i.s.c. and convex.

Let us denote by $\Gamma_0(\mathbf{R}^n)$ the set of all functions F: $\mathbf{R}^n \to]-\infty,\infty]$ which are convex, i.s.c. and such that $D(F) \neq \varnothing$.

The following theorem holds

THEOREM 2. *If* $F \in \Gamma_0(\mathbf{R}^n)$, *then* $F^* \in \Gamma_0(\mathbf{R}^n)$ *and* $F^{**} = F$, *so the Fenchel transform is an involution of* $\Gamma_0(\mathbf{R}^n)$.

Proof. The function F^* being i.s.c. and convex, we have to prove only that $D(F^*) \neq \varnothing$. From theorem 1 it follows the existence of $(v,a) \in \mathbf{R}^{n+1}$ so that

$$F(u) \geq \langle v, u \rangle - a, \ \forall\, u \in \boldsymbol{R}^n,$$

so $a \geq \langle v, u \rangle - F(u), \ \forall\, u \in \boldsymbol{R}^n$. Then $a \geq F^*(v)$ and $v \in D(F^*) \neq \emptyset$.

It is clear that $(v,a) \in \operatorname{epi} F$ if and only if $F(u) \geq \langle v, u \rangle - a, \ \forall\, u \in \boldsymbol{R}^n$.

So

$$F(u) = \sup_{\substack{(v,a) \in R^{n+1} \\ (v,\cdot) - a \leq F}} \{\langle v, u \rangle - a\} = \sup_{\substack{v \in D(F^*) \\ a \geq F^*(v)}} \{\langle v, u \rangle - a\} =$$
$$= \sup_{v \in D(F^*)} \{\langle v, u \rangle - F^*(v)\} = F^{**}(\ ), \ \forall\, u \in \boldsymbol{R}^n,$$

and the equality $F = F^{**}$ is proved. □

We give now some examples for $n = 1$.

Example 1. For $F(u) = |u|^p/p$, $p \in]1, \infty[$ we have

$$F^*(v) = |v|^q/q,$$

with q such that $1/p + 1/q = 1$ (q is the conjugate of p). The Fenchel inequality becomes in this case

$$uv \leq |u|^p/p + |u|^q/q,$$

which is the well-known Young's inequality from which some classical inequalities of calculus may be derived.

Example 2. For $F(u) = |u|$, we have

$$F^*(v) = \begin{cases} 0, & |v| \leq 1 \\ +\infty, & |v| > 1. \end{cases}$$

Example 3. For $F(u) = \alpha |u|^p/p$, $\alpha > 0$, $p \in]1, \infty[$ we have

$$F^*(v) = \alpha^{-q/p} |v|^q/q,$$

with q the conjugate of p.

Example 4. For $p \in]1,\infty[$ and $c, c' > 0$, if

$$c|u|^p \leq F(u) \leq c'|u|^p,$$

then

$$k|v|^q \leq F^*(v) \leq k'|v|^q,$$

with q the conjugate of p and $k = (c'p)^{-q/p}q^{-1}$, $k' = (cp)^{-q/p}q^{-1}$.

For a function F: $\mathbf{R}^n \to]-\infty,\infty]$ such that $D(F) \neq \varnothing$, the *sub-differential* of F at u is the set

$$\partial F(u) = \{v \in \mathbf{R}^n\colon F(w) \geq F(u) + \langle v, w-u\rangle, \ \forall\, w \in D(F)\}.$$

The function F is said *sub-differentiable* at u if $\partial F(u) \neq \varnothing$.

It is clear that if F is sub-differentiable at u, then $u \in D(F)$; F is subdifferentiable at $u \in D(F)$ iff there exists an affine function which is equal to F at u and is less than F on $\mathbf{R}^n$; the set $\partial F(u)$ is closed and convex in $\mathbf{R}^n$. The function F has a global minimum at u iff $0 \in \partial F(u)$.

THEOREM 3. *If $F \in \Gamma_0(\mathbf{R}^n)$, the following assertions are equivalent*

(a) $v \in \partial F(u)$;

(b) $F(u) + F^*(v) = \langle v, u\rangle$;

$$\text{(c)} \quad u \in \partial F^*(v).$$

Proof. (a) $\Leftrightarrow$ (b) follows from the fact that

$$v \in \partial F(u) \Leftrightarrow \langle v, u\rangle - F(u) \geq \langle v, w\rangle - F(w),\ \forall\, w \in D(F)$$

$$\Leftrightarrow \langle v, u\rangle - F(u) \geq \sup_{w \in D(F)} \{\langle v, w\rangle - F(w)\}$$

$$\Leftrightarrow \langle v, u\rangle - F(u) = F^*(v).$$

Then, using theorem 2 we have

$$u \in \partial F^*(v) \Leftrightarrow \langle v, u\rangle = F^*(v) + F^{**}(u) = F^*(v) + F(u),$$

so (b) $\Leftrightarrow$ (c) and the theorem is proved. □

The next result shows the relation between sub-differentiability and convexity.

THEOREM 4. *If* $F\colon \boldsymbol{R}^n \to]-\infty,\infty]$ *is convex and continuous at* $u \in D(F)$, *then* F *is sub-differentiable at* u.

It the function is convex and differentiable, the sub-differential coincides with the gradient, as the following theorem shows.

THEOREM 5. *Let* $F\colon \boldsymbol{R}^n \to]-\infty,\infty]$ *be a convex function. If* F *is differentiable at* $u \in \operatorname{int} D(F)$, *then*

$$\partial F(u) = \{\nabla F(u)\}.$$

Proof. We show at first that $\nabla F(u) \in \partial F(u)$. The function F being convex, we have

$$F((1-a)u + aw) \le (1-a)F(u) + aF(w)$$

for each $w \in \mathbf{R}^n$, $a \in]0,1[$, or

$$[F(u + a(w-u)) - F(u)]/a \le F(w) - F(u).$$

Letting $a \to 0^+$ one has

$$<\nabla F(u), w-u> \le F(w) - F(u),$$

hence $\nabla F(u) \in \partial F(u)$.

We prove now that the unique element in $\partial F(u)$ is $\nabla F(u)$. Let $v \in \partial F(u)$. Then, for each $w \in \mathbf{R}^n$

$$F(u) - <v,u> \le F(w) - <v,w>,$$

so the function $F - <v,\cdot>$ has at u a global minimum. From its differentiability at u, we obtain

$$0 = \nabla F(u) - v,$$

hence $v = \nabla F(u)$ and the theorem is proved. □

COROLLARY 6. *The gradient of a convex function* $F: \mathbf{R}^n \to]-\infty,\infty]$ *which is differentiable at* $u \in \text{int } D(F)$ *satisfies*

$$F(w) \ge F(u) + <\nabla F(u), u-w>, \textit{ for each } w \in \mathbf{R}^n.$$

If $\nabla F(u) = 0$, *then* F *admits a global minimum at* u.

Proof. The inequality follows from the fact that $\partial F(u) = \{\nabla F(u)\}$. If $\nabla F(u) = 0$, we have $F(w) \geq F(u)$ for each $w \in \boldsymbol{R}^n$, hence u is a global minimum of F.□

The next theorem gives conditions on F in order to assure the differentiability of F^*.

THEOREM 7. *If $F \in \Gamma_0(\boldsymbol{R}^n)$ is strictly conv and satisfies a coercivity condition*

$$F(u)/|u| \to \infty \text{ for } |u| \to \infty,$$

then $F^* \in C^1(\boldsymbol{R}^n, \boldsymbol{R})$.

Proof. Let $v \in \boldsymbol{R}^n$ be fixed; we define G_v: $I^{\cdot} \to \boldsymbol{R}$,

$$G_v(w) = <v, w> - F(w).$$

The function $-G_v$ is strictly convex and $-G_v(w) \to \infty$, as $|w| \to \infty$, so there is one and only one point $u \in \boldsymbol{R}^n$ where $-G_v$ attains its infimum. Theorem 3 implies $\partial F^*(v) = \{u\}$.

The function ∂F^*: $\boldsymbol{R}^n \to \boldsymbol{R}^n$, $v \mapsto u$ where $\{u\} = \partial F^*(v)$ is continuous. We have from theorem 2 that F^* is i.s.c., hence ∂F^* will have a closed graph. To prove the continuity of ∂F^* it suffices to show that the image of any bounded set is bounded. Let $r > 0$ be given and $|v| \leq r$, $\{u\} = \partial F^*(v)$. Theorem 3 implies

$v \in \partial F(u)$, hence

$$F(0) \geq F(u) - \langle v, u \rangle.$$

Supposing withous loss of generality that $F(0) < +\infty$, from

$$r \geq |v| \geq (F(u) - F(0))/|u|,$$

we obtain using the coercivity condition the existence of $R > 0$ so that $|u| \leq R$ for each v with $|v| \leq r$.

Let us prove that ∂F^* is also differentiable. Let $\{u\} = \partial F^*(v)$ and $\{u_h\} = \partial F^*(v+h)$, $v \in \mathbf{R}^n$, $h \in \mathbf{R}^n \setminus \{0\}$. Then

$$\langle h, u \rangle \leq F^*(v+h) - F^*(v) \leq \langle h, u_h \rangle$$

and

$$0 \leq [F^*(v+h) - F^*(v) - \langle h, u \rangle]/|h| \leq \langle h, u_h - u \rangle/|h| \leq |u_h - u|.$$

The continuity of ∂F^* implies $|u_h - u| \to 0$ for $|h| \to 0$, so F^* is differentiable at v and $\{\nabla F^*(v)\} = \{u\} = \partial F^*(v)$. It follows that $F^* \in C^1(\mathbf{R}^n, \mathbf{R})$.□

Let now be given a convex function $F \in C^1(\mathbf{R}^n, \mathbf{R})$. Using theorems 3 and 5, F^* can be defined implicitly by

$$\begin{cases} F^*(v) = \langle v, u \rangle - F(u) \\ v = \nabla F(u). \end{cases}$$

It the gradient ∇F is locally invertible, these relations define indeed a function of v, considering $u = (\nabla F)^{-1}(v)$. The function F^* is known as the *Legendre transform* of F. If F is strictly convex and $F(u)/|u| \to \infty$ for $|u| \to \infty$, then by theorem 7 the Legendre transform F^* is in the class $C^1(\mathbf{R}^n, \mathbf{R})$.

It is known that for $F \in C^2(\mathbf{R}^n, \mathbf{R})$, F is convex if and only if $D^2F(x)$ is positive semi-definite for every $x \in \mathbf{R}^n$ (i.e. $< D^2F(x)y, y> \geq 0$ for each $y \in \mathbf{R}^n$); if $D^2F(x)$ is positive definite for every $x \in \mathbf{R}^n$ (i.e. $<D^2F(x)y,y> > 0$, for each $y \in \mathbf{R}^n \setminus \{0\}$), then F is strictly convex. For $F \in C^2(\mathbf{R}^n, \mathbf{R})$ with D^2F uniformly positive definite (i.e. there exists $\alpha > 0$ such that $< D^2F(x)y, y> \geq \alpha\|y\|^2$ for each $x,y \in \mathbf{R}^n$), then for every $y \in \mathbf{R}^n$, the equation

$$\nabla F(x) = y$$

has a unique solution.

We obtain now the following theorem for C^2 - class functions.

THEOREM 8. *Let $F \in C^2(\mathbf{R}^n, \mathbf{R})$ be given such that D^2F is uniformly positive definite. Then the following statements are true:*

(i) *The transform given by* (1) *has the form*

$$F^*(v) = < v, u > - F(u),$$

u being the solution of $v = \nabla F(u)$;

(ii) *$F^* \in C^2(\mathbf{R}^n, \mathbf{R})$, F^* is stricly convex and $\nabla F^* = (\nabla F)^{-1}$;*

(iii) $F(u) + F^*(v) \geq \langle u, v \rangle$ *for each* $u, v \in \boldsymbol{R}^n$ *and* $F(u) + F^*(v) = \langle u, v \rangle$ *iff* $\nabla F(u) = v$;

(iv) $F^{**} = F$.

Proof. It remains to prove that $F^* \in C^2(\boldsymbol{R}^n, \boldsymbol{R})$. This follows from the equality $\nabla F^* = (\nabla F)^{-1}$ and the theorem of implicit functions.□

3. Euler-Lagrange and Hamiltonian systems. The Legendre transform is of great importance in Mechanics, as it is specified in [2] or [4]. Indeed, it is useful in transforming the implicit Euler-Lagrange systems in the explicit Hamiltonian ones in a very simple way. The following theorem presents this equivalence.

THEOREM 9. *Let* $I \subset \boldsymbol{R}$ *be an open interval and* $D \subset \boldsymbol{R}^n$ *a domain. Consider* $L \in C^2(\boldsymbol{R}^n \times D \times I, \boldsymbol{R})$ *such that for each value of the argument* $(\dot{q}_0, q_0, t_0) \in \boldsymbol{R}^n \times D \times I$, $L_{\dot{q}\dot{q}}(\dot{q}_0, q_0, t_0) \in \mathfrak{L}(\boldsymbol{R}^n)$ *is uniformly positive define. Then the Euler-Lagrange equation*

$$\frac{d}{dt} L_{\dot{q}} = L_q \tag{3}$$

is equivalent to the Hamiltonian system

$$\begin{cases} \dot{p} = -H_q \\ \dot{q} = H_p, \end{cases} \tag{4}$$

where the Hamiltonian $H \in C^2(\boldsymbol{R}^n \times D \times I, \boldsymbol{R})$ *is the Legendre transform of the Lagrangian L with respect to the variable* $\dot{q}$, i.e.

$$H(p,q,t) = \langle p,\dot{q}\rangle - L(\dot{q},q,t), \tag{5}$$

on the right-hand side $\dot{q}$ *being obtained from the equation*

$$p = L_{\dot{q}}, \tag{6}$$

where $H_q := \nabla_q H$ *denotes the gradient with respect to the variable q for fixed t and p.*

Proof. We apply theorem 8 considering L as a function of $\dot{q}$, (and q, t as parameters). Then $H = L^*$ is given by (5), where $\dot{q} = \dot{q}(p,q,t)$ is obtained from the unique solution of (6). We have then $H \in C^2(\boldsymbol{R}^n \times D \times I)$, $H_p = (L_{\dot{q}})^{-1}$ and $L^{**} = L$.

Let now q: $I \to \boldsymbol{R}^n$ be a solution of the Euler-Lagrange equations. From (6) and (3) we obtain $\dot{p} = L_q$. But using (5) we get immediately $L_q = -H_q$, so the first line in (4) is obtained. Because of $H_p = (L_{\dot{q}})^{-1}$ we have from (6) $\dot{q} = H_p$, the second line in (4). It follows that if q is a solution of (3), then (p,q) is a solution of (4).

Conversely, let (p,q) be a solution of (4). Because $L^{**} = L$, we have

$$L(\dot{q},q,t) = <p,\dot{q}> - H(p,q,t),$$

where $\dot{q} = H_p$, i.e. $p = p(\dot{q},q,t) = (H_p)^{-1} = L_{\dot{q}}$. Then $\dot{p} = \frac{d}{dt}(L_{\dot{q}}) = -H_q$ because of (4). But $L_q = -H_q$ and q is a solution of the Euler-Lagrange equations.□

The following theorem states some important properties of a Hamiltonian system.

THEOREM 10. *In the conditions of theorem 9 we have*

a) $\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$;

b) *if the system* (4) *is autonomous,* $H(p,q)$ *is a first integral which is by definition the energy*;

c) *if* $\frac{\partial H}{\partial q_i} = 0$, *then* p_i *is a first integral* (q_i *is a cyclic variable*);

d) *if all the* q_i *are cyclic, the system is integrable by quadratures.*

Proof. a) $\frac{dH}{dt} = \frac{\partial H}{\partial t} + <H_p,\dot{p}> + <H_q,\dot{q}> = \frac{\partial H}{\partial t}$, and using the form of H as a Legendre transform, $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$.

b) If the system (4) is autonomous, $\frac{\partial H}{\partial t} = 0$, hence $\frac{dH}{dt} = 0$ and $H(p,q)$ = const for the solutions p and q.

c) Using (4), we have $\dot{p}_i = 0$, hence $p_i = c_i$ gives a first integral.

d) Applying c), we obtain $p_i = c_i$, $i = \overline{1,n}$. From (4) we have $\dot{q}_i = \frac{\partial H}{\partial p_i}(c_1, \ldots, c_n, t)$ and $q_i(t) = q_i(t_0) + \int_{t_0}^{t} \frac{\partial H}{\partial p_i}(c_1, \ldots, c_n, \tau)\, d\tau$.□

4. Applications. In the problems of mechanics, the Lagrangian function has usually the form

$$L(\dot{q}, q, t) = T(\dot{q}, q, t) - U(q, t) = E_{kin} - E_{pot},$$

where

$$T(\dot{q}, q, t) = \frac{1}{2} < A(q,t)\,\dot{q}, \dot{q} >, \tag{7}$$

$A(q,t)$ being a symmetric uniformly positive matrix with entries of C^2-class. Then theorem 9 applies and the Hamiltonian obtained as a Legendre transform will be

$$H(p, q, t) = <p, q> - L(\dot{q}, q, t),$$

where $p = A(q,t)\dot{q}$, hence $\dot{q} = A(q,t)^{-1} p$.

It this case

$$H(p, q, t) = <p, A^{-1}p> - \frac{1}{2} <AA^{-1}p, A^{-1}p> + U(q,t) =$$

$$= \frac{1}{2} <p, A^{-1}p> + U(q,t). \tag{8}$$

So, if the kinetic energy (7) is given by a uniformly positive definite matrix A, then the Hamiltonian is the total energy, expressed in terms of the space and momentum variables. In the case of autonomous systems, we have by theorem 10 b) the energy integral

$$H(p,q) = \text{const.}$$

If $A(q,t) = I_m$, i.e. $T(\dot{q}) = \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^2$, it follows $\dot{q}_i = p$ and

$$H(p,q,t) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + U(q,t). \tag{9}$$

Special problems of this type are, for example, those of particles in Newtonian central field of the harmonic osçillator.

1. A particle in the Newtonian central field. The motion of a punctual mass is described by system of equations

$$\begin{cases} m\ddot{x} = kxr^{-3} \\ m\ddot{y} = kyr^{-3} \\ m\ddot{z} = kzr^{-3} \end{cases},$$

where $k > 0$ is the gravitational constant and $r = (x^2 + y^2 + z^2)^{1/2}$. This system is of the type (3) with $L\colon \boldsymbol{R}^3 \times (\boldsymbol{R}^3\backslash\{0\}) \to \boldsymbol{R}$ given by

$$L(\dot{q},q) = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - kr^{-1},$$

where $q = (x,y,z)$. The Hamiltonian will be of the form given in (8),

$$H(p,q) = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + kr^{-1}$$

and the initial system has the Hamiltonian form

$$\begin{cases} \dot{p}_i = kq_i r^{-3} \\ \dot{q}_i = p_i/m \end{cases}, \quad i = 1,2,3.$$

2. The harmonic oscillator. The equation for the harmonic oscillator is

$$m\ddot{q} = -kq,$$

$q \in \boldsymbol{R}$, the constants $m,k > 0$ (which has the known solution $q(t) = C\sin(\omega t + \alpha)$, with the frequency $\omega = \sqrt{k/m}$). The Lagrange function $L = T - U$ is $L: \boldsymbol{R}^2 \to \boldsymbol{R}$,

$$L(\dot{q},q) = (m\dot{q}^2 - kq^2)/2.$$

Hence $L_{\dot{q}\dot{q}} = m > 0$ and theorem 9 applies. From $p = L_{\dot{q}}$ we obtain the momentum $p = m\dot{q}$. The Hamiltonian $H = p\dot{q} - L$ will be $H: \boldsymbol{R}^2 \to \boldsymbol{R}$

$$H(q,p) = \frac{1}{2m}p^2 + \frac{k}{2}q^2,$$

and the Hamilton equations (4)

$$\begin{cases} \dot{p} = -kq \\ \dot{q} = p/m. \end{cases}$$

3. A punctual mass on a torus. The motion of a punctual mass on a torus is governed by a system of the type (3) with $L: \boldsymbol{R}^2 \times (0,2\pi)^2 \to \boldsymbol{R}$ given by

$$L(\dot{\theta},\dot{\phi},\theta,\phi) = \frac{m}{2}(r^2\dot{\theta}^2 + (R + r\cos\theta)^2\dot{\phi}^2) - mgr\sin\theta,$$

with $m > 0$, $R > r > 0$. Denoting $q = (\theta,\phi)$, we have

$$L_{\dot{q}\dot{q}} = \begin{pmatrix} mr^2 & 0 \\ 0 & m(R + r\cos q_1)^2 \end{pmatrix}$$

a n d $<D^2Ly,y> = m(r^2y_1^2 + (R + r\cos q_1)^2y_2^2) \geq m\min\{r^2,(R-r)^2\}(y_1^2 + y_2^2)$,

hence $L_{\dot{q}\dot{q}}$ is uniformly positive definite. From

$$p_1 = mr^2\dot{q}_1$$

$$p_2 = m(R + r\cos q_1)^2\dot{q}_2,$$

we obtain the Hamiltonian H: $\boldsymbol{R}^2 \times (0, 2\pi)^2 \to \boldsymbol{R}$,

$$H(p_1,p_2,q_1,q_2) = \frac{1}{2mr^2}p_1^2 + \frac{1}{2m}p_2^2/(R + r\cos q_1)^2 + mgr\sin q_1.$$

The system corresponding to (4) is

$$\begin{cases} \dot{p}_1 = -\frac{1}{m}p_2^2(R + r\cos q_1)^{-3}r\sin q_1 - mgr\cos q_1 \\ \dot{p}_2 = 0 \\ \dot{q}_1 = \frac{1}{mr^2}p_1 \\ \dot{q}_2 = \frac{1}{m}(R + r\cos q_1)^2 p_2. \end{cases}$$

In several problems we have to deal with generalized Lagrangian functions having the form

$$L_1(\dot{q},q,t) = T(\dot{q},q,t) - <f,\dot{q}> - U(q,t), \qquad (10)$$

with T given by (7) and f a function of q. Applying the Legendre transform we shall get by theorem 9 the Hamiltonian function

$$H_1(p,q,t) = <p,\dot{q}> - L_1(\dot{q},q,t),$$

where $p = A(q,t)\dot{q} - f$, hence $\dot{q} = A(q,t)^{-1}(p+f)$.

Then

$$H_1(p,q,t) = <p,A^{-1}(p+f)> - \frac{1}{2}<AA^{-1}(p+f),A^{-1}(p+f)> +$$
$$+ <f,A^{-1}(p+f)> + U(q,t),$$

hence

$$H_1(p,q,t) = \frac{1}{2}<p,A^{-1}p> + <p,A^{-1}f> + \frac{1}{2}<f,A^{-1}f> + U(q,t). \quad (11)$$

Therefore the transform of a generalized Lagrangian of type (10) is the Hamiltonian (11), the corresponding systems (3) and (4) being equivalent.

For autonomous systems we have in this case an energy integral

$$H_1(p,q) = const$$

given by theorem 10 b).

If $A(q,t) = I_n$, i.e. $T(\dot{q}) = \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i^2$, it follows $\dot{q} = p + f$ and

$$H_1(p,q,t) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i f_i + \frac{1}{2}\sum_{i=1}^{n} f_i^2 + U(q,t). \quad (12)$$

The next applications contain problems having generalized Lagrangian functions of type (10).

4. The photogravitational three-body problem. Let us consider the three-dimensional photogravitational three-body problem given by the system of equations

$$\begin{cases} \ddot{x} - 2\dot{y} = \Omega_x \\ \ddot{y} + 2\dot{x} = \Omega_y \\ \ddot{z} = \Omega_z \end{cases}$$

with Ω: $D = \boldsymbol{R}^3 \backslash \{(\mu,0,0), (\mu - 1,0,0)\} \to \boldsymbol{R}$ given by

$$\Omega(x,y,z) = (x^2 + y^2)/2 + A_1/r_1 + A_2/r_2,$$

$$A_1 = a_1(1 - \mu),\ A_2 = a_2\mu,$$

$$r_1^2 = (x - \mu)^2 + y^2 + z^2,\ r_2^2 = (x - \mu + 1)^2 + y^2 + z^2,$$

where $\mu \in [0, 1/2]$ and $a_1, a_2 \in]-\infty,1]$.

The system may be written as

$$\frac{d}{dt} L_{\dot{q}} = L_q,$$

where $q = (q_1, q_2, q_3) = (x,y,z)$ and L: $\boldsymbol{R}^3 \times D \to \boldsymbol{R}$ is a generalized Lagrangian of the form

$$L(\dot{q}, q) = T(\dot{q}) - Z(\dot{q}, q).$$

The kinetic energy is given by

$$T(\dot{q}) = (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)/2$$

and the generalized potential Z by

$$Z(\dot{q}, q) = \dot{q}_1 q_2 - \dot{q}_2 q_1 - \Omega(q).$$

The Lagrangian is of the type (10) with $A(q,t) = I_3, f(q,t) = (q_2, -q_1, 0)$ and $U(q,t) = -\Omega(q)$.

We have $\dot{q} = A(q,t)^{-1}(p+f)$, hence

$$\dot{q}_1 = p_1 + q_2$$

$$\dot{q}_2 = p_2 - q_1$$

$$\dot{q}_3 = p_3.$$

It follows

$$H(p,q) = (p_1^2 + p_2^2 + p_3^2)/2 + p_1 q_2 - p_2 q_1 + \frac{1}{2}(q_1^2 + q_2^2) - \Omega(q) =$$
$$= (p_1^2 + p_2^2 + p_3^2)/2 + p_1 q_2 - p_2 q_1 - A_1/r_1 - A_2/r_2.$$

The Hamiltonian system (4) is in this case

$$\begin{cases} \dot{p}_1 = p_2 - q_1 + \Omega_{q_1} \\ \dot{p}_2 = -p_1 - q_2 + \Omega_{q_2} \\ \dot{p}_3 = \Omega_{q_3} \\ \dot{q}_1 = p_1 + q_2 \\ \dot{q}_2 = p_2 - q_1 \\ \dot{q}_3 = p_3. \end{cases}$$

5. A charged particle in a magnetic field $B(r) = \operatorname{curl} A(r)$.

The equations are given by

$$m\ddot{q} = \frac{e}{c}(\dot{q} \times \operatorname{curl} A),$$

where $m > 0$ is the mass, e the charge of the particle and $A = (A_1, A_2, A_3)$, with $A_i \in C^1(\boldsymbol{R}^3)$, $i = 1, 2, 3$.

In this case $q = (q_1, q_2, q_3)$ and L: $\boldsymbol{R}^6 \to \boldsymbol{R}$,

$$L(q,\dot{q}) = \frac{m}{2}\dot{q}^2 + \frac{e}{c} < A(q), \dot{q} >,$$

so we have a generalized Lagrangian of the type (10) with

$$A = mI_3, f = -\frac{e}{c}A, \ U \equiv 0.$$

It follows for $p = (p_1, p_2, p_3)$ that

$$p = m\dot{q} + \frac{e}{c}A,$$

hence

$$\dot{q} = \frac{1}{m}\left(p - \frac{e}{c}A\right)$$

and from (11) we obtain H: $\boldsymbol{R}^2 \to \boldsymbol{R}$,

$$H(p,q) = \frac{1}{2m}\sum_{i=1}^{3}(p_i - \frac{e}{c}A_i(q))^2 = \frac{1}{2m} < p - \frac{e}{c}A(q), p - \frac{e}{c}A(q) >.$$

The Hamiltonian system is

$$\begin{cases} \dot{p}_i = \dfrac{e}{mc} < p - \dfrac{e}{c}A, A_{q_i} > \\ \dot{q}_i = \dfrac{1}{m}(p_i - \dfrac{e}{c}A_i). \end{cases}$$

6. A charged particle in an electromagnetic field $(E(q,t), B(q,t))$.

In this case, the motion of a particle of mass m and charge e is governed by the equations

$$m\ddot{q} = eE + \frac{e}{c}(\dot{q} \times B),$$

$q = (q_1, q_2, q_3)$ being the coordinates of the particle. These equations admit the Lagrangian L: $\boldsymbol{R}^7 \to \boldsymbol{R}$,

$$L(\dot{q}, q, t) = \frac{m}{2}\dot{q}^2 - e\phi(q,t) + \frac{e}{c}<A(q,t), \dot{q}>,$$

the fields E and B being related to the scalar potential ϕ and the vector potential A by

$$E = -\operatorname{grad}\phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

$$B = \operatorname{curl} A.$$

The Lagrangian is of the type (10) with

$$A = mI_3,\ f = -\frac{e}{c}A,\ U = e\phi.$$

It follows that p is given by

$$p = m\dot{q} + \frac{e}{c}A,$$

hence

$$\dot{q} = \frac{1}{m}\left(p - \frac{e}{c}A\right)$$

and the corresponding Hamiltonian is

$$H(p,q) = \frac{1}{2m}<p - \frac{e}{c}A(q,t),\ p - \frac{e}{c}A(q,t)> + e\phi(q,t).$$

The Hamiltonian system is

$$\begin{cases} \dot{p}_i = -e\phi_{q_i} + \dfrac{e}{mc}<p - \dfrac{e}{c}A, A_{q_i}> \\ \dot{q}_i = \dfrac{1}{m}(p_i - \dfrac{e}{c}A_i) \end{cases},\quad i = 1,2,3.$$

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