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On a Steffensen–Hermite method of order three

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ABSTRACT

In this paper we study a third order Steffensen type method obtained by controlling the interpolation nodes in the Hermite inverse interpolation polynomial of degree 2. We study the convergence of the iterative method and we provide new convergence conditions which lead to bilateral approximations for the solution; it is known that the bilateral approximations have the advantage of offering a posteriori bounds of the errors. The numerical examples confirm the advantage of considering these error bounds.

1. Introduction

The iterative methods play a crucial role in approximating the solutions of nonlinear equations. The methods with superlinear convergence offer good approximations with a reduced number of steps. In a series of papers [1–11] the authors obtain different methods or modifications of some known methods, in order to achieve iterative methods with higher convergence orders.

The Steffensen, Aitken or Aitken–Steffensen methods lead to sequences having at least order 2 of convergence. A natural approach to generalize such methods can be obtained with the aid of inverse polynomial interpolation (Lagrange, Hermite, Taylor, etc.), with controlled interpolation nodes [12–17]. One of the advantages of such methods is the fact that the interpolation nodes may be controlled such that the methods offer sequences with bilateral approximations (both from above and from below) of the solutions. This aspect offers the control of the error at each step [14,16].

In this paper we shall extend a Steffensen type method using the Hermite inverse interpolatory polynomial of degree 2 with two nodes. In [13] we have shown that among all the Steffensen–Hermite methods with two nodes of arbitrary orders, the optimal efficiency index is attained in the case when one node is simple and the other one is double (see [18] for definitions of efficiency index); we have also shown there that the convergence order of this method is 3. Here we provide new convergence conditions, which offer bilateral approximations of the solution; these are very useful for controlling the error at each iteration step. In Section 2, we shall study the convergence of this method, and in Section 3 we shall indicate a method of constructing the auxiliary functions used for controlling the interpolations nodes. Some numerical examples will be shown in Section 4.

Let $c, d \in \mathbb{R}, c < d, f : [c, d] \to \mathbb{R}, g : [c, d] \to [c, d]$ and consider the following equivalent equations:

$$f(x) = 0, \tag{1}$$
$$g(x) = x. \tag{2}$$

As usually, the first order divided difference of f at $a, b \in [c, d]$ will be denoted by $[a, b; f]$; if a is double, then $[a, a; f] = f'(a)$. For the second order divided differences we have

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$$[a,b,e;f]=\frac{[b,e;f]-[a,b;f]}{e-a},\quad a,b,e\in[c,d]$$

and if a is double, then

$$[a,a,b;f]=\frac{[a,b;f]-f'(a)}{b-a}.$$

Let $F=f([c,d])$ and assume the following conditions hold:

(**A**) $f\in C^3([c,d])$ and $f'(x)\neq 0\ \forall x\in[c,d]$.

By **A** it follows that $f:[c,d]\to F$ is invertible so there exists $f^{-1}:F\to[c,d]$.

Let $a_i\in[c,d], i=1,2$ and $b_i=f(a_i), i=1,2,$ i.e. $a_i=f^{-1}(b_i)$, and denote $a_1'=(f^{-1}(b_1))'=\frac{1}{f'(a_1)}$. Consider now the inverse interpolatory Hermite polynomial having b_1 as double node and b_2 as simple node, i.e. the second degree polynomial H determined such that

$$\begin{aligned}&H(b_1)=a_1,\\&H'(b_1)=a_1',\\&H(b_2)=a_2.\end{aligned}\tag{3}$$

Using the divided differences on multiple nodes, the resulted Hermite polynomial may be expressed in one of the following equivalent ways [17]:

$$H(y)=a_1+[b_1,b_2;f^{-1}](y-b_1)+[b_1,b_2,b_1;f^{-1}](y-b_1)(y-b_2),\tag{4}$$

$$H(y)=a_1+[b_1,b_1;f^{-1}](y-b_1)+[b_1,b_1,b_2;f^{-1}](y-b_1)^2\tag{5}$$

or

$$H(y)=a_2+[b_2,b_1;f^{-1}](y-b_2)+[b_2,b_1,b_1;f^{-1}](y-b_2)(y-b_1).\tag{6}$$

The remainder is given by

$$f^{-1}(y)-H(y)=[y,b_1,b_1,b_2;f^{-1}](y-b_1)^2(y-b_2),\quad y\in F.\tag{7}$$

It can be easily seen that the representations given by (4)–(6) verify condition (3).

(**B**) Assume that Eq. (1) has a solution $\bar{x}\in[c,d]$.

By **A** it follows that the solution $\bar{x}$ is unique in $[c,d]$.

One has $\bar{x}=f^{-1}(0)$, whence, by (4)–(7), one obtains the following representations for $\bar{x}$:

$$\bar{x}=a_1-[b_1,b_2;f^{-1}]b_1+[b_1,b_2,b_1;f^{-1}]b_1b_2-r,\tag{8}$$

$$\bar{x}=a_1-[b_1,b_1;f^{-1}]b_1+[b_1,b_1,b_2;f^{-1}]b_1^2-r\tag{9}$$

or

$$\bar{x}=a_2-[b_2,b_1;f^{-1}]b_2+[b_2,b_1,b_1;f^{-1}]b_2b_1-r,\tag{10}$$

where

$$r=[0,b_1,b_1,b_2;f^{-1}]b_1^2b_2.\tag{11}$$

If in (8), (9) or (10) we neglect the remainder r, one may obtain an approximation for $\bar{x}$, denoted by a_3:

$$a_3=a_1-[b_1,b_2;f^{-1}]b_1+[b_1,b_2,b_1;f^{-1}]b_1b_2\tag{12}$$

or

$$a_3=a_1-[b_1,b_1;f^{-1}]b_1+[b_1,b_1,b_2;f^{-1}]b_1^2\tag{13}$$

or

$$a_3=a_2-[b_1,b_2;f^{-1}]b_2+[b_1,b_1,b_2;f^{-1}]b_1b_2.\tag{14}$$

It can be easily seen that

$$[b_1, b_1; f^{-1}] = \frac{1}{f'(a_1)}, \tag{15}$$

$$[b_1, b_2; f^{-1}] = \frac{1}{[a_1, a_2; f]}, \tag{16}$$

$$[b_1, b_1, b_2; f^{-1}] = -\frac{[a_1, a_1, a_2; f]}{[a_1, a_2; f]^2 f'(a_1)}. \tag{17}$$

For the divided difference $[0, b_1, b_1, b_2; f^{-1}]$ we take into account hypothesis **A**, i.e. $f^{-1} \in C^3(F)$ and apply the mean value formula. There exists $\eta \in \mathrm{int}E_0$ such that

$$[0, b_1, b_1, b_2; f^{-1}] = \frac{(f^{-1}(\eta))'''}{3!}, \tag{18}$$

where E_0 is the smallest interval containing b_1, b_2 and 0.

Since f is invertible it follows that there exists $\xi \in \mathrm{int}e_0$ such that $\eta = f(\xi)$, where e_0 is the smallest interval containing a_1, a_2 and $\bar{x}$.

It can be easily seen that [15,17,19]

$$(f^{-1}(y))''' = \frac{3(f''(x))^2 - f'(x)f'''(x)}{(f'(x))^5}. \tag{19}$$

Relations (15) and (16) lead to the following representation for the third order divided difference

$$[0, b_1, b_1, b_2; f^{-1}] = \frac{3(f''(\xi))^2 - f'(\xi)f'''(\xi)}{6(f'(\xi))^5}. \tag{20}$$

Let $x_n \in [c, d]$ be an approximation of the solution $\bar{x}$. If we set in (12)–(14) $a_1 = x_n$, $a_2 = g(x_n)$ and we take into account relations (15)–(17) then we obtain a new approximation x_{n+1} for $\bar{x}$ [16,13]:

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]} - \frac{[x_n, x_n, g(x_n); f]f(x_n)f(g(x_n))}{[x_n, g(x_n); f]^2 f'(x_n)}, \quad n = 0, 1 \ldots, \tag{21}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[x_n, x_n, g(x_n); f]f^2(x_n)}{[x_n, g(x_n); f]^2 f'(x_n)}, \quad n = 0, 1 \ldots, \tag{22}$$

$$x_{n+1} = g(x_n) - \frac{f(g(x_n))}{[x_n, g(x_n); f]} - \frac{[x_n, x_n, g(x_n); f]f(x_n)f(g(x_n))}{[x_n, g(x_n); f]^2 f'(x_n)}, \quad n = 0, 1 \ldots \tag{23}$$

It can be easily seen that (21)–(23) yield in fact a same approximation x_{n+1}.

Analogously, if in (12)–(14) we set $a_1 = g(x_n), a_2 = x_n$ we obtain the approximation x_{n+1} in one of the following (equivalent) forms

$$x_{n+1} = g(x_n) - \frac{f(g(x_n))}{[x_n, g(x_n); f]} - \frac{[x_n, g(x_n), g(x_n); f]f(x_n)f(g(x_n))}{[x_n, g(x_n); f]^2 f'(g(x_n))}, \quad n = 0, 1, \ldots \tag{24}$$

$$x_{n+1} = g(x_n) - \frac{f(g(x_n))}{f'(g(x_n))} - \frac{[x_n, g(x_n), g(x_n); f]f^2(g(x_n))}{[x_n, g(x_n); f]^2 f'(g(x_n))}, \quad n = 0, 1, \ldots \tag{25}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]} - \frac{[x_n, g(x_n), g(x_n); f]f(x_n)f(g(x_n))}{[x_n, g(x_n); f]^2 f'(g(x_n))}, \quad n = 0, 1, \ldots \tag{26}$$

Obviously, (24)–(26) represent the same approximation of $\bar{x}$.

The approximation x_{n+1} given by (21)–(23) obeys

$$\bar{x} - x_{n+1} = -[0, f(x_n), f(x_n), f(g(x_n)); f^{-1}]f^2(x_n)f(g(x_n)). \tag{27}$$

Analogously, for x_{n+1} given by (24)–(26) one obtains:

$$\bar{x} - x_{n+1} = -[0, f(x_n), f(g(x_n)), f(g(x_n)); f^{-1}]f(x_n)f^2(g(x_n)). \tag{28}$$

In this paper we shall study both the convergence of the sequence $(x_n)_{n \geqslant 0}$ given by any of relations (21)–(23) and of the sequence given by any of relations (24)–(26).

Under some reasonable conditions on f regarding monotonicity and convexity on $[c, d]$, we shall show that de above methods yield monotone sequences. Moreover, these iterations provide bilateral sequences, approximating the solution both from above and from below, which lead so a more precise control of the error [13–15].

2. Study of convergence

For the study of method given by (21)–(23), besides **A** and **B** we consider the following hypotheses:

(**C**) Eqs. (1) and (2) are equivalent;
(**D**) function g is continuous and decreasing on $[c,d]$;
(**E**) function $E_f(x) = 3(f''(x))^2 - f'(x)f'''(x) \leqslant 0, \forall x \in [c,d]$.

In the following we shall study the 4 situations when f does not change the monotony and convexity on $[c,d]$.

Theorem 1. *If functions f, g and* $x_0 \in [c,d]$ *obey* **A**–**E** *and, moreover,*

$\mathbf{i}_1$. $f'(x) > 0, \forall x \in [c,d]$;
$\mathbf{ii}_1$. $f''(x) \geqslant 0, \forall x \in [c,d]$;
$\mathbf{iii}_1$. $g(x_0) \in [c,d]$,

then the following properties hold:

$\mathbf{j}_1$. *If* $f(x_0) < 0$ *then sequence* $(x_n)_{n\geqslant 0}$ *given by* (21)–(23) *is increasing and bounded, while the sequence* $(g(x_n))_{n\geqslant 0}$ *is decreasing and bounded;*
$\mathbf{jj}_1$. *If* $f(x_0) > 0$ *then sequence* $(x_n)_{n\geqslant 0}$ *is decreasing and bounded, while* $(g(x_n))_{n\geqslant 0}$ *is increasing and bounded;*
$\mathbf{jjj}_1$. $\lim x_n = \lim g(x_n) = \bar{x}$;
$\mathbf{jv}_1$. $|\bar{x} - x_{n+1}| \leqslant |g(x_{n+1}) - x_{n+1}|, n = 0, 1, \ldots$

Proof. Let $x_n \in [c,d]$ be an approximation for $\bar{x}$ such that $f(x_n) < 0$ and $g(x_n) \in [c,d]$. By $\mathbf{i}_1$ it follows that $x_n < \bar{x}$, while by (**D**) $g(x_n) > \bar{x}$, i.e., $f(g(x_n)) > 0$. From $\mathbf{i}_1, \mathbf{ii}_1$, and (21) it can be easily seen that $x_{n+1} > x_n$ and $g(x_{n+1}) < g(x_n)$. By (18)–(20), (27) it follows that exists $\xi_n \in]x_n, g(x_n)[$ such that

$$\bar{x} - x_{n+1} = -\frac{E_f(\xi_n)}{6(f'(\xi_n))^5} f^2(x_n) f(g(x_n)). \tag{29}$$

The above equality, together with (**E**) and $\mathbf{i}_1$ imply $x_{n+1} < \bar{x}$, which in turn attracts $g(x_{n+1}) > \bar{x}$. The property $\mathbf{j}_1$ is proved. Let $x_n \in [c,d]$ such that $f(x_n) > 0$ and $g(x_n) \in [c,d]$. Obviously, $x_n > \bar{x}$ and $g(x_n) < \bar{x}$, i.e. $f(g(x_n)) < 0$. By $\mathbf{i}_1, \mathbf{ii}_1$ and (22) it follows $x_{n+1} < x_n$. From (29) we get $\bar{x} < x_{n+1}$, i.e. $g(x_{n+1}) < \bar{x}$. These prove $\mathbf{jj}_1$. For proving $\mathbf{jjj}_1$, let $\lim_{n\to\infty} x_n = \ell$. Passing to limit $n \to \infty$ in (21) and taking into account the continuity of f and g leads to $f(\ell) = 0$. Since the solution $\bar{x}$ is unique on $[c,d]$, it follows $\ell = \bar{x}$. Relation $\lim_{n\to\infty} g(x_n) = \bar{x}$ is obvious, as well as property $\mathbf{iv}_1$. □

Theorem 2. *If* $x_0 \in [c,d]$ *and functions f, g verify* **A**–**E** *and, moreover,*

$\mathbf{i}_2$. $f'(x) < 0, \forall x \in [c,d]$;
$\mathbf{ii}_2$. $f''(x) \geqslant 0, \forall x \in [c,d]$;
$\mathbf{iii}_2$. $g(x_0) \in [c,d]$,

then

$\mathbf{j}_2$. *If* $f(x_0) > 0$, *then the sequence* $(x_n)_{n\geqslant 0}$, *generated by any of the relations* (21)–(23), *is increasing and bounded, while sequence* $(g(x_n))_{n\geqslant 0}$ *is decreasing and bounded;*
$\mathbf{jj}_2$. *If* $f(x_0) < 0$, *then the sequence* $(x_n)_{n\geqslant 0}$ *generated by any of the relations* (21)–(23) *is decreasing and bounded, while sequence* $(g(x_n))_{n\geqslant 0}$ *is increasing and bounded;*
$\mathbf{jjj}_2$. *Properties* $\mathbf{jjj}_1$ *and* $\mathbf{jv}_1$ *of Theorem 1 hold true.*

Proof. Let $x_n \in [c,d]$ such that $f(x_n) > 0$ and $g(x_n) \in [c,d]$. By $\mathbf{i}_2$ it follows $x_n < \bar{x}$ i.e. $g(x_n) > \bar{x}$ and $f(g(x_n)) < 0$. If we take into account $\mathbf{i}_2, \mathbf{ii}_2$ and (22), we get $x_{n+1} > x_n$. Hypotheses (**E**) and $\mathbf{i}_1$, together with (29) ensure relation $x_{n+1} < \bar{x}$, i.e. $g(x_{n+1}) > \bar{x}$. Since $x_{n+1} > x_n$ it follows $g(x_{n+1}) < g(x_n)$. Property $\mathbf{j}_2$ is proved. Let $x_n \in [c,d]$ such that $f(x_n) < 0$ and $g(x_n) \in [c,d]$. Obviously, from $\mathbf{i}_2$ it follows $x_n > \bar{x}$ and from (**D**) it follows $g(x_n) < \bar{x}$, i.e. $f(g(x_n)) > 0$. By $\mathbf{i}_2, \mathbf{ii}_2$ and (21) we get $x_{n+1} < x_n$. By (29), taking into account (**E**) and $\mathbf{i}_2$ it follows $x_{n+1} > \bar{x}$, i.e. $g(x_{n+1}) < \bar{x}$. Property $\mathbf{jj}_2$ is proved. Property $\mathbf{jjj}_2$ is obvious. □

In the following, instead of Eq. (1) we shall consider equation

$$h(x) = 0, \tag{30}$$

where $h : [c,d] \to \mathbb{R}, h(x) = -f(x)$.

We notice that if function $E_f(x)$ verifies (**E**), then function $E_h(x) = 3(h''(x))^2 - h'(x)h'''(x) = E_f(x)$ verifies hypothesis (**E**).

Taking into account the above relations, the following assertions are consequences of Theorem 1 respectively Theorem 2:

Corollary 3. *If $x_0 \in [c,d]$ and functions f, g verify hypotheses* **A**–**E** *and, moreover,*

c$_1$. $f'(x) < 0, \forall x \in [c,d]$;
cc$_1$. $f''(x) \leqslant 0, \forall x \in [c,d]$;
ccc$_1$. $g(x_0) \in [c,d]$,

then the following properties hold:

k$_1$. *If $f(x_0) > 0$, then sequence $(x_n)_{n\geqslant 0}$ generated by (21)–(23) is increasing and bounded, while sequence $(g(x_n))_{n\geqslant 0}$ is decreasing and bounded;*
kk$_1$. *If $f(x_0) < 0$, then sequence $(x_n)_{n\geqslant 0}$ generated by (21)–(23) is decreasing and bounded, while sequence $(g(x_n))_{n\geqslant 0}$ is increasing and bounded;*

kkk$_1$. *The properties* **jjj**$_1$ *and* **jv**$_1$ *of Theorem 1 hold true.*

Proof. For the proof of this theorem we notice that if f verifies hypotheses of this Corollary, then function h verifies the assumptions of Theorem 1, so properties **k**$_1$–**kkk**$_1$ are obvious. □

Corollary 4. *If $x_0 \in [c,d]$ and functions f, g satisfy assumptions* **A**-**E** *and, moreover,*

c$_2$. $f'(x) > 0, \forall x \in [c,d]$;
cc$_2$. $f''(x) \leqslant 0, \forall x \in [c,d]$;
ccc$_2$. $g(x_0) \in [c,d]$,

then the following are true:

k$_2$. *If $f(x_0) < 0$ then sequence $(x_n)_{n\geqslant 0}$ generated by (21)–(23) is increasing and bounded while sequence $(g(x_n))_{n\geqslant 0}$ is decreasing and bounded;*
kk$_2$. *If $f(x_0) > 0$ then sequence $(x_n)_{n\geqslant 0}$ generated by (21)–(23) is decreasing and bounded, while sequence $(g(x_n))_{n\geqslant 0}$ is increasing and bounded;*

kkk$_2$. *The properties* **jjj**$_1$ *and* **jv**$_1$ *of Theorem 1 hold true.*

Proof. The proof is analogous to the proof of Corollary 3. □

In the following we shall study the convergence of the sequence $(x_n)_{n\geqslant 0}$ generated by (24)–(26). Instead of hypothesis (**E**) we shall assume further hypothesis

(**E**$'$) $E_f(x) \geqslant 0, \quad \forall x \in [c,d]$.

Theorem 5. *If $x_0 \in [c,d]$ and functions f, g verify assumptions* **A**–**D**, **E**$'$ *and moreover*

i$_5$. $f'(x) > 0, \forall x \in [c,d]$;
ii$_5$. $f''(x) \geqslant 0, \forall x \in [c,d]$;
iii$_5$. $g(x_0) \in [c,d]$,

then the following are true:

j$_5$. *If $f(x_0) < 0$ then sequence $(x_n)_{n\geqslant 0}$ generated by any of (24)–(26) is increasing and bounded while sequence $(g(x_n)_{n\geqslant 0}$ is decreasing and bounded;*
jj$_5$. *Properties* **jjj**$_1$ *and* **jv**$_1$ *of Theorem 1 hold true.*

Proof. Let $x_n \in [c,d]$ such that $f(x_n) < 0$ and $g(x_n) \in [c,d]$. Obviously, by $\mathbf{i}_5$ it follows $x_n < \bar{x}$ and by 2) we get $g(x_n) > \bar{x}$, i.e. $f(g(x_n)) > 0$. Properties $\mathbf{i}_5, \mathbf{ii}_5$ and (26) imply $x_{n+1} > x_n$. Properties (18)–(20), (28) attract the existence of $\xi_n \in]x_n, g(x_n)[$ such that

$$\bar{x} - x_{n+1} = -\frac{E_f(\xi_n)}{6(f'(\xi_n))^5} f(x_n) f^2(g(x_n)). \tag{31}$$

This equality, together with $(\mathbf{E}')$, $\mathbf{i}_5$ and inequality $f(x_n) < 0$ lead to $\bar{x} > x_{n+1}$ and $g(x_{n+1}) > \bar{x}$. From $x_{n+1} > x_n$ and $(\mathbf{D})$ we get $g(x_{n+1}) < g(x_n)$. Property $\mathbf{j}_5$ is proved. Property $\mathbf{jj}_5$ is obvious if we take into account the proof of Theorem 1. □

Theorem 6. *If $x_0 \in [c,d]$ and functions f, g obey* **A–D**, **E**$'$ *and, moreover,*

$\mathbf{i}_6$. $f'(x) < 0, \forall x \in [c,d]$;
$\mathbf{ii}_6$. $f''(x) \geqslant 0, \forall x \in [c,d]$;
$\mathbf{iii}_6$. $g(x_0) \in [c,d]$,

then the following are true:

$\mathbf{j}_6$. *If $f(x_0) < 0$ then sequence $(x_n)_{n\geqslant 0}$ generated by (24)–(26) is decreasing and bounded, while sequence $(g(x_n))_{n\geqslant 0}$ is increasing and bounded;*
$\mathbf{jj}_6$. *Properties $\mathbf{jjj}_1$ and $\mathbf{jv}_1$ of Theorem 1 hold true.*

Proof. The proof is analogous to the proof of Theorem 5. □

If instead of Eq. (1) we consider Eq. (30) then the following consequences of Theorem 5 and respectively Theorem 6 are true:

Corollary 7. *If $x_0 \in [c,d]$ and functions f, g verify* **A–D**, **E**$'$ *and*

$\mathbf{c}_7$. $f'(x) < 0, \forall x \in [c,d]$;
$\mathbf{cc}_7$. $f''(x) \leqslant 0, \forall x \in [c,d]$;
$\mathbf{ccc}_7$. $g(x_0) \in [c,d]$,

then

$\mathbf{k}_7$. *If $f(x_0) > 0$ then sequence $(x_n)_{n\geqslant 0}$ generated by (24)–(26) is increasing and bounded, while $(g(x_0))_{n\geqslant 0}$ is decreasing and bounded;*
$\mathbf{kk}_7$. *The properties $\mathbf{jjj}_1$ and $\mathbf{jv}_1$ of Theorem 1 are true.*

Corollary 8. *If $x_0 \in [c,d]$ and functions f, g verify* **A–D**, **E**$'$ *and*

$\mathbf{c}_8$. $f'(x) > 0, \forall x \in [c,d]$;
$\mathbf{cc}_8$. $f''(x) \leqslant 0, \forall x \in [c,d]$;
$\mathbf{ccc}_8$. $g(x_0) \in [c,d]$,

then

$\mathbf{k}_8$. *If $f(x_0) > 0$ then sequence $(x_n)_{n\geqslant 0}$ generated by (24)–(26) is decreasing and bounded while sequence $(g(x_n))_{n\geqslant 0}$ is increasing and bounded;*
$\mathbf{kk}_8$. *Properties $\mathbf{jjj}_1$ and $\mathbf{jv}_1$ of Theorem 1 are true.*

The convergence order of the sequences studied above is given in the following results.

Theorem 9. *Under the hypotheses of Theorem 1 if, moreover,*

$\mathbf{i}_9$. *The function g is derivable at $\bar{x}$, and the divided differences $[x, \bar{x}; g]$ are bounded on $[c,d]$,*

then the q-convergence order of the sequence $(x_n)_{n\geqslant 0}$ generated by any of the relations (21)–(23) is equal to 3.

Proof. Taking into account equalities $f(\bar{x}) = 0, f(g(\bar{x})) = 0$, relation (29) may be written in the form

$$\bar{x} - x_{n+1} = \frac{E_f(\xi_n)}{6(f'(\xi_n))} \cdot \left(\frac{f(x_n) - f(\bar{x})}{x_n - \bar{x}}\right)^2 \frac{f(g(x_n)) - f(g(\bar{x}))}{g(x_n) - g(\bar{x})} \cdot \frac{g(x_n) - g(\bar{x})}{x_n - \bar{x}} \cdot (x_n - \bar{x})^3.$$

Now, by hypotheses of Theorem 1 there exist $v_n, \mu_n \in]x_n, g(x_n)[$ such that

$$\bar{x} - x_{n+1} = -\frac{E_f(\xi_n)}{6(f'(\xi_n))^5} \cdot (f'(\mu_n))^2 \cdot f'(g(v_n)) \cdot [x_n, \bar{x}; g] \cdot (x_n - \bar{x})^3.$$

By $\mathbf{A}, \mathbf{i}_1$ and $\mathbf{i}_9$ it follows the existence of a constant $C > 0$ such that

$$|\bar{x} - x_{n+1}| \leqslant C \cdot |\bar{x} - x_n|^3,$$

which shows the q-convergence order 3 of the sequence. □

Theorem 10. *Under the hypotheses of Theorem 5 if, moreover, hypothesis* $\mathbf{i}_9$ *of Theorem 9 hold, then the convergence order of the sequence generated by any of relations (24)–(26) is 3.*

Proof. The proof is obtained in a similar manner as in Theorem 9. □

3. Choosing the auxiliary function

We notice that in all the results of the previous section, the continuity and monotony of g are essential (hypothesis (**D**) requires that g is continuous and decreasing).

In the following we shall point out a simple modality of obtaining such a function in accordance with the monotony and convexity of f.

1. If function f verifies: $f'(x) > 0$ and $f''(x) \geqslant 0, \forall x \in [c, d]$, then it can be easily seen that g can be chosen in the following way:

$$g(x) = x - \frac{f(x)}{f'(a)}.$$

More generally, if $\lambda_1 \in [0, f'(a)]$ then we may take

$$g(x) = x - \frac{f(x)}{\lambda_1}.$$

It can be easily seen that $g'(x) \leqslant 0$, i.e., g is decreasing.

2. When $f'(x) > 0, f''(x) \leqslant 0, \ \forall x \in [c, d]$, we may take

$$g(x) = x - \frac{f(x)}{f'(b)};$$

more generally, if $\lambda_2 \in]0, f'(b)[$ then we may take

$$g(x) = x - \frac{f(x)}{\lambda_2}$$

and hypothesis (**D**) is fulfilled.

3. If $f'(x) < 0, f''(x) \geqslant 0, \ \forall x \in [c, d]$ one may take

$$g(x) = x - \frac{f(x)}{f'(b)}$$

or, more generally, for $\lambda_3 \in [f'(b), 0)$ one may take

$$g(x) = x - \frac{f(x)}{\lambda_3}.$$

4. Finally, if $f'(x) < 0, f''(x) \leqslant 0, \ \forall x \in [c, d]$ then one may take

$$g(x) = x - \frac{f(x)}{f'(a)}$$

or more generally, if $\lambda_4 \in [f'(a), 0)$ one may take

$$g(x) = x - \frac{f(x)}{\lambda_4}.$$

4. Numerical examples

(**a**) Let $f(x) = e^x + 10x - 6$. We notice that equation $f(x) = 0$ has a solution $\bar{x} \in]0,1[$, $f(0) = -5$, $f(1) = e + 4$. This solution is unique since $f'(x) = e^x + 10 > 0, \forall x \in [0,1]$. The second and third derivatives of f are $f''(x) = f'''(x) = e^x$, while $E_f(x)$ is given by $E_f(x) = 2e^x(e^x - 5)$, such that for $x \in [0,1]$ we have $E_f(x) < 0$. Hypotheses of Theorem 1 are considered. Let g be given by

$$g(x) = x - \frac{f(x)}{f'(0)},$$

i.e.,

$$g(x) = \frac{-e^x + x + 6}{11}.$$

We take $x_0 = 0$ and we have $g(0) = \frac{5}{11} \in [0,1]$. If we consider $x_0 = 1$ then $g(1) = \frac{7-e}{11} \in [0,1]$. The numerical results in Tables 1 and 2 are in accordance with the properties of the sequences (x_i) and $(g(x_i))$ proved in Theorem 1. In Tables 1–6 we have written only the first two digits for the mantissa of $f(x_i)$ and $g(x_i) - x_i$, since only the magnitude of these quantities is important.

(**b**) Consider the equation

$$f(x) = xe^x + 6x + 6 = 0$$

for $x \in [-1,0]$. Since $f(0) = 6, f(-1) = -\frac{1}{e}$ it follows that the above equation has a solution $\bar{x} \in [-1,0]$. The derivatives of f are

$$f'(x) = e^x(x+1) + 6,$$
$$f''(x) = e^x(x+2),$$
$$f'''(x) = e^x(x+3).$$

It can be easily seen that if $x \in [-1,0]$ then $f'(x) > 0$ and $f''(x) > 0$. Function E_f has the form

$$E_f(x) = e^x(x+3)\left(\frac{2x^2 + 8x + 9}{x+3}e^x - 6\right).$$

Elementary considerations on the function

$$h(x) = \frac{2x^2 + 8x + 9}{x+3}e^x - 6$$

Table 1
Numerical results for $f(x) = e^x + 10x - 6, x_0 = 0$.

i	x_i	$f(x_i)$	$g(x_i)$	$g(x_i) - x_i$
0	0	−5	4.545454545454545e−1	4.5e−01
1	4.440664289515356e−1	−3.0e−04	4.440938528883854e−1	2.7e−05
2	4.440925265279589e−1	−8.8e−16	4.440925265279590e−1	1.1e−16

Table 2
Numerical results for $f(x) = e^x + 10x - 6, x_0 = 1$.

i	x_i	$f(x_i)$	$g(x_i)$	$x_i - g(x_i)$
0	1	6.7e+00	3.892471065037231e−1	6.1e−1
1	4.443161590489098e−1	2.5e−03	4.440811568660437e−1	2.3e−4
2	4.440925265279666e−1	8.7e−14	4.440925265279586e−1	8.0e−15

Table 3
Numerical results for $f(x) = xe^x + 6x + 6, x_0 = -1$.

i	x_i	$f(x_i)$	$g(x_i)$	$g(x_i) - x_i$
0	−1	−3.6e−1	−9.386867598047596e−1	6.1e−2
1	−9.388063596878438e−1	−5.2e−8	−9.388063510191005e−1	8.6e−9
2	−9.388063510535405e−1	0	−9.388063510535405e−1	0

Table 4
Numerical results for $f(x) = xe^x + 6x + 6, x_0 = 0$.

i	x_i	$f(x_i)$	$g(x_i)$	$x_i - g(x_i)$
0	0	6	−1	1
1	−9.373133790648003e−1	8.9e−03	−9.388123833083162e−1	1.4e−03
2	−9.388063510532724e−1	1.6e−12	−9.388063510535415e−1	2.6e−13
3	−9.388063510535405e−1	0	−9.388063510535405e−1	0

Table 5
Numerical results for $f(x) = x^2 + x + e^x - 2, x_0 = 0$.

i	x_i	$f(x_i)$	$g(x_i)$	$g(x_i) - x_i$
0	0	−1	5.000000000000000e−1	5.0e−01
1	3.812436839992096e−1	−9.3e−3	3.858962983331455e−1	4.6e−03
2	3.841231457070055e−1	−1.4e−8	3.841231530080986e−1	7.3e−09
3	3.841231502186257e−1	0	3.841231502186258e−1	5.5e−17

Table 6
Numerical results for $f(x) = x^2 + x + e^x - 2, x_0 = 1$.

i	x_i	$f(x_i)$	$g(x_i)$	$x_i - g(x_i)$
0	1	2.7e+00	−3.591409142295228e−1	1.3e+00
1	8.171724311528673e−1	1.7e+00	−5.734363097371054e−2	8.7e−01
2	4.455499951929994e−1	2.0e−01	3.428432514870640e−1	1.0e−01
3	3.841760770231760e−1	1.7e−04	3.840904238727148e−1	8.5e−05
4	3.841231502186540e−1	9.1e−14	3.841231502186082e−1	4.5e−14
5	3.841231502186256e−1	−4.4e−16	3.841231502186259e−1	2.2e−16

lead to inequality $h(x) < 0, \forall x \in [-1, 0]$, i.e. $E_f(x) < 0$ for $x \in [-1, 0]$. We consider g given by

$$g(x) = x - \frac{f(x)}{f'(-1)} = -\frac{xe^x + 6}{6}.$$

We have

$$g'(x) = -\frac{(x+1)e^x}{6} \leqslant 0, \quad \forall x \in [-1, 0].$$

For $x_0 = -1$ one has $g(-1) = -\frac{6-\frac{1}{e}}{6} \in [-1, 0]$, so the hypotheses of Theorem 1 are satisfied. The approximations from Tables 3 and 4 are again in accordance with Theorem 1.

(**c**) We consider equation

$$f(x) = x^2 + x + e^x - 2 = 0, \quad x \in [0, 1].$$

Since $f(0) = -1, f(1) = e$ it follows that the above equation has a solution in $[0, 1]$. The derivatives of f are given by

$$f'(x) = 2x + 1 + e^x > 0, \quad \text{for } x \in [0, 1],$$
$$f''(x) = 2 + e^x > 0, \quad \text{for } x \in [0, 1],$$
$$f'''(x) = e^x.$$

Function $E_f(x)$ is given by

$$E_f(x) = 12 + (11 - 2x)e^x + 2e^{2x} > 0 \quad \text{for } x \in [0, 1].$$

We consider

$$g(x) = x - \frac{f(x)}{f'(0)} = \frac{1}{2}(x - x^2 - e^x + 2)$$

with

$$g'(x) = \frac{1}{2}(1 - 2x - e^x) < 0, \quad x \in [0, 1].$$

For $x_0 = 0$ we have $g(0) = \frac{1}{2} \in [0, 1]$. Hypotheses of Theorem 5 are satisfied. The approximations of $\bar{x}$ from Tables 5 and 6 are in accordance with the statement of Theorem 5.

The numerical examples show here that the use of $|g(x_n) - x_n|$ offer better error bounds than $|f(x_n)|$.

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