

FAMILIES OF STRAIGHT LINES IN PLANAR POTENTIALS

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Abstract. It is shown that all planar potentials $V(x, y)$ that can give rise, among other conditions, to monoparametric families of straight lines (FSL) satisfy a condition, expressed in the form of a second order nonlinear partial differential equation in V . To each such potential there corresponds just one FSL, whereas each preassigned monoparametric FSL can be created by infinitely many potentials. Various types of potentials (e.g., separable, homogeneous, etc.) producing FSL are studied. A necessary and sufficient condition is found, satisfied by all "adelphic" families of orbits, i.e. families that can coexist with a given FSL.

Key words: inverse problem of Dynamics – planar potentials – families of straight lines.

1. INTRODUCTION

The finding of potentials creating preassigned families of orbits, traced by a material point of unit mass, is the goal of the classical inverse problem of Dynamics. For the two-dimensional version of this problem, the available mathematical equipment consists of

(i) Szebehely's (1974) first order PDE (relating monoparametric families $f(x, y) = c$, energy dependence functions $E = E(c)$, and potentials $V(x, y)$), and

(ii) a free of energy second order PDE, which relates merely orbits (more specifically slope functions $\gamma = f_y / f_x$) to potentials $V = V(x, y)$ (see Bozis 1984).

On the basis of the above two equations, a richness of compatible pairs of potentials and families has been obtained. Several examples can be found in a review

paper by Bozis (1995). More recent pertinent work refers to programmed motion (Anisiu and Bozis 2000) and also to the solvability (Grigoriadou et al. 1999), to the integrability (Grigoriadou 1999), and to the nonintegrability problem (Bozis and Meletlidou 1998).

In all these studies, there was no systematic approach to the case of families of *straight lines*. In fact, this case, being special from the mathematical point of view, was tacitly or deliberately excluded. Some simple families of straight lines, namely those of parallel lines or of pencils of lines, were studied by Grigoriadou (1999) in connection with the problem of integrability in the sense of Darboux.

On the other hand, *isolated straight line solutions* have been found, e.g. for the Hénon-Heiles model, by Antonov and Timoshkova (1993), with the aid of Szebehely's equation, and by Van der Merwe (1991), with the aid of Rajaraman's integro-differential equation. Isolated straight lines have been also found in galactic models by Contopoulos and Zikides (1980) or by Caranicolas and Innanen (1992).

These findings contribute to the tracing of solvable cases for model potentials including certain parameters. Besides that, such isolated straight lines are also used in the reasoning of Yoshida's (1987) theorem, which offers a sufficient condition of nonintegrability of a given homogeneous potential. All the above may be considered as a motivation for our present work also, in addition, of course, to our goal of covering this particular and interesting case of the inverse problem of Dynamics.

In the present paper we study planar potentials $V = V(x, y)$ that can produce (among other orbits) monparametric *families* of straight lines (FSL)

$$a_1(c)x + a_2(c)y + a_3(c) = 0$$

parameterized by c . Assuming that the dependence of the coefficients on c leads (by means of an implicit function theorem) to a monparametric family $f(x, y) = c$, this can be studied with the tools of the inverse problem.

In Section 2 we give Szebehely's equation and we derive a differential condition (equation (12) below) satisfied by all potentials associated with FSL. To each solution of this equation (12) there corresponds one family of straight lines, whereas any preassigned FSL can be generated by infinitely many potentials found from (11). All these potentials satisfy the condition (12).

In Section 3 we find some obvious solutions of the condition (12) and we prove a theorem offering a class of potentials producing FSL.

In Section 4 we study some special forms of potentials (separable, homogeneous, etc.) giving rise to FSL.

In Section 5 we study the so-called family boundary curves (Bozis and Ichtiaroglou 1994) for the case at hand.

Finally, in Section 6, we find a condition for all the other families $\gamma(x, y)$ which can coexist with a preassigned FSL $\gamma^*(x, y)$.

2. A CONDITION FOR POTENTIALS GENERATING FSL

The two-dimensional inverse problem considers a family of orbits

$$f(x, y) = c \quad (1)$$

traced by a material point of unit mass. This family is determined by the "slope" function

$$\gamma = \frac{f_y}{f_x}. \quad (2)$$

To each f corresponds obviously one γ , but also to each γ corresponds just one monoparametric family (1). We define also the function

$$\Gamma = \gamma \gamma_x - \gamma_y, \quad (3)$$

It was proved by Szebehely (1974) that the potentials producing the family (1) (or, equivalently, associated with the slope function (2)) must satisfy the first order partial differential equation

$$V_x + \gamma V_y + \frac{2\Gamma}{1 + \gamma^2} (E - V) = 0, \quad (4)$$

where $E = E(f(x, y))$ is the energy assigned to the given family (1).

Remark 1. In what follows (except in Section 6) we shall consider $\gamma \neq 0$. For $\gamma = 0$ we obtain $V_x = 0$, hence $V = V(y)$; this potential, among other planar orbits, produces the family of vertical straight lines $x = c$.

Considering $\Gamma \neq 0$ and denoting

$$k = \frac{1}{\gamma} - \gamma, \quad (5)$$

$$\lambda = \frac{1}{\gamma \Gamma} (\Gamma_y - \gamma \Gamma_x), \quad (6)$$

$$\mu = \lambda \gamma + \frac{3\Gamma}{\gamma}, \quad (7)$$

Bozis (1984) proved that V must satisfy the second order (free of energy) partial differential equation

$$-V_{xx} + kV_{xy} + V_{yy} = \lambda V_x + \mu V_y. \quad (8)$$

The motion of the material point is allowed only in the region of the xy plane where

$$E(f(x, y)) - V(x, y) \geq 0. \quad (9)$$

This region is delimited by the so-called family boundary curves.

By (2) and (3), Γ is expressed in terms of the derivatives of f as

$$\Gamma = \frac{2f_{xy}f_xf_y - f_{xx}f_y^2 - f_{yy}f_x^2}{f_x^3},$$

hence the curvature of the orbits in (1) is given by $K = \Gamma / (1 + \gamma^2)^{3/2}$. It follows that the family (1) consists of straight lines if and only if $\Gamma = 0$; in view of (3), this condition may be written as

$$\gamma\gamma_x - \gamma_y = 0. \quad (10)$$

For conservative force fields, Szebehely's equation for a family of straight lines becomes

$$V_x + \gamma V_y = 0. \quad (11)$$

Remark 2. If V depends only on one variable x or y , equation (11) implies that it must be constant (By Remark 1, we consider families with $\gamma \neq 0$). This trivial case will be excluded from our study.

Since $\Gamma = 0$, the energy does not appear in (11). Expressing γ from (11) and introducing its value into (10), we get the nonlinear partial differential equation

$$V_x V_y (V_{xx} - V_{yy}) = V_{xy} (V_x^2 - V_y^2), \quad (12)$$

which must be satisfied by all potentials creating (among other orbits) a family of straight lines.

Remark 3. Given an adequate γ (i.e. satisfying (10)), there exist infinitely many potentials $V(x, y)$, found from (11), creating the monoparametric family of straight lines. But for adequate V (i.e. satisfying (12) and depending on both

variables x and y), there corresponds to it exactly one γ , found from (11), hence one family of straight lines.

Some potentials are described in a simpler way using polar coordinates. Equation (11) can be written in polar coordinates as

$$V_r r \cos \theta - V_\theta \sin \theta + \gamma(V_r r \sin \theta + V_\theta \cos \theta) = 0, \quad (13)$$

while equation (12) becomes

$$V_{r\theta} V_r^2 r^3 - V_{r\theta} V_\theta^2 r - V_{rr} V_r V_\theta r^3 + V_{\theta\theta} V_r V_\theta r + V_\theta^3 = 0. \quad (14)$$

3. PARTICULAR POTENTIALS AND CLASSES OF POTENTIALS PRODUCING FSL

Focusing our attention on equation (12), we record some of its obvious solutions:

- (i) $V = V(x)$ or $V = V(y)$;
- (ii) $V = V(c_1 x + c_2 y)$, c_1 and c_2 constants;
- (iii) $V = x^2 + y^2$.

The FSL corresponding to the above potentials and having $\gamma \neq 0$ are the following: in case (i), equation (11) can be satisfied only for the trivial case, $V = \text{constant}$, as pointed out in Remark 2, and each $\gamma = \text{constant}$ will represent a FSL. For the class of potentials in case (ii), we obtain from (11) the family of parallel straight lines given by $\gamma = -c_1 / c_2 = \text{constant}$. In case (iii), the family of straight lines is a pencil of lines through the origin of the axes, having the homogeneous of order zero slope function $\gamma = -x / y$. These particular solutions of equation (12) are a good reason for a further study of classes of solutions.

We now prove an important property of equation (12), which is also shared by equation (11).

Theorem 1. *If $v = v(x, y)$ is a solution of equation (12), then all functions $V = F(v)$, with F an arbitrary C^2 -function, are solutions of (12); for a potential v associated with the family of straight lines γ (according to (11)), $F(v)$ will be associated with the same family of straight lines.*

Proof. Let v be a solution of (12) and $V = F(v)$. We prepare the values of first order derivatives of V :

$$V_x = F'(v)v_x, \quad V_y = F'(v)v_y,$$

and of second order derivatives

$$V_{xx} = F''(v)v_x^2 + F'(v)v_{xx},$$

$$V_{xy} = F''(v)v_xv_y + F'(v)v_{xy},$$

$$V_{yy} = F''(v)v_y^2 + F'(v)v_{yy};$$

inserting them in (12) gives

$$v_xv_y(v_{xx} - v_{yy}) - v_{xy}(v_x^2 - v_y^2) = 0,$$

which is true because v was supposed to be a solution for (12). It is now obvious that, according to (11), both v and $F(v)$ lead to the same γ . $\square$

Remark 4. The result in Theorem 1 enables us to state that if $V(r, \theta)$ is a solution of equation (14), respectively (13), $F(V)$ is also a solution of the corresponding equation, for every arbitrary C^2 -function F .

Apart from this, a direct proof of the above statement can be made, similar to the proof of Theorem 1.

Corollary 2. *Additive and multiplicative constants in the expression of the solutions of (12) and (11) may be omitted.*

Corollary 3. *The special solution in (iii), $V = x^2 + y^2$, allows us to state that all central potentials $V = V(r)$ are solutions of (12).*

Remark 5. Out of all central potentials, which are solutions of equation (12), only the logarithmic potential $V = \ln r$ is also solution of the two-dimensional equation of Laplace.

4. SPECIAL TYPES OF POTENTIALS PRODUCING FSL

In what follows we shall look for other classes of solutions for equation (12), respectively (14), if polar coordinates are used.

4.1. SEPARABLE POTENTIALS $V = u(x) + w(y)$

Substituting $V = u(x) + w(y)$ in (12) gives $u'v'(u'' - v'') = 0$, the primes denoting the corresponding derivative with respect to the one variable of the functions. Omitting the one-variable potentials considered in (i), we have $u'' = v'' = a$, so, the general form of separable solutions for (12), in view of Corollary 2 also, is

$$V = a(x^2 + y^2) + c_1x + c_2y. \quad (15)$$

Cases (ii) and (iii) at the beginning of Section 3 are included in (15). The potential (15) is central, centered at the point $(-c_1/(2a), -c_2/(2a))$.

Remark 6. From the relation (15), in view of Theorem 1 we obtain the potentials of the form $V = u(x)w(y)$ which are solutions of equation (12). These potentials are given by

$$V = \exp[a(x^2 + y^2) + c_1x + c_2y], \quad (16)$$

the corresponding slope function γ being $\gamma = (2ax + c_1)/(2ay + c_2)$. For $a = 0$, a family of parallel lines is obtained, and for $a \neq 0$ a pencil of lines.

4.2. HOMOGENEOUS OF ORDER m POTENTIALS

Let us consider a homogeneous of order m potential having the form $V(x, y) = x^m R(z)$, where $z = y/x$. Expressing the derivatives $V_x, \dots, V_{yy}$ in terms of the function R and derivatives R', R'' with respect to z , and substituting in (12), we obtain

$$[mR(z)R''(z) - (m-1)R'^2(z)][(1+z^2)R'(z) - mzR(z)] = 0,$$

hence at least one of the two factors must be zero.

a) Let us consider at first the equation

$$mR(z)R''(z) - (m-1)R'^2(z) = 0.$$

For $m = 0$, it follows $R = \text{constant}$ and we are in the trivial case of $V(x, y)$ constant.

For $m = 1$, we obtain $R(z) = c_1z + c_2$ and $V(x, y) = c_1x + c_2y$. Solution which was already mentioned in (ii) at the beginning of Section 3 and enters also in the class of separable potentials considered in Subsection 4.1.

For $m \in \mathbb{Z} \setminus \{0, 1\}$, integration of

$$m \frac{R''(z)}{R'(z)} = (m-1) \frac{R'(z)}{R(z)}$$

gives $R(z) = (k_1 + k_2 z)^m$, hence $V(x, y) = (k_1 x + k_2 y)^m$. This solution is again of the form (ii).

b) The equation $(1 + z^2)R'(z) = mzR(z)$ has the solution $R = k_0(1 + z^2)^{m/2}$. This gives the potential $V(x, y) = k_0(x^2 + y^2)^{m/2}$, which can be written as $V(r) = k_0 r^m$, i.e. a central potential considered in Corollary 3.

In conclusion, the homogeneous of order m potentials satisfying (12) are of the form $V(x, y) = (k_1 x + k_2 y)^m$ (a special case of (ii)), or $V(r) = k_0 r^m$, hence a type of central potential.

Studying the motion of a star in a galaxy, Hénon and Heiles (1964) considered a potential having cylindrical symmetry; the corresponding planar potential consisted of a sum of two homogeneous of second and third order functions.

Remark 7. It can be shown that the Hénon-Heiles potential

$$V = (x^2 + by^2)/2 + ax^3 + xy^2 \quad (17)$$

does not admit any family of straight lines, i.e. there are no values of the constants a and b for which (17) satisfies the condition (12). The rectilinear motions for this potential, found by Antonov and Timoshkova (1993), are isolated straight lines. More generally, it can be shown that families of straight lines do not exist for any potential that includes the harmonic oscillator terms $(x^2 + y^2)/2$ and third order terms in x and y . This also happens in the case of the genuine two-fixed-centers problem (for equal masses, there exists, of course, the isolated solution $x = 0$) (Whittaker 1904).

4.3. SEPARABLE POTENTIALS IN POLAR COORDINATES

The substitution of $V(r, \theta) = A(r)B(\theta)$ in (14) gives

$$B' \left(A' \left(\frac{A'}{A} \right)' r^3 - A' \left(\frac{B'}{B} \right)' r - A \left(\frac{B'}{B} \right)^2 \right) = 0,$$

where the prime denotes the derivative with respect to the pertinent argument of the functions.

Excluding the case $B' = 0$ (leading again to central potentials), we are left with the equation

$$\frac{A'}{A} \left(\frac{A'}{A} \right)' r^3 - \frac{A'}{A} \left(\frac{B'}{B} \right)' r - \left(\frac{B'}{B} \right)^2 = 0$$

The equality is possible only if both expressions containing B and its derivatives are constant, or both expressions containing r and A and its derivatives are constant.

Let us suppose at first that $B'/B = b$, which implies $(B'/B)' = 0$. If $b = 0$, it is necessary that $B = \text{constant}$, case leading to central potentials. For $b \neq 0$ we are left with the equation

$$\frac{A'}{A} \left(\frac{A'}{A} \right)' r^3 = b,$$

with the solution

$$A = A_0 \exp \left(\pm \int \sqrt{a^2 - b^2 / r^2} dr \right),$$

defined for $r > |b/a|$. We get $V(r, \theta) = \exp \left(b\theta \pm \int \sqrt{a^2 - b^2 / r^2} dr \right)$, solution that can be written as

$$V(r, \theta) = \exp \left[b \left(\theta \mp \arctan_0 \left(\frac{\sqrt{a^2 r^2 - b^2}}{b} \right) \right) \pm \sqrt{a^2 r^2 - b^2} \right], \quad (18)$$

where by the notation $\arctan_0$ we mean the principal value of the multivalued function $\arctan$.

Supposing now that $(A'/A)r = \alpha$, we get

$$\frac{A'}{A} \left(\frac{A'}{A} \right)' r^3 = -\alpha^2$$

and $A = A_0 r^\alpha$. The equation in θ will be $\alpha(B'/B)' + (B'/B)^2 + \alpha^2 = 0$, with the solution $B = B_0 \cos^\alpha(\theta - \theta_0)$, leading to the (obvious) potential

$$V(r, \theta) = r \cos(\theta - \theta_0), \quad (19)$$

which is of the form (ii) of Section 3.

Remark 8. To the expression (18) there correspond, according to Theorem 1 of Section 3, solutions of (14) separable in the sense of addition, i.e. of the form $V(r, \theta) = A(r) + B(\theta)$. Indeed, from (18) we find the solution

$$V(r, \theta) = b\theta \pm \sqrt{a^2 r^2 - b^2} \mp b \arctan_0 \left(\frac{\sqrt{a^2 r^2 - b^2}}{b} \right), \quad (20)$$

to which, according to (11), there corresponds the slope function

$$\gamma = \frac{yb \mp x\sqrt{T}}{xb \pm y\sqrt{T}}, \quad (21)$$

with $T = a^2(x^2 + y^2) - b^2$. The potential (20) can be written in Cartesian coordinates as

$$V(x, y) = b \arctan_0 \left(\frac{yb - x\sqrt{T}}{xb + y\sqrt{T}} \right) \pm \sqrt{T}.$$

Some members of the family (21) generated by this potential are plotted in Fig. 1.

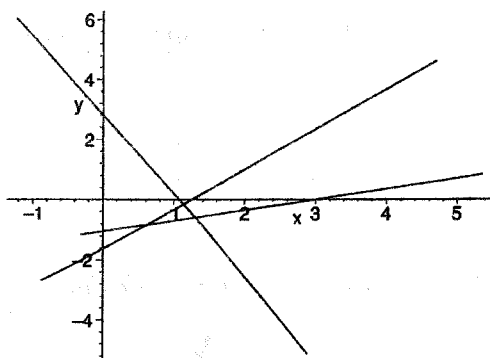


Fig. 1 – Members of the family (21) produced by the potential (20) written in Cartesian coordinates.

4.4. CIRCLE-PRODUCING AND FSL-PRODUCING POTENTIALS

The general form of the potentials that produce the family of circles $x^2 + y^2 = c$ was given by Broucke and Lass (1977) and it is

$$V(r, \theta) = g(r) + \frac{1}{r^2} h(\theta), \quad (22)$$

g and h being arbitrary functions.

Preparing the partial derivatives V_r , V_θ , V_{rr} , $V_{r\theta}$, $V_{\theta\theta}$ and inserting them into (14), we find

$$(2hh'' - 3h'^2 - 4h^2)h' = r^3h'[g'(h'' + 2h) + 2rg''h - 2r^3g'^2 - r^4g''g'],$$

where primes denote the derivatives of the functions with respect to their unique argument.

If $h'(\theta) = 0$, then $h = h_0$ and the potential (22) is central, as such, it will produce also a family of straight lines.

If $h'(\theta) \neq 0$, we obtain the equation

$$2hh'' - 3h'^2 - 4h^2 = r^3[g'(h'' + 2h) + 2rg''h - 2r^3g'^2 - r^4g''g']. \quad (23)$$

The left-hand side is independent of r and the differentiation with respect to r leads to

$$h''(rg'' + 3g') + 2h(r^2g''' + 5rg'' + 3g') = r^3(r^2g'g''' + r^2g''^2 + 11g'g'' + 12g'^2). \quad (24)$$

The right-hand side of (24) depends merely on r , therefore

$$h'''(rg'' + 3g') + 2h'(r^2g''' + 5rg'' + 3g') = 0.$$

If $rg'' + 3g' = 0$, g must have the expression $g(r) = g_0 + c/r^2$; then we obtain from (23) $h(\theta) = -c + c_1/\cos^2(\theta - \theta_0)$. The potential

$$V(r, \theta) = g_0 + \frac{c_1}{r^2 \cos^2(\theta - \theta_0)}$$

will produce both the family of circles and a family of straight lines. It can be seen that, apart the constants g_0 and c_1 , V is obtained as the -2 power of the potential (19), so it follows also from Theorem 1 that this potential produces families of straight lines.

If $rg'' + 3g' \neq 0$, we must have

$$-\frac{h'''}{h'} = \frac{2(r^2g''' + 5rg'' + 3g')}{rg'' + 3g'} = \text{constant}. \quad (25)$$

If the common constant value is 0, equations (25) give $g = g_0 + g_1/r^2 + g_2 \ln r$ and $h = h_0 + h_1\theta + h_2\theta^2$, but these solutions do not satisfy equation (24). If the

common constant value in (25) is k^2 , we must have $g = g_0 + g_1/r^2 + g_2 r^{k^2/2}$ and $h = h_0 + h_1 \cos(k\theta) + h_2 \sin(k\theta)$, while for $-k^2$ we obtain the solutions $g = g_0 + g_1/r^2 + g_2 r^{-k^2/2}$ and $h = h_0 + h_1 \exp(k\theta) + h_2 \exp(-k\theta)$. These solutions fail to satisfy equation (23), so there are not other potentials producing both a family of circles and a FSL, except central potentials and those having the form $V(r, \theta) = g_0 + c_1/[r^2 \cos^2(\theta - \theta_0)]$.

We mention that we have considered only real potentials. There are complex potentials as

$$V = \frac{1}{r^2} \exp(2i\theta)$$

producing both circles and the family of "parallel" straight lines with $\gamma = -i$, i.e. the family given by $f(x, y) = x - iy = c$.

Comment. This family of complex lines has the property of being an isotach family of orbits. Isotach orbits have been considered by Szebehely (1963), Nahon (1964), and – within the framework of the inverse problem – by Bozis and Borghero (1998); along these orbits the kinetic energy of the moving unit mass is constant.

It is known the fact that a potential producing isotach orbits satisfies the equation $\gamma V_x - V_y = 0$, so such a nonconstant potential will satisfy also equation (11) if and only if $\gamma^2 + 1 = 0$. If we consider only real orbits, there is no nonconstant potential producing an isotach family of straight lines. But if we allow for the orbits and potentials to be complex, as it was already done by Ramani et al. (1982), Hietarinta (1984), Bozis and Grigoriadou (1993), Contopoulos and Bozis (2000), for $\gamma = \pm i$ we obtain the potential $V = v(x \pm iy)$.

The classes of solutions for equation (12), in Cartesian or polar coordinates, show that this equation has an interest in itself.

5. FAMILY BOUNDARY CURVES FOR POTENTIALS CREATING FSL

As it was settled in Remark 3, a nonconstant potential satisfying equation (12) and depending on both variables x and y will provide a unique γ given from (11) by

$$\gamma = -\frac{V_x}{V_y},$$

which obviously satisfies the condition (10).

The families

$$f(x, y) = c \quad (26)$$

of straight lines are found as solutions of the differential equation

$$\frac{dy}{dx} = -\frac{1}{\gamma} = \frac{V_y}{V_x}. \quad (27)$$

The total energy distribution may be preassigned to the family. Let it be $E = E(c)$. If $V(x, y)$ is a straight-lines-producing potential, the pertinent family (26) is real in the region, delimited by the *family boundary curves*, where

$$V(x, y) \leq E(f(x, y)). \quad (28)$$

It is possible to *program the motion* in a specific region by choosing a suitable energy function.

Example. Let $E = (1/8)c^2$ be the energy distribution along the straight lines family $f(x, y) = -\frac{x}{y} = c$, produced by the two-dimensional harmonic oscillator potential $V = (x^2 + y^2)/2$.

The allowed region (28) is defined by $(x^2 + y^2)/2 \leq x^2/(8y^2)$, or

$$|y| \leq \sqrt{\frac{\sqrt{x^4 + x^2} - x^2}{2}}.$$

Some straight lines from the given family, traced with the mentioned energy, in the above allowed region are plotted in Fig. 2.

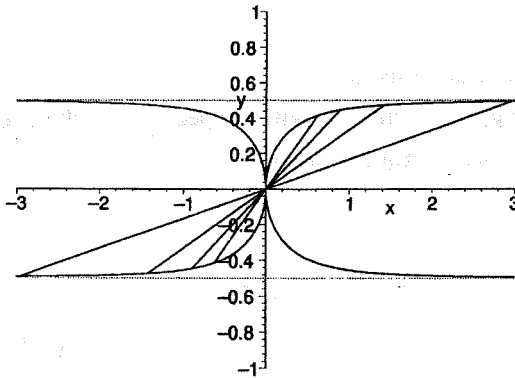


Fig. 2 – Members of the family $-x/y = c$, for $c = -6, -3, -2, -1.5$.

6. FAMILIES OF ORBITS ADELPHIC TO PREASSIGNED FSL

Let us consider a family of straight lines (1) with the slope function γ^* . The potential producing this family will satisfy Szebehely's equation

$$V_x + \gamma^* V_y = 0. \quad (29)$$

The same potential can create, of course, other families of curves. Let us denote by $\gamma \neq \gamma^*$ the slope function of any family coexisting with the specific straight lines family, i.e. *adelphic* to it. For the function γ we shall have $\Gamma = \gamma\gamma_x - \gamma_y \neq 0$. In view of (29), Bozis' second order equation (8) for V and γ reads

$$-V_{xx} + kV_{xy} + V_{yy} = (\mu - \lambda\gamma^*)V_y, \quad (30)$$

equation in which the quantities k , λ and μ are provided by (5)-(7), whereas γ^* satisfies

$$\gamma^* \gamma_x^* - \gamma_y^* = 0. \quad (31)$$

For the sake of completeness, we shall consider also the case $\gamma^* = 0$, when we are in the presence of the straight lines family parallel to the Oy axis $x = c$; equation (29) gives $V = v(y)$ and (30) becomes

$$\frac{v''}{v'} = \mu.$$

Such a potential V will exist if and only if

$$\mu_x = 0, \quad (32)$$

μ being given by (7).

From now on we shall consider $\gamma^* \neq 0$.

By differentiating equation (29) with respect to x , then to y , and bringing together equation (30), we obtain a system in V_{xx} , V_{xy} , V_{yy}

$$\begin{aligned} V_{xx} + \gamma^* V_{xy} &= -\gamma_x^* V_y, \\ V_{xy} + \gamma^* V_{yy} &= -\gamma_y^* V_y, \\ -V_{xx} + kV_{xy} + V_{yy} &= (\mu - \lambda\gamma^*)V_y. \end{aligned} \quad (33)$$

Substituting in the third equation V_{xx} , respectively V_{yy} , calculated from the first and second equation, we obtain

$$(1 - k\gamma^* - \gamma^{*2})V_{xy} = -\gamma^*(\mu - \lambda\gamma^*)V_y. \quad (34)$$

Case 1: $\delta = 1 - k\gamma^* - \gamma^{*2} \neq 0$.

It can be easily seen that δ is the discriminant of the system (33) and the following relations hold

$$\frac{V_{xx}}{V_y} = \frac{\delta_1}{\delta}, \quad \frac{V_{xy}}{V_y} = \frac{\delta_2}{\delta}, \quad \frac{V_{yy}}{V_y} = \frac{\delta_3}{\delta}, \quad (35)$$

in which we have denoted

$$\begin{aligned} \delta_1 &= (\mu - \lambda\gamma^*)\gamma^{*2} - \delta\gamma_x^*, \\ \delta_2 &= -(\mu - \lambda\gamma^*)\gamma^*, \\ \delta_3 &= \mu - \lambda\gamma^* - \delta\gamma_x^*. \end{aligned} \quad (36)$$

It is allowed to divide by V_y , because, by virtue of (29), $V_y = 0$ would imply $V_x = 0$, hence the trivial case $V = \text{constant}$.

The compatibility conditions for (35), in view of (29), are

$$\left(\frac{\delta_1}{\delta\gamma^*} \right)_y = \left(\frac{\delta_2}{\delta\gamma^*} \right)_x \quad \text{and} \quad \left(\frac{\delta_2}{\delta} \right)_y = \left(\frac{\delta_3}{\delta} \right)_x. \quad (37)$$

These two conditions (37) lead to the same equation

$$\left(\frac{\mu - \lambda\gamma^*}{\delta} \right)_x = \left(\frac{\gamma_x^*}{\gamma^*} - \frac{(\mu - \lambda\gamma^*)\gamma^*}{\delta} \right)_y.$$

Substituting k , λ and μ from (5)–(7), after tedious calculations performed with the aid of symbolic algebra (Mathematica), we obtain an equation of the form

$$\begin{aligned} & p_1\gamma_{xxx} + p_2\gamma_{xxy} + p_3\gamma_{xyy} + p_4\gamma_{yyy} + \\ & q_1\gamma_{xx}^2 + q_2\gamma_{xy}^2 + q_3\gamma_y^2 + q_4\gamma_{xx}\gamma_{yy} + q_5\gamma_{xx}\gamma_{xy} + q_6\gamma_{xy}\gamma_{yy} + \\ & r_1\gamma_{xx} + r_2\gamma_{xy} + r_3\gamma_{yy} + s_1\gamma_x^4 + s_2\gamma_x^3\gamma_y + s_3\gamma_x^2\gamma_y^2 + s_4\gamma_x\gamma_y^3 + s_5\gamma_y^4 + \\ & t_1\gamma_x^3 + t_2\gamma_x^2\gamma_y + t_3\gamma_x\gamma_y^2 + t_4\gamma_y^3 + u_1\gamma_x^2 + u_2\gamma_x\gamma_y + u_3\gamma_y^2 = 0, \end{aligned} \quad (38)$$

where the coefficients p , q , r , s , t , u depend on γ , γ_x , γ_y and on γ^* and its derivatives up to the second order.

If we consider a function γ satisfying (38), from (35) we obtain V_y (up to a multiplicative constant) and then, from (29), V_x . The potential determined in this way will satisfy both equations (29) and (30), so the two families γ^* (straight lines) and γ will coexist.

Case 2: $\delta = 0$.

Factoring δ shows that the condition is equivalent with $\gamma = -1/\gamma^*$. From (34) it follows that, by necessity, $\mu - \lambda\gamma^* = 0$. Substituting in this $\gamma = -1/\gamma^*$, we obtain

$$\gamma_{xx}^*(1 + \gamma^{*2})^2 = 0.$$

The condition $\gamma_{xx}^* = 0$, in view of (31), gives a special FSL (a pencil of lines through the point $(-c_1, -c_2)$)

$$\gamma^* = -\frac{x + c_1}{y + c_2}. \quad (39)$$

From (11) we obtain that the potential compatible with this family must be of the form

$$V = v((x^2 + y^2)/2 + c_1x + c_2y). \quad (40)$$

It is easily seen that this potential is compatible with the family of circles centered at $(-c_1, -c_2)$

$$\gamma = -\frac{y + c_2}{x + c_1}. \quad (41)$$

besides the FSL (39).

To summarize, a potential producing a given family of straight lines γ^* will be compatible with the family $\gamma = -1/\gamma^*$ if and only if γ^* satisfies (39) and the potential is of the form (40); for a given family of straight lines γ^* , a potential compatible with $\gamma \neq -1/\gamma^*$ can be found from (35) if the function γ satisfies the condition (38).

Examples. Let us consider the family $\gamma^* = -x/y$ (produced by the potential $V = v(x^2 + y^2)$). The function $\gamma = -1/\gamma^*$ (representing a family of

circles centered in the origin of the axes) is of the form (39) with $c_1 = 0 = c_2$, hence it is compatible with the central potential V (case 2). If $\gamma \neq -1/\gamma^*$ is compatible with the central potential V , it has to verify the condition (38). Such a function is $\gamma = y/(x + \sqrt{x^2 + y^2})$, which produces the family of parabolas $r \cos^2(\theta/2) = c$ (Case 1).

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