SPECIAL FAMILIES OF ORBITS FOR THE HÉNON - HEILES TYPE POTENTIAL

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Abstract. For a Hénon-Heiles type potential, with undetermined coefficients, the possibility for a material point to describe a family of curves of the form $x^{-p}y = q$ is studied. Two admissible cases, $p = 4$ and $p = -2$, are found; for these the potential and the family boundary curves are specified, and the regions in which the curves of the given family can be traced are determined.

Key words: inverse problem, family boundary curves, Hénon-Heiles potential.

1. INTRODUCTION

The Hénon-Heiles potential was introduced by Hénon and Heiles (1964) in order to model the motion of a star in a galaxy. The considered potential had cylindrical symmetry and was constructed adding to the potential of a planar oscillator two terms of third degree in the coordinates. In fact, the added terms induced the character of non-linearity and non-integrability of the problem. The Hénon-Heiles potential appears also expanding the potential of an integrable system (resulting after some canonical transformations applied to a system representing the motion of three particles on a circle acted upon by exponentially decreasing forces) up to the third degree terms (Boccaletti and Pucacco, 1996, p. 123).

The orbits that can be described by a material point of unit mass under the influence of Hénon-Heiles potential are of great interest. This problem was considered by van der Merwe (1991), who used an integro-differential equation relating configuration space trajectories to the potential, for obtaining several possible orbits. Another task was fulfilled by Antonov and Timoshkova (1993), namely, finding rectilinear trajectories and those given by second-order curves.

The two-dimensional inverse problem of Dynamics, which is as old as Newton's *Principia* (1687), consists of finding a potential V that generates a family of curves

$$f(x, y) = q \tag{1}$$

in the xy Euclidean space. From this point of view, not isolated trajectories are given, but entire families. We are going to find whether a Hénon-Heiles type potential can produce special families of planar orbits.

The problem faced in this paper is:

Considering the Hénon-Heiles type potential V given by

$$V(x, y) = ax^2 + bx^2 + cx^2y + dy^3, \tag{2}$$

determine the coefficients $a, b, c, d \in \mathbf{R}$, $a, b > 0$, so that the system of differential equations

$$\begin{aligned} \ddot{x} &= -V_x, \\ \ddot{y} &= -V_y \end{aligned} \tag{3}$$

admits as solutions some specific families of orbits (1). If such a potential V is determined, a material point of unit mass could describe, under suitable initial conditions, the specified family of orbits, in a certain region of the xy plane.

2. SZEBEHELY'S EQUATION

For solving the problem of determining Hénon-Heiles type potentials producing specific families of orbits, we can consider the equation given by Szebehely, relating the potential V and the family of curves (1), which is

$$f_xV_x + f_yV_y - \frac{2(E(f) - V)}{f_x^2 + f_y^2}\left(f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2\right) = 0. \tag{4}$$

The subscripts denote partial derivatives and

$$E = E(f(x, y)) \tag{5}$$

represents the energy on each family of orbits (1).

From equation (4) one can obtain the expression of $E(f)$; the energy has the partial derivatives

$$E_x = E'(f)f_x \qquad E_y = E'(f)f_y, \tag{6}$$

so the condition

$$E_x f_y - E_y f_x = 0 \tag{7}$$

holds.

Relation (7) contains only the partial derivatives of V and f, unlike equation (4) in which the energy E appeared explicitly, but its form is rather complicate. Actually, the task of obtaining a partial differential equation, which does not contain the energy, was already accomplished by Bozis (1984).

3. BOZIS' SECOND ORDER EQUATION

Denoting

$$\gamma = \frac{f_y}{f_x},\ \Gamma = \gamma\gamma_x - \gamma_y, \tag{8}$$

Bozis (1984) obtained the second order linear partial differential equation

$$-V_{xy} + kV_{xy} + V_{yy} = \lambda V_x + \mu V_y \tag{9}$$

where

$$k = \frac{1-\gamma^2}{\gamma},\ \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma},\ \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}.$$

This was done writing Szebehely's equation (4) in the form given by Bozis (1983)

$$V_x + \gamma V_y + \frac{2\Gamma}{1+\gamma^2}(E - V) = 0, \tag{10}$$

solving it for E and inserting the result into the relation $E_y = \gamma E_x$, which is the equivalent of (7).

For the potential of Hénon-Heiles type (2), equation (9) becomes

$$2(b-a) + 2(3d-c)y + 2cxk = 2(a+cx)x\lambda + \left(2by + cx^2 + 3dy^2\right)\mu \tag{11}$$

and will be our main tool in the following.

4. SPECIAL FAMILIES OF CURVES

Consider the special family of orbits

$$f(x, y) = x^{-p} y, \tag{12}$$

with $p \in \mathbf{Z}\setminus\{0,1\}$, in which case we have $\gamma = \frac{x}{py}$.

From equation (9) we obtain

$$3\left(2cp^2 - dp - 2d\right)y^2 + c(p-4)x^2 + 4\left(p^2a - b\right)y = 0$$

and, equating the coefficients of y^2, x^2 and y to 0, we obtain, besides the trivial case with $c = d = 0$ and $b = ap^2$, two significant cases.

The first case is that of the family (12) with $p = 4$,

$$x^{-4}y = q \tag{13}$$

and of the potential (2) with $b = 16\,a$ and $d = \frac{16}{3}c$. The expression of the potential is

$$V_1(x, y) = a\left(x^2 + 16y^2\right) + c\left(x^2 + \frac{16}{3}y^2\right)y, \tag{14}$$

with $a > 0, c \in \mathbf{R}, c \neq 0$.

This is in concordance with Example B of Bozis et al. (1997), obtained while searching for inhomogeneous potentials producing families of orbits having homogeneous analytic expressions, and also with the result of van der Merwe (1991), obtained looking for single orbits of the form $y = \alpha x^p + \beta$ for the Hénon-Heiles potential.

The second case holds for $p = -2$, i.e. for the family

$$x^2 y = q \tag{15}$$

which can be produced by a potential (2) with $b = 4a\ c = 0$. This potential has then the form

$$V_2(x, y) = a\left(x^2 + 4y^2\right) + dy^3, \tag{16}$$

with $a > 0, d \in \mathbf{R}, d \neq 0$.

5. FAMILY BOUNDARY CURVES

As stated in the paper of Bozis and Ichtiaroglou (1994), during the motion of a material point of unit mass along an orbit of the family (1), the inequality

$$B(x, y) \geq 0 \tag{17}$$

must be observed, with

$$B = E(f(x, y)) - V(x, y). \tag{18}$$

The motion is allowed along those curves (or parts of them) of the family (1) which are lying in some regions limited by the *family boundary curves* (FBC) of equation $B(x,y) = 0$.

We shall restrict ourselves to the families (13) and (15) with $q > 0$, because the case $q < 0$ can be handled in a similar way. From the analytic expression of the function f generating the two families, it is obvious that the motion on the given curves of the two families is then allowed only in the upper halfplane.

For V_1 given by (14) and the family of curves (13), relation (10) gives the value of the energy $E_1 = -\frac{cx^4}{24y}$. In this case we obtain for B given by (18) the expression depending on the constants $a > 0, c \in \mathbf{R}, c \neq 0$,

$$B_1 = -\frac{1}{24y}\left(x^2 + 16y^2\right)\left(cx^2 + 8cy^2 + 24ay\right)$$

This means that, for $c > 0$, the family (13) cannot be traced in the upper halfplane, because $B_1(x, y) < 0$ for $y > 0$. For $c < 0$, the family (13) will be traced in the upper semiplane outside the ellipse given by

$$x^2 + 8\left(y + \frac{3a}{2c}\right)^2 - 18\frac{a^2}{c^2} = 0, \tag{19}$$

which represents the FBC. In Fig. 1 this situation is illustrated for $a = 0.5$ and $c = -1$, where the curves with $q = \frac{1}{100}, \frac{1}{32}, \frac{1}{20}, \frac{1}{10}$ from the family (13) are traced. The curve $y = \frac{1}{32}x^4$ is tangent to the ellipse.

In the second case, for V_2 given by (16) and the family of curves (15), from relation (10) we get the value of the energy $E_2 = \frac{dx^2 y}{4}$. To obtain the admissible region we calculate B given by (18)

$$B_2 = -\frac{1}{4}\left(x^2 + 4y^2\right)\left(dy + 4a\right),$$

which depends on the constants $a > 0, d \in \mathbf{R}$, $d \neq 0$. It follows that for $d > 0$ the curves of the family (15) cannot be described in the halfplane $y > 0$, while for $d < 0$ they will be traced in the region above the FBC

$$y = (4a / d). \qquad (20)$$

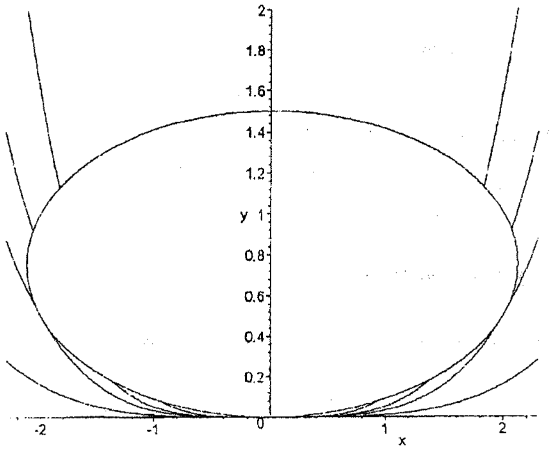

Fig. 1

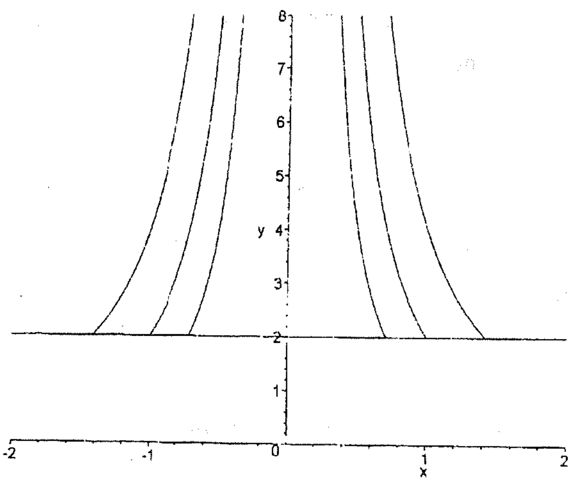

Fig. 2

In Fig. 2 the curves of the family (15) with $q = 1,2,4$ are represented for $a = 0.5$ and $d = -1$.

So, for the Hénon-Heiles type potential (2), the families of curves (13) and (15) can be traced under the action of potentials (14) and (16), respectively. This cannot be done in the entire upper halfplane, but only outside the ellipse given by (19) in the first case, and above the FBC (20) in the case of the family (15).

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