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Estimating the radius of an attraction ball[☆]

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ABSTRACT

Given a nonlinear mapping G differentiable at a fixed point x^*, the Ostrowski theorem offers the sharp sufficient condition

$$\rho(G'(x^*)) < 1$$

for x^* to be an attraction point, where ρ denotes the spectral radius. However, no estimate for the size of an attraction ball is known.

We show in this note that such an estimate may be readily obtained in terms of $\|G'(x^*)\| < 1$ (with $\|\cdot\|$ an arbitrary given norm) and of the Hölder (in particular Lipschitz) continuity constant of G'. An elementary example shows that this estimate may be sharp.

Our assumptions do not necessarily require G to be of contractive type on the whole estimated ball.

1. Estimation of the radius

Let $G : D \subseteq \mathbb{R}^n \to D$ be a nonlinear mapping which has a fixed point $x^* \in \operatorname{int} D$:

$$x^* = G(x^*).$$

This point is an *attraction point* [1, Def. 10.1.1] if, given a norm on $\mathbb{R}^n$, there exists an open ball $B_r := B_r(x^*) = \{x \in \mathbb{R}^n : \|x - x^*\| < r\} \subseteq D$ such that for any initial approximation $x_0 \in B_r$, the successive approximations

$$x_{k+1} = G(x_k), \quad k = 0, 1, \ldots \tag{1}$$

remain in D and converge to x^*. Note that a finite number of iterates are allowed to lie outside B_r.

The following result is well known.

Theorem 1 (*Ostrowski; see, e.g., [2, Th.22.1], [1, Th.10.1.3] and [3, Th.3.5]*). *If G is differentiable at the fixed point x^* and the spectral radius satisfies*

$$\sigma := \rho\left(G'(x^*)\right) = \max\left\{|\lambda| : \lambda \in \mathbb{C}, \lambda \text{ eigenvalue of } G'(x^*)\right\} < 1 \tag{2}$$

then x^* is an attraction point.

Remark 1. According to [1, N.R.10.1-2], this result holds also when instead of $\mathbb{R}^n$ one considers an arbitrary Banach space X, with the observation that in defining the spectral radius of a linear continuous operator from X to X one takes its resolvent and spectrum (see, e.g., [4, p. 795]).

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Condition (2) is sharp:

Example 1 (*[1, E.10.1.2]*). The function $G : \mathbb{R} \to \mathbb{R}$, $G(x) = x + x^3$, is differentiable on $\mathbb{R}$, and $\sigma = 1$ at the fixed point $x^* = 0$, which is not an attraction point.

The spectral radius also offers some global information regarding the convergence rate of all the sequences of the successive approximations converging toward x^*, while the spectral elements of $G'(x^*)$ characterize the convergence rate of each individual such sequence. Indeed, σ yields in fact the worst convergence rate among all the sequences converging to x^* (see [1, Th.10.1.4] and [3, Th.3.5]), while the zero eigenvalue and its corresponding eigenvectors characterize the high convergence orders (more precisely, the q-superlinear convergence and with q-orders $1 + p$, $p \in (0, 1]$) of a single such sequence [5].[1]

Theorem 1 says nothing about the size r of the attraction ball. It is interesting to note that such an estimate can be deduced by applying some existing results. Indeed, under some additional differentiability assumptions on G and defining $F(x) = x - G(x)$, the successive approximations may be regarded as inexact Newton iterates for solving the nonlinear system $F(x) = 0$ [5]:

$$\begin{aligned} &F'(x_k)s_k = -F(x_k) + r_k \\ &x_{k+1} = x_k + s_k, \quad k = 0, 1, \ldots. \end{aligned} \tag{3}$$

Ypma [8] has obtained an estimate for the radius of attraction of this method, and we can deduce the corresponding one for the successive approximations.

However, instead of conditions in terms of the Hölder continuity constant of $I - G'$ required by this approach, we obtain below (sharp) estimates in a direct manner, in terms of the Hölder continuity constant of G'. Before doing that, we notice that the successive approximations may also be regarded as quasi-Newton iterates (or, more generally, as inexactly solved perturbed Newton iterates [9]) [10], but not as (exact) Newton iterates; in the first case there do not exist estimates for the radius of the convergence, while in the second they do exist (see, e.g., [11–14]).

Theorem 2. *Suppose there exist $r_1 > 0$, $p \in (0, 1]$, $K_p > 0$ and a norm $\|\cdot\|$ in $\mathbb{R}^n$ such that G is differentiable on B_{r_1}, with G' Hölder continuous at exponent p:*

$$\|G'(x) - G'(y)\| \le K_p \|x - y\|^p, \quad \forall x, y \in B_{r_1}.$$

Moreover, assume that

$$\|G'(x^*)\| \le q < 1 \tag{4}$$

and define

$$\begin{aligned} r_2 &= \left(\frac{(1+p)(1-q)}{K_p} \right)^{\frac{1}{p}}, \\ r &= \min\{r_1, r_2\}. \end{aligned} \tag{5}$$

Then, for any initial approximation $x_0 \in B_r$, the successive approximations remain in B_r and converge (q-)linearly:

$$\|x_{k+1} - x^*\| \le t\|x_k - x^*\|, \quad k = 0, 1, \ldots, \tag{6}$$

where $t = \frac{K_p}{1+p}\|x_0 - x^*\|^p + q < 1$.

Therefore,

$$\|x_k - x^*\| \le t^k \|x_0 - x^*\|, \quad k = 1, 2, \ldots.$$

Proof. The continuity hypothesis on G' implies (see, e.g., [1, 3.2.12]):

$$\left\| G(x) - G(x^*) - G'(x^*)(x - x^*) \right\| \le \frac{K_p}{1+p} \|x - x^*\|^{1+p}, \quad \forall x \in B_{r_1}.$$

Next,

$$\begin{aligned} \|x_1 - x^*\| &\le \|G(x_0) - G(x^*) - G'(x^*)(x_0 - x^*)\| + \|G'(x^*)(x_0 - x^*)\| \\ &\le \left(\frac{K_p}{1+p} \|x_0 - x^*\|^p + q \right) \|x_0 - x^*\| := t\|x_0 - x^*\|. \end{aligned}$$

The key condition is $t < 1$, which yields $\|x_0 - x^*\| < r_2$. The rest of the proof follows by induction. ∎

[1] For the rigorous definitions of the convergence rates and for different related results we refer the reader to [1, ch.9] and [6] (see also [3, ch.3] and [7]).

Remark 2. (a) The relationship between condition (2) ($\sigma < 1$) and the existence of a norm such that (4) ($\|G'(x^*)\| < 1$) holds is the following. Condition (4) implies (2), since the spectral radius satisfies

$$\rho\left(G'(x^*)\right) \leq \|G'(x^*)\|, \quad \text{for any norm } \|\cdot\| \text{ on } \mathbb{R}^n. \tag{7}$$

Condition (2) does not imply (4) in any norm, but for any $\varepsilon > 0$ there exists a norm $\|\cdot\|_{(\varepsilon)}$ on $\mathbb{R}^n$ such that (see, e.g., [1, 2.2.8],[2, Th.19.3]):

$$\sigma \leq \|G'(x^*)\|_{(\varepsilon)} \leq \sigma + \varepsilon. \tag{8}$$

In the case of a Banach space instead of $\mathbb{R}^n$, the statements regarding relations (7) and (8) remain true, with the remark that the norms involved must be equivalent to the initial one (see, e.g., [4, p. 795]). Therefore, the relationship between (2) and (4) remains the same.

Corollary 1. *Under the hypotheses of Theorem 2, if G' is Lipschitz continuous, i.e., Hölder continuous with $p = 1$, $L := K_1$, then instead of (5) one may take*

$$r_2 = \frac{2(1-q)}{L} \tag{9}$$

and, correspondingly, $t = \frac{L}{2}\|x_0 - x^*\| + q$.

Corollary 2. *The assumptions of Theorem 2 imply the necessary condition*

$$\|G'(x)\| < 1 + p(1-q), \quad \forall x \in B_r, \tag{10}$$

regardless of the value of the Hölder or Lipschitz continuity constant.

If G' is Lipschitz continuous, then (10) becomes

$$\|G'(x)\| < 2 - q, \quad \forall x \in B_r. \tag{11}$$

Proof. The statement is obtained by taking into account the triangle inequality

$$\begin{aligned}\|G'(x)\| &\leq \|G'(x) - G'(x^*)\| + \|G'(x^*)\| \leq K_p r^p + q = K_p \left(\frac{(1+p)(1-q)}{K_p}\right)^{\frac{p}{p}} + q \\ &= 1 + p(1-q). \quad \blacksquare\end{aligned}$$

The assumptions that we have considered do not necessarily require contractive-type nonlinear mappings on the whole estimated ball, as the following example shows.

Example 2. Let $G : \mathbb{R} \to \mathbb{R}$, $G(x) = x^2$, having $x^* = 0$ as attraction point; $r_1 > 0$ may be chosen arbitrarily large, the derivative G' is Lipschitz continuous on $\mathbb{R}$, with $L = 2$. Since $G'(0) = 0$, one obtains from (9) that $r = r_2 = 1$. Moreover,

$$\|G'(x)\| = 2, \quad \text{for } \|x - x^*\| = r = 1.$$

The above example also shows that estimates (9) and (11) are sharp.

Remark 3. We notice that the predicted radius r_2 may vary inversely proportionally with r_1, since the Hölder continuity constant may increase with r_1. In the previous example, if we take $r_1 = \frac{1}{2}$ we get $L = 1$ and then $r_2 = 2$, which is too large. Analogously, we can obtain too small values for r_2 if we take for instance $G(x) = x^3$ and large values for r_1.

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