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On an Aitken–Newton type method

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Abstract We study the solving of nonlinear equations by an iterative method of Aitken type, which has the interpolation nodes controlled by the Newton method. We obtain a local convergence result which shows that the q-convergence order of this method is 6 and its efficiency index is $\sqrt[5]{6}$, which is higher than the efficiency index of the Aitken or Newton methods. Monotone sequences are obtained for initial approximations farther from the solution, if they satisfy the Fourier condition and the nonlinear mapping satisfies monotony and convexity assumptions on the domain.

Keywords Nonlinear equations · Aitken method · Newton method · Monotone convergence

1 Introduction

In this note we study an Aitken type method, for which the interpolation nodes are given by two iterations of Newton type. We show that this method has the q-convergence order 6 and it requires 5 function evaluations at each step. This implies that the efficiency index of this method is $\sqrt[5]{6}$, which is greater than $\sqrt{2}$ (the efficiency index of the Newton and of the Aitken method) [10, 14, 20].

Consider the equation

$$f(x) = 0 \tag{1}$$

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where $f : [a, b] \to \mathbb{R}$, $a < b$ and assume

α) this equation has a solution $x^* \in]a, b[$.

We consider two more equations

$$x - g_1(x) = 0$$
$$x - g_2(x) = 0 \tag{2}$$

$g_1, g_2 : [a, b] \to [a, b]$, and we assume they are equivalent to (1).

The Aitken method consists in constructing the sequence $(x_n)_{n\geq 0}$ given by [1–3, 6–8, 10–19],

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n), g_2(x_n);\ f]},\ n = 0, 1, ...,\ x_0 \in [a, b], \tag{3}$$

where $[x, y;\ f]$ stands for the first order divided difference of f at x and y. We suppose that f is derivable on $[a, b]$ and we consider the function

$$g_1(x) = x - \frac{f(x)}{f'(x)}. \tag{4}$$

Denoting

$$g_2(x) = g_1(g_1(x)),$$

we are lead to the following Aitken-type iterative method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(y_n)}{f'(y_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{[y_n, z_n;\ f]},\quad n = 0, 1, ..., x_0 \in [a, b], \tag{5}$$

which we call the Aitken–Newton method.

In Section 2 we provide a local convergence result for this method, and we show a similar result which holds for the Newton method: if f maintains its monotony and convexity on a larger domain, and the initial approximation obeys the Fourier condition, then the iterates converge monotonically to the solution. These properties, together with the fact that the efficiency index of this method is shown that is higher than the efficiency index of the Newton or Aitken methods, justify the study of this method.

2 Convergence of the method

We obtain the following local convergence result.

Theorem 1 *Assume* α) *and*

β) *there exists an open interval* I, $x^* \in I \subseteq]a, b[$ *such that* f *is two times differentiable on* I, *with* f'' *continuous at* x^*.

Then there exists an interval $J \subseteq I$ *such that for any initial approximation* $x_0 \in J$, *the iterations* (5) *are well defined, remain in* J *and converge to* x^* *with q-order at least* 6.

Proof The first and second relations in (5) imply the existence of θ_n and η_n in the interior of the intervals determined by x_n and x^*, resp. y_n and x^* such that

$$x^* - y_n = -\frac{f''(\theta_n)}{2f'(x_n)}(x^* - x_n)^2, \ n = 0, 1, ... \tag{6}$$

$$x^* - z_n = -\frac{f''(\eta_n)}{2f'(y_n)}(x^* - y_n)^2, \ n = 0, 1, ... \tag{7}$$

The third relation in (5) and the Newton identity imply that

$$x^* - x_{n+1} = -\frac{[x^*, y_n, z_n; f]}{[y_n, z_n; f]}(x^* - z_n)(x^* - y_n), \ n = 0, 1, ... \tag{8}$$

Relations (6)–(8) lead to

$$x^* - x_{n+1} = -\frac{[x^*, y_n, z_n; f] f''(\eta_n) \cdot (f''(\theta_n))^3}{16[y_n, z_n; f] f'(y_n)(f'(x_n))^3}(x^* - x_n)^6, \ n = 0, 1, ..., \tag{9}$$

which shows the assertion, provided that x_0 is sufficiently close to x^*. □

Under supplementary conditions on f we obtain the following result.

Theorem 2 *If* f *and* x_0 *verify* α), *and*

β') f *is two times differentiable on* $[a, b]$, *with* f'' *continuous at* x^*;
γ) $x_0 \in [a, b]$ *verifies the Fourier condition:* $f(x_0) f''(x_0) > 0$ (*see* [10]),

and, moreover,

i$_1$. $f'(x) > 0, \ \forall x \in [a, b]$;
ii$_1$. $f''(x) \geq 0, \ \forall x \in [a, b]$,

then the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ *and* $(z_n)_{n\geq 0}$ *generated by* (5), *remain in* $[a, b]$ *and obey*

j$_1$. $x_n > y_n > z_n > x_{n+1} > x^*$, $n = 0, 1, ...$;
jj$_1$. $\lim x_n = \lim y_n = \lim z_n = x^*$.

Proof By α) and i_1 it follows that x^* is the unique solution of (1). Let $x_n \in]a, b[$ be an approximation which verifies the relation $f(x_n) f''(x_n) > 0$. Then by ii_1 it follows that $f(x_n) > 0$, which, together with i_1 lead to $x_n > x^*$.

From i_1, ii_1 and relation (6) we have $x^* - y_n \leq 0$, i.e. $y_n \geq x^*$. Analogously, from (7) and (8) it follows $z_n \geq x^*$ and $x_{n+1} \geq x^*$.

The first relation in (5) and $f(x_n) > 0$, $f'(x_n) > 0$ imply that $y_n < x_n$. Analogously, the second relation in (5) leads to $z_n < y_n$, while the third relation in (5) to $x_{n+1} < z_n$. Conclusion j_1 is therefore proved. Moreover, it is clear that the elements of the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ remain in the interval $[x^*, x_0] \subset [a, b]$. By j_1 it follows that these three sequences are convergent.

Let $\lim x_n = \ell$. Relation j_1 implies that $\lim y_n = \lim z_n = \ell$. The first relation in (5) attracts that $\ell = \ell - \frac{f(\ell)}{f'(\ell)}$, so $f(\ell) = 0$, i.e., $\ell = x^*$. □

The following immediate consequence holds.

Corollary 3 *If f and $x_0 \in [a, b]$ verify $\alpha)$, $\beta')$, $\gamma)$ and, moreover*

i$_2$. $f'(x) < 0, \forall x \in [a, b]$;
ii$_2$. $f''(x) \leq 0, \forall x \in [a, b]$,

then the elements of the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$, and $(z_n)_{n\geq 0}$ generated by (5) remain in the interval $[a, b]$ and satisfy the conclusions j_1 and jj_1 of Theorem 2.

It is easy to see that if instead of (1) we consider

$$-f(x) = 0 \tag{10}$$

then function $h : [a, b] \to \mathbb{R}$ given by relation

$$h(x) = -f(x)$$

verifies hypothesis of Theorem 2.

Theorem 4 *If f and $x_0 \in [a, b]$ verify $\alpha)$, $\beta')$, $\gamma)$ and, moreover,*

i$_3$. $f'(x) > 0, \forall x \in [a, b]$;
ii$_3$. $f''(x) \leq 0, \forall x \in [a, b]$,

then the elements of the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ generated by (5) remain in $[a, b]$ and, moreover, obey

j$_3$. $x_n < y_n < z_n < x_{n+1} < x^*$, $n = 0, 1, ..., ;$
jj$_3$. $\lim x_n = \lim y_n = \lim z_n = x^*$.

The proof of this result is similar to the proof of Theorem 2.

If we replace (1) by (10) then we obtain:

Corollary 5 *If f and $x_0 \in [a, b]$ verify hypothesis $\alpha)$, $\beta')$, $\gamma)$ and, moreover,*

i$_n$. $f'(x) < 0, \ \forall x \in [a, b]$;
ii$_n$. $f''(x) \geq 0, \ \forall x \in [a, b]$,

then the elements of the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ generated by (5) remain in $[a, b]$ and satisfy the statements j_3 and jj_3 of Theorem 4.

Remark 6 Relations (9) show us that the Aitken–Newton method has the q-convergence order at least 6 (it is exactly 6 if $f''(x^*) \neq 0$, see [9]). In order to obtain x_{n+1} from x_n in (5) we need to perform the following function evaluations: $f(x_n)$, $f'(x_n)$, $f(y_n)$, $f'(y_n)$ and $f(z_n)$, i.e., 5 function evaluations. This shows that the efficiency index of this method is $\sqrt[5]{6}$ which is greater than of Aitken or Newton method.

Remark 7 Under additional information on the bounds of the size of derivatives, one can obtain obtain some a posteriori error estimations of the error:

$$\left|x^* - x_{n+1}\right| \leq \frac{M}{2m} \left|x_{n+1} - y_n\right| \left|x_{n+1} - z_n\right|, \; n = 0, 1, ... \tag{11}$$

where

$$m \leq \min_{x \in [a,b]} \left|f'(x)\right|, \; M \geq \max_{x \in [a,b]} \left|f''(x)\right|.$$

In order to prove them, we consider the Newton identity,

$$\begin{aligned} f(x_{n+1}) = & \; f(y_n) + [y_n, z_n; f](x_{n+1} - y_n) + \\ & + [x_{n+1}, y_n, z_n; f](x_{n+1} - y_n)(x_{n+1} - z_n), \; n = 0, 1, ... \end{aligned}$$

whence, taking into acount (5), we get

$$f(x_{n+1}) - f(x^*) = [x_{n+1}, y_n, z_n; f](x_{n+1} - y_n)(x_{n+1} - z_n), \; n = 0, 1, ...,$$

or

$$x_{n+1} - x^* = \frac{[x_{n+1}, y_n, z_n; f]}{[x^*, x_{n+1}; f]}(x_{n+1} - y_n)(x_{n+1} - z_n), \; n = 0, 1, ...$$

The mean value formulas for divided differences lead to (11).

These estimations can be applied in connection to any of the results proved above.

3 Numerical examples

Example 8 Consider the equation

$$f(x) = e^x + \sin x - 2, \; x \in [0, 1].$$

The derivatives of f are given by

$$\begin{aligned} f'(x) &= e^x + \cos x > 0, \quad x \in [0, 1], \\ f''(x) &= e^x - \sin x. \end{aligned}$$

Some elementary considerations on f'' show that $f''(x) > 0$, $x \in [0, 1]$. Since f is continuous, $f'(x) > 0$, $x \in [0, 1]$ and $f(0) = -1$, $f(1) = e + \sin 1 - 2 > 0$, it follows that f has a unique solution on [0, 1].

Taking $x_0 = 1$, the hypotheses of Theorem 2 are satisfied.

Table 1 Numerical results when solving $e^x + \sin x - 2 = 0$

n	x_n	y_n	z_n	$f(x_n)$
0	1.000000000000000e+0	5.213403278939761e−1	4.498799895489901e−1	1.5e+0
1	4.486920253023863e−1	4.486719164440748e−1	4.486719163512726e−1	4.9e−5
2	4.486719163512727e−1			0

In Table 1 we present the results obtained in double precision using MATLAB.

One can easily verify that $\min_{x\in[0,1]} \left|f'(x)\right| = 2$ and $\max_{x\in[0,1]} \left|f''(x)\right| \le e$.

Taking into account (11) we get

$$\left|x^* - x_2\right| \le \frac{e}{4} \left|x_2 - y_1\right| \cdot \left|x_2 - z_1\right|.$$

The quantity in the right hand side is majorized by 7.0e−27, which, together with the fact that $f(x_2) = 0$, shows that in this particular case, x^* can be computed with accuracy higher than the machine epsilon.

Example 9

$$f(x) = \ln(x^2 + x + 2) - x + 1 = 0,\ \ x \in [4, 5].$$

We have

$$f'(x) = \frac{-x^2 + x - 1}{x^2 + x + 2} < 0,\ \text{ for } x \in [4, 5];$$

$$f''(x) = \frac{-2x^2 - 2x + 1}{(x^2 + x + 2)^2} < 0,\ \text{ for } x \in [4, 5];$$

$f(4) = \ln 22 - 3 > 0$ and $f(5) = \ln 32 - 4 < 0$.

We take $x_0 = 5$, so hypotheses of Corollary 3 are verified. The obtained results are presented in Table 2.

In this case we have $\min_{x\in[4,5]} \left|f'(x)\right| > \frac{1}{3}$ and $\max_{x\in[4,5]} \left|f''(x)\right| \le \frac{1}{8}$, whence, by (11) we get

$$\left|x^* - x_2\right| \le \frac{3}{16} \left|x_2 - y_1\right| \cdot \left|x_2 - z_1\right|.$$

The quantity in the right hand side is majorized by 1.4e−31, which, together with the fact that $f(x_2) = 0$ shows that in this example too x^* is computed with accuracy higher than the machine epsilon.

Table 2 Numerical results when solving $\ln(x^2 + x + 2) - x + 1 = 0$

n	x_n	y_n	z_n	$f(x_n)$
0	5.000000000000000e+0	4.185883280456726e+0	4.152656878948953e+0	−5.3e−1
1	4.152590868900850e+0	4.152590736757159e+0	4.152590736757158e+0	−7.9e−8
2	4.152590736757158e+0			0

We make the following comments regarding the convergence order of the obtained sequences. Since they are monotonic in these two examples, the quotient convergence factors [9, ch. 9] can be determined by (9), taking into account the mean value formulas for divided differences, as

$$Q_6\{x_k\} = \lim_{k\to\infty} \frac{|x_{k+1}-x^*|}{|x_k-x^*|^6} = \frac{\left(f''(x^*)\right)^5}{32\left(f'(x^*)\right)^5}. \tag{12}$$

Therefore, since in both examples $f''(x^*) \neq 0$, the q-convergence order is exactly 6. However, the above quantity requires the knowledge of the solution x^*. In [4] and [5], the asymptotical constant $Q_6\{x_k\}$ was approximated by some quantities computable at each step:

$$\frac{|x_{k+1}-x_k|}{|x_k-x_{k-1}|^6},\ k = 1, 2, \ldots \tag{13}$$

In Example 8, formula (12) (with x^* considered as x_2) yields the value 6.3e−4, while formula (13) for $k = 1$ yields 7.1e−4, i.e., two close quantities. These values are also close in Example 9, where we obtain 1.0e−6, respectively 3.5e−7.

Some other formulas to determine the convergence order were considered in [21]:

$$p \approx \frac{\ln|(x_{k+1}-x^*)/(x_k-x^*)|}{\ln|(x_k-x^*)/(x_{k-1}-x^*)|},\ k = 1, 2, \ldots$$

subsequently approximated in [5] by

$$p \approx \frac{\ln|(x_{k+1}-x_k)/(x_k-x_{k-1})|}{\ln|(x_k-x_{k-1})/(x_{k-1}-x_{k-2})|},\ k = 2, 3, \ldots$$

but since in the presented examples the solution was approximated in only three steps (the convergence order is high), we cannot use these formulas.

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