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# On the Chebyshev method, with numerical applications to the eigenpair problem

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**Abstract.** We present a semilocal convergence result for the Chebyshev method applied to a polynomial system of equations of degree 2.

We apply the method in order to approximate the eigenpairs of matrices. The norming function we have proposed in a previous paper of us shows on test matrices better convergence properties of the iterates than the classical norming function.

**Key words:** systems of polynomial equations of degree 2, Chebyshev method, eigenvalue/eigenvector problem.

**AMS Subject Classification:** 65H10.

**1. INTRODUCTION** Let $F : X \to X$ be a nonlinear mapping, where $(X, \|\cdot\|)$ is a Banach space, and consider the equation

$$F(x) = 0. \tag{1}$$

We shall assume that $F$ is a polynomial of degree 2, i.e., it is indefinitely differentiable on $X$, with $F^{(i)}(x) = \theta_i$, for all $x \in X$ and $i \geq 3$, where $\theta_i$ is the $i$-linear null operator.

In the present paper we shall study the convergence of the Chebyshev method

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k) - \tfrac{1}{2} F'(x_k)^{-1} F''(x_k) \left(F'(x_k)^{-1} F(x_k)\right)^2.$$

We shall apply this study to the approximation of the eigenpairs of the linear operators in Banach spaces, and we shall consider some numerical examples for some test matrices.

**2. A semilocal convergence result** The iterative Chebyshev method for solving equation (1) consists in the successive construction of the elements of the sequence $(x_k)_{k\geq 0}$ given by

$$x_{k+1} = x_k - \Gamma_k F(x_k) - \tfrac{1}{2}\Gamma_k F''(x_k)(\Gamma_k F(x_k))^2, \; k = 0, 1, ..., \; x_0 \in X, \tag{2}$$

where $\Gamma_k = F'(x_k)^{-1}$.

Let $x_0 \in X$ and $\delta > 0$, $\beta > 0$ be two real numbers. Denote $B = B_\delta(x_0) = \{x \in X : \|x - x_0\| \leq \delta\}$. If $K = \sup_{x\in B}\|F''(x)\|$, then $\sup_{x\in B}\|F'(x)\| \leq \|F'(x_0)\| + K\delta$ and

$$\sup_{x\in B}\|F(x)\| \leq \|F(x_0)\| + \delta\|F'(x_0)\| + K\delta^2 = m_0.$$

Consider the numbers

$$\begin{aligned} \mu &= \tfrac{1}{2}K^2\beta^4\left(1 + \tfrac{1}{4}Km_0\beta^2\right) \\ \nu &= \beta\left(1 + \tfrac{1}{2}Km_0\beta^2\right) \end{aligned} \tag{3}$$

With the above notations, the following theorem holds:

**Theorem 1.** *If the operator $F$ is three times differentiable with $F'''(x) \equiv \theta_3$ for all $x \in B$ and if, moreover, there exist $x_0 \in X$, $\delta > 0$, $\beta > 0$ such that the following relations hold*

***i.*** *the operator $F'(x)$ has a bounded inverse for all $x \in B$, and*

$$\|F'(x)^{-1}\| \leq \beta;$$

***ii.*** *the numbers $\mu$ and $\nu$ given by (3) satisfy the relations*

$$\rho_0 = \sqrt{\mu}\,\|F(x_0)\| < 1$$

*and*

$$\frac{\nu\rho_0}{\sqrt{\mu}(1-\rho_0)} \leq \delta,$$

*then the following properties hold:*

***j.*** *the sequence $(x_k)_{k\geq 0}$ given by (2) is convergent;*
***jj.*** *denoting $x^* = \lim_{x\to\infty} x_k$, then $x^* \in B$ and $F(x^*) = \theta_1$;*
***jjj.*** $\|x_{k+1} - x_k\| \leq \frac{\nu\rho_0^{3^k}}{\sqrt{\mu}}$, $k = 0, 1, \ldots;$

***jv.*** $\|x^* - x_k\| \leq \frac{\nu \rho_0^{3^k}}{\sqrt{\mu}(1-\rho_0^{3^k})}$, $k = 0, 1, \ldots$.

*Proof.* Denote by $G : B \to X$ the following mapping:

$$G(x) = -\Gamma(x) F(x) - \tfrac{1}{2}\Gamma(x) F''(x) [\Gamma(x) F(x)]^2, \tag{4}$$

where $\Gamma(x) = F'(x)^{-1}$.

It can be easily seen that for all $x \in B$ the following identity holds

$$\begin{aligned} &F(x) + F'(x) G(x) + \tfrac{1}{2} F''(x) G^2(x) = \\ &= \tfrac{1}{2} F''(x) \left[ F'(x)^{-1} F(x), F'(x)^{-1} F''(x) \left[ F'(x)^{-1} F(x) \right]^2 \right] + \\ &\quad + \tfrac{1}{8} F''(x) \left[ F'(x)^{-1} F''(x) \left[ F'(x)^{-1} F(x) \right]^2 \right]^2 \end{aligned}$$

whence we obtain

$$\begin{aligned} &\left\| F(x) + F'(x) G(x) + \tfrac{1}{2} F''(x) G^2(x) \right\| \leq \\ &\leq \tfrac{1}{2} K^2 \beta^4 \left(1 + \tfrac{1}{4} m_0 K \beta^2\right) \|F(x)\|^3, \end{aligned} \tag{5}$$

or

$$\left\| F(x) + F'(x) G(x) + \tfrac{1}{2} F''(x) G^2(x) \right\| \leq \mu \|F(x)\|^3, \text{ for all } x \in B. \tag{6}$$

Similarly, using (4) and taking into account the notations we made, we get

$$\|G(x)\| \leq \nu \|F(x)\|, \text{ for all } x \in B. \tag{7}$$

From the hypotheses of the theorem, the inequality (6) and the fact that $F'''(x) = \theta_3$, we obtain the following inequality:

$$\begin{aligned} \|F(x_1)\| \leq &\left\| F(x_1) - F(x_0) - F'(x_0) G(x_0) - \tfrac{1}{2} F''(x_0) G^2(x_0) \right\| + \\ &+ \left\| F(x_0) + F'(x_0) G(x_0) + \tfrac{1}{2} F''(x_0) G^2(x_0) \right\| \\ \leq &\mu \|F(x_0)\|^3. \end{aligned}$$

Since $x_1 - x_0 = G(x_0)$, by (6) we have

$$\|x_1 - x_0\| \leq \nu \|F(x_0)\| = \tfrac{\nu \sqrt{\mu} \|F(x_0)\|}{\sqrt{\mu}} < \tfrac{\nu \rho_0}{\sqrt{\mu}(1-\rho_0)} \leq \delta,$$

whence it follows that $x_1 \in B$.

Suppose now that the following properties hold:

a) $x_i \in B$, $i = 0, \dots, k$;
b) $\|F(x_i)\| \le \mu \|F(x_{i-1})\|^3$, $i = 1, \dots, k$.

By the fact that $x_k \in B$, using (6) it follows

$$\|F(x_{k+1})\| \le \mu \|F(x_k)\|^3, \tag{8}$$

and from relation $x_{k+1} - x_k = G(x_k)$

$$\|x_{k+1} - x_k\| < \nu \|F(x_k)\|. \tag{9}$$

The inequalities b) and (8) lead us to

$$\|F(x_i)\| \le \tfrac{1}{\sqrt{\mu}} \left(\sqrt{\mu} \|F(x_0)\|\right)^{3^i}, \ i = 1, \dots, k+1. \tag{10}$$

We have that $x_{k+1} \in B$:

$$\|x_{k+1} - x_0\| \le \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \le \sum_{i=1}^{k+1} \nu \|F(x_{i-1})\| \le \tfrac{\nu}{\sqrt{\mu}} \sum_{i=1}^{k+1} \rho_0^{3^{i-1}} \le \tfrac{\nu \rho_0}{(1-\rho_0)\sqrt{\mu}}.$$

Now we shall prove that the sequence $(x_k)_{k\ge 0}$ is Cauchy. Indeed, for all $m, k \in \mathbb{N}$ we have

$$\|x_{k+m} - x_k\| \le \sum_{i=0}^{m-1} \|x_{k+i+1} - x_{k+i}\| \le \nu \sum_{i=0}^{m-1} \|F(x_{k+i})\| \le \tag{11}$$
$$\le \tfrac{\nu}{\sqrt{\mu}} \sum_{i=0}^{m-1} \rho_0^{3^{k+i}} = \tfrac{\nu}{\sqrt{\mu}} \rho_0^{3^k} \sum_{i=0}^{m-1} \rho_0^{3^{k+i} - 3^k} \le \tfrac{\nu \rho_0^{3^k}}{\sqrt{\mu}(1-\rho_0^{3^k})},$$

whence, taking into account that $\rho_0 < 1$, it follows that $(x_k)_{k\ge 0}$ converges. Let $x^* = \lim_{k\to\infty} x_k$. Then, for $m \to \infty$ in (11) it follows jv. The consequence jjj follows from (9) and (10).

**3. Application and numerical examples** We shall study this method when applied to approximate the eigenpairs of matrices.

Denote $V = \mathbb{K}^n$ and let $A \in \mathbb{K}^{n\times n}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For computing the eigenpairs of $A$ one may consider a norming function $G : V \to \mathbb{K}$ with $G(0) \ne 1$. The eigenvalues $\lambda \in \mathbb{K}$ and eigenvectors $v \in V$ of $A$ are the solutions of the nonlinear system

$$F(x) = \begin{pmatrix} Av - \lambda v \\ G(v) - 1 \end{pmatrix} = 0,$$

where $x = \binom{v}{\lambda} \in V \times \mathbb{K} = \mathbb{K}^{n+1}$, $\left(x^{(1)}, x^{(2)}, ..., x^{(n)}\right) = v$ and $x^{(n+1)} = \lambda$. The first $n$ components of $F$, $F_i$, $i = 1, ..., n$, are given by

$$\begin{aligned} F_i\left(x\right) = & a_{i1}x^{(1)} + \cdots + a_{i,i-1}x^{(i-1)} + \left(a_{ii} - x^{(n+1)}\right)x^{(i)} \\ & + a_{i,i+1}x^{(i+1)} + \cdots + a_{in}x^{(n)}. \end{aligned}$$

The standard choice for $G$ is

$$G\left(v\right) = \alpha \left\|v\right\|_2,$$

with $\alpha = \frac{1}{2}$. We have proposed in [ 4] (see also [ 7]), the choice $\alpha = \frac{1}{2n}$, which has shown a better behavior for the iterates than the standard choice.

In both cases we can write

$$F_{n+1}\left(x\right) = \alpha\Big(\left(x^{(1)}\right)^2 + \cdots + \left(x^{(n)}\right)^2\Big) - 1.$$

The first and the second order derivatives of $F$ are given by

$$F'\left(x\right)h = \begin{pmatrix} a_{11} - x^{(n+1)} & a_{12} & \ldots & a_{1n} & -x^{(1)} \\ a_{21} & a_{22} - x^{(n+1)} & \ldots & a_{2n} & -x^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - x^{(n+1)} & -x^{(n)} \\ 2\alpha x^{(1)} & 2\alpha x^{(2)} & \ldots & 2\alpha x^{(n)} & 0 \end{pmatrix} \begin{pmatrix} h^{(1)} \\ h^{(2)} \\ \vdots \\ h^{(n)} \\ h^{(n+1)} \end{pmatrix},$$

and

$$F''\left(x\right)hk = \begin{pmatrix} -k^{(n+1)} & 0 & \ldots & 0 & -k^{(1)} \\ 0 & -k^{(n+1)} & \ldots & 0 & -k^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & -k^{(n+1)} & -k^{(n)} \\ 2\alpha k^{(1)} & 2\alpha k^{(2)} & \ldots & 2\alpha k^{(n)} & 0 \end{pmatrix} \begin{pmatrix} h^{(1)} \\ h^{(2)} \\ \vdots \\ h^{(n)} \\ h^{(n+1)} \end{pmatrix},$$

where $x = \left(x^{(i)}\right)_{i=\overline{1,n+1}}$, $h = \left(h^{(i)}\right)_{i=\overline{1,n+1}}$, $k = \left(k^{(i)}\right)_{i=\overline{1,n+1}} \in \mathbb{K}^{n+1}$.

We shall consider two test matrices from the Harwell Boeing collection[1] in order to study the behavior of the Chebyshev method for approx-

[1]These matrices are available from MatrixMarket at the following address: `http://math.nist.gov/MatrixMarket/`.

imating the eigenpairs. The programs were written in Matlab. As in [ 20], we used the Matlab operator '\' for solving the linear systems.

FIDAP002 MATRIX. This real symmetric matrix of dimension $n = 441$ arises from finite element modelling. Its eigenvalues (contained on the diagonal of `D`, after using the Matlab command `[V,D]=eig(A)`) are all simple and range from $-7 \cdot 10^8$ to $3 \cdot 10^6$. As in [ 20], we have chosen to study the smallest eigenvalue, $\lambda^* =$`D(1,1)`, which is well separated. Its corresponding eigenvector `V(:,1)` has the Euclidean norm equal to 1, and therefore we have $v^* =$`sqrt(2) * V(:,1)` for the first choice, respectively $v^* =$`sqrt(2*n) * V(:,1)` for the second choice.

It is interesting to note that $\|F(x^*)\|_2 = 3.7e - 7$ in the first case, and $\|F(x^*)\|_2 = 9.1e - 6$ in the second case.

The initial approximations were taken $\lambda_0 = \lambda^* + 10^2 = -6.999\,6 \cdot 10^8 + 100$, and for the initial vector $v_0$ we perturbed the solution $v^*$ with random vectors having the components uniformly distributed on $(-\varepsilon,\varepsilon)$, $\varepsilon = 0.3$. This lead in a relative error $\frac{\|x_0 - x^*\|}{\|x^*\|} = 7.0 \cdot 10^1$ in the first case, and $\frac{\|x_0 - x^*\|}{\|x^*\|} = 3.3 \cdot 10^0$ in the second one.

The following results are typical for the runs made (we have considered a common perturbation vector); Table 1 contains the norms of the vectors $F(x_k)$.

**Table 1.** The Fidap002 matrix.

| | Choice $\alpha = \frac{1}{2}$ | Choice $\alpha = \frac{1}{2n}$ |
|---|---|---|
| $k$ | $\|F(x_k)\|$ | $\|F(x_k)\|$ |
| 0 | $2.48 \cdot 10^{+9}$ | $2.48 \cdot 10^{+9}$ |
| 1 | $5.10 \cdot 10^{+4}$ | $2.41 \cdot 10^{+1}$ |
| 2 | $6.81 \cdot 10^{+3}$ | $4.70 \cdot 10^{-6}$ |
| 3 | $9.34 \cdot 10^{+2}$ | |
| 4 | $7.06 \cdot 10^{+1}$ | |
| 5 | $4.86 \cdot 10^{-2}$ | |
| 6 | $2.81 \cdot 10^{-7}$ | |

The values of $\varepsilon$ must be decreased to 0.03 for the iterates with first choice to attain the solution at step $k = 2$ (as it does with choice two), while the same $\varepsilon$ may be increased to 7 for the iterates with the second choice to attain the solution at step $k = 6$.

Because of the scalings, the components of the eigenvector when taken with the second choice may be affected by larger errors than with the first choice.

Sherman1 matrix. This matrix arises from oil reservoir simulation. It is real, unsymmetric, of dimension 1000 and all its eigenvalues are real. We have chosen to study the smallest eigenvalue $\lambda^* = -5.0449$, which is not well separated (the closest eigenvalue is $-4.9376$). The initial approximation was taken $\lambda_0 = \lambda^* + 0.05$ and for the initial vector $v_0$ we considered $\varepsilon = 0.01$. The following results are typical for the runs made (we have considered again a same random perturbation vector for the initial approximations). The relative errors $\frac{\|x_0 - x^*\|}{\|x^*\|}$ for the two choices were $2.65 \cdot 10^{-1}$ resp. $8.38 \cdot 10^{-3}$.

**Table 2.** Sherman1 matrix.

| | Choice $\alpha = \frac{1}{2}$ | Choice $\alpha = \frac{1}{2n}$ |
|---|---|---|
| $k$ | $\|F(x_k)\|$ | $\|F(x_k)\|$ |
| 0 | $1.57 \cdot 10^{+00}$ | $2.73 \cdot 10^{+00}$ |
| 1 | $1.18 \cdot 10^{-02}$ | $6.39 \cdot 10^{-03}$ |
| 2 | $2.69 \cdot 10^{-03}$ | $1.72 \cdot 10^{-09}$ |
| 3 | $4.02 \cdot 10^{-09}$ | $2.58 \cdot 10^{-14}$ |
| 4 | $8.08 \cdot 10^{-16}$ | |

For this particular matrix and eigenvalue, the Chebyshev method has shown a strong sensitivity to the size of the perturbations: increasing $\varepsilon$ leads to the loss of the convergence of the Chebyshev iterates.

As it can be seen from both examples, the second choice presents a better behavior than the first one.

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