

ON THE CURVATURE OF THE TRAJECTORIES IN THE RESTRICTED PROBLEM OF THREE BODIES

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Abstract. It is proved that for the restricted problem of three bodies there are at most three osculating solutions at one point. Points where there are three or two osculating solutions are specified.

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We consider the motion of a body of negligible mass in the gravitational field due to two bodies, named primaries, of masses $1 - \mu$ and μ revolving in circular orbits around their common center of mass. The body whose motion is studied does not influence the motion of the primaries and it moves in the plane determined by their orbits.

In the synodic coordinate system, the dimensionless form of the equations governing the motion of the body of negligible mass in the restricted problem of three bodies is :

$$\begin{cases} \ddot{x} = 2\dot{y} + F_x \\ \ddot{y} = -2\dot{x} + F_y \end{cases} \quad (1)$$

where

$$F(x, y) = (x^2 + y^2)/2 + (1 - \mu)/r_1 + \mu/r_2, \quad (2)$$

$$r_1 = [(x + \mu)^2 + y^2]^{1/2}, \quad r_2 = [(x + \mu - 1)^2 + y^2]^{1/2}. \quad (3)$$

By F_x, F_y we denote the partial derivatives of F and in fact we have

$$F_x = x - (1 - \mu)(x + \mu)/r_1^3 - \mu(x + \mu - 1)/r_2^3$$

$$F_y = y(1 - (1 - \mu)/r_1^3 - \mu/r_2^3)$$

Many results on the system (1) are to be found in the book of V. Szebehely (1967). In a paper of A. Breazna (1989), some considerations on the number of osculating trajectories at one point are made and it is said that this number is at most two. In this note we show that there are points through which three such trajectories are passing.

Let the point $(X, Y) \in \mathbb{R}^2 \setminus \{(-\mu, 0), (1-\mu, 0)\}$ be given such that at this point $v \neq 0$ (where $v = (\dot{x}^2 + \dot{y}^2)^{1/2}$ is the modulus of the velocity at (X, Y)). Two or more solutions passing through (X, Y) are called *osculating solutions* if their velocities at (X, Y) are collinear and their curvature at the point (X, Y) is the same.

The curvature k at (X, Y) is given by

$$k = (\dot{x}\ddot{y} - \dot{y}\ddot{x})/v^3. \quad (4)$$

Let $\mathbf{v} = (\dot{X}, \dot{Y})$ such that $v \neq 0$. We obtain from (4)

$$k = -2/v + (\dot{X}F_y(X, Y) - \dot{Y}F_x(X, Y))/v^3, \text{ where } v = (\dot{X}^2 + \dot{Y}^2)^{1/2}.$$

We now look for $\lambda \in \mathbb{R} \setminus \{0\}$ such that $k = k(\lambda)$, where $k(\lambda)$ is the value given by (4) for the initial conditions (X, Y) and $\lambda(\dot{X}, \dot{Y})$. The relation $k = k(\lambda)$ gives the following equation:

$$\begin{aligned} -2/v + (\dot{X}F_y - \dot{Y}F_x)/v^3 &= \\ &= [-2/v + (\dot{X}F_y - \dot{Y}F_x)/(\lambda v^3)]/|\lambda| \end{aligned} \quad (5)$$

Let us denote

$$(\dot{X}F_y - \dot{Y}F_x)/v^3 = a; \quad -2/v = b < 0. \quad (6)$$

The equation (5) has then the form:

$$(a + b) |\lambda| \cdot \lambda - b\lambda - a = 0. \quad (7)$$

If $a + b = 0$, the equation (7) gives $\lambda = 1$. In the following we suppose $a + b \neq 0$.

Case 1°. Let us consider $\lambda > 0$. The equation (7) has two positive distinct solutions:

$$\lambda_1 = 1, \quad \lambda_2 = -a/(a + b) \quad (8)$$

if $a \in (0, -b/2) \cup (-b/2, -b)$, and only one positive solution, namely $\lambda = 1$, if $a \in (-\infty, 0] \cup \{-b/2\} \cup (-b, \infty)$.

Case 2°. Let $\lambda < 0$, The equation (7) becomes

$$(a + b) \lambda^2 + b\lambda + a = 0.$$

and has $\Delta \geq 0$ if and only if $|b - 2a| \geq 2\sqrt{2} |a|$.

Denote $b_1 = b(\sqrt{2} - 1)/2 < 0$.

The equation has two negative distinct solutions if $a \in (b_1, 0)$, and only one negative solution if $a \in [0, -b)$ or $a = b_1$.

The conclusion is that the equation (7) has :

- three distinct solutions if $a \in (b_1, -b) \setminus \{0, -b/2\}$;
- two distinct solutions if $a \in \{b_1, 0, -b/2\}$;
- only the solution $\lambda = 1$ if $a \in (-\infty, b_1) \cup [-b, +\infty)$.

We obtain the following

THEOREM. Let the initial conditions $x(0) = X, y(0) = Y, \dot{x}(0) = \dot{X}$ and $\dot{y}(0) = \dot{Y}$ be given for the system (1) ($(X, Y) \in \mathbb{R}^2 \setminus \{(-\mu, 0), (1 - \mu, 0)\}, \dot{X}^2 + \dot{Y}^2 \neq 0$) and denote

$$a = (\dot{X}F_y(X, Y) - \dot{Y}F_x(X, Y))/(\dot{X}^2 + \dot{Y}^2)^{3/2}$$

$$b = -2/(\dot{X}^2 + \dot{Y}^2)^{1/2}.$$

If $a \in (b(\sqrt{2} - 1)/2, -b) \setminus \{0, -b/2\}$, then there are three distinct osculating solutions through (X, Y) .

If $a \in \{b(\sqrt{2} - 1)/2, 0, -b/2\}$, then there are two distinct osculating solutions through (X, Y) .

If $a \in (-\infty, b_1) \cup [-b, +\infty)$, then for the solution given by the above initial conditions there are no osculating solutions through (X, Y) . ■

We show now that the three mentioned possibilities occur indeed.

Let $X = 1/2 - \mu, \dot{X} = 1, \dot{Y} = 0$. The value $Y = y \in \mathbb{R}$ is not fixed at this moment.

We have $a = y [1 - (1/4 + y^2)^{-3/2}]$ and $b = -2$.

The continuous function

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(y) = y[1 - (1/4 + y^2)^{-3/2}]$$

satisfies the conditions $f(-\infty) = -\infty, f(+\infty) = \infty$, hence it is surjective. Then for each value $a \in \mathbb{R}$, there is $y \in \mathbb{R}$ such that $f(y) = a$.

Applying the theorem we obtain :

1°. Let $y \in \mathbb{R}$ be given such that $a \in (1 - \sqrt{2}, 0)$. Then the solutions of (1) given by the initial conditions

$$(1/2 - \mu, 1, 0), \quad (1/2 - \mu, y, \lambda_1, 0), \quad (1/2 - \mu, y, \lambda_2, 0),$$

with λ_1, λ_2 the solutions of $(a - 2)\lambda^2 - 2\lambda + a = 0$, are osculating at $(1/2 - \mu, y)$.

2°. Let $y \in \mathbb{R}$ be given such that $a \in (0, 2) \setminus \{1\}$. Then the solutions of (1) given by the initial conditions

$$(1/2 - \mu, y, 1, 0), \quad (1/2 - \mu, y, a/(2 - a), 0),$$

$$\left(1/2 - \mu, y, \frac{-1 - \sqrt{1 + 2a - a^2}}{2 - a}, 0\right)$$

are osculating at $(1/2 - \mu, y)$.

3°. Let $y \in \mathbb{R}$ be given such that $a = 1 - \sqrt{2}$, respectively $a = 0$ or $a = 1$. Then the solutions of (1) given by the initial conditions

$$(1/2 - \mu, y, 1, 0), (1/2 - \mu, y, 1 - \sqrt{2}, 0)$$

respectively

$$(1/2 - \mu, y, 1, 0), (1/2 - \mu, y, -1, 0)$$

or

$$(1/2 - \mu, y, 1, 0), (1/2 - \mu, y, -1 - \sqrt{2}, 0)$$

are osculating at $(1/2 - \mu, y)$.

For $y \in \mathbb{R}$ such that $a \in (-\infty, 1 - \sqrt{2}) \cup [2, +\infty)$ there is no osculating solution to that corresponding to the initial conditions $(1/2 - \mu, y, 1, 0)$.

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