ROMANIAN ACADEMY

ROMANIAN ASTRONOMICAL JOURNAL

Vol. 5, No. 2
1995

EDITURA ACADEMIEI ROMÂNE

ROMANIAN ACADEMY

ROMANIAN ASTRONOMICAL JOURNAL

The ROMANIAN ASTRONOMICAL JOURNAL appears twice a year. Orders from abroad (issues or subscriptions) should be sent to: **RODIPET S.A.** Piaţa Presei Libere nr. 1, P. O. Box 33–57, Fax: 401-222 6407, Telephone: 401-618 5103, 401-222 4126, Bucureşti, România; **ORION PRESS INTERNATIONAL, SRL**, Şos. Olteniţei, 35–37, Sect. 4, P.O. Box 61–170 Bucharest 4, Fax: 401-312 2425, 401-634 7145, Telephone: 401-634 6345, Bucureşti, România; **AMCO PRESS SRL**, Bd. Nicolae Grigorescu 29 A, ap. 66, sect. 3, P.O. Box 57–88, Fax: 401-312 5109, Telephone: 401-643 9390, 401-312 5109, Bucureşti, România.

The manuscripts, the books and journals proposed in exchange and the mail should be sent to the Editorial Board.

Editorial Board's Address:

ROMANIAN ASTRONOMICAL JOURNAL
Institutul Astronomic,
Str. Cuţitul de Argint 5,
Cod 75212, Bucureşti 28,
Fax: (401) 337 33 89,
Telephone: (401) 335 68 92,
ROMÂNIA

EDITURA ACADEMIEI ROMÂNE

Calea 13 Septembrie 13, sector 5, Bucureşti, tel. 410.38.46

ROMANIAN ASTRONOMICAL JOURNAL

Vol. 5, No. 2, 1995

CONTENTS

ISSN 1210–5168

Rom. Astron. J., Vol. 5, No. 2, p. 101–195, Bucharest, 1995

Lei 2000

ON THE INVERSE PROBLEM OF DYNAMICS FOR GENERALIZED LAGRANGIAN SYSTEMS

MIRA-CRISTIANA ANISIU

Mathematical Institute
P. O. Box 68, 3400 Cluj, Romania

Abstract. A system of partial differential equations for the force function U (hence for the potential function $-U$) is given when one knows a family of orbits for the system

$$\frac{d}{dt} L_{\dot{q}} = L_q$$

where $L : D' \times D \subseteq \mathbb{R}^{2n} \to \mathbb{R}$,

$$L(\dot{q}, q) = \left(\sum_{k=1}^{n} \sum_{j=1}^{n} g_{jk}(q) \dot{q}_j \dot{q}_k \right) / 2 + \sum_{j=1}^{n} f_j(q) \dot{q}_j + U(q);$$

this is a generalization not only of the results of Szebehely for 2-dimensional conservative systems but also of those of Drâmbă for the planar circular restricted three-body problem.

Key words: generalized Lagrangian systems, potential function, inverse problem.

The inverse problem of Dynamics requires the determination of the potential (or force) function when an orbit or a family of orbits is known. In a natural way, the topic was considered for 2-dimensional conservative systems with $V : D \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$\begin{aligned} \ddot{x} &= V_x \\ \ddot{y} &= V_y \end{aligned} \tag{1}$$

by Szebehely (1974). A similar problem was studied by Drâmbă (1963) for the planar circular restricted three-body problem with $\Omega : D \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x \\ \ddot{y} + 2\dot{x} &= \Omega_y. \end{aligned} \tag{2}$$

Recently, the result of Szebehely was extended by Stavinschi and Mioc (1993) for n-dimensional conservative systems with $W : D \subseteq \mathbb{R}^n \to \mathbb{R}$,

$$\ddot{x}_i = W_{x_i}, \quad i = 1, \ldots, n. \tag{3}$$

The roots of the problem are to be found before the beginning of our century, in the paper of Dainelli (1880), described by Whittaker (1904) in his well-known

Rom. Astron. J., Vol. 5, No. 2, p. 149–155, Bucharest, 1995

book *Analytical Dynamics,* for 2-dimensional systems of the type

$$\ddot{x} = F \qquad\qquad (4)$$
$$\ddot{y} = G,$$

where $F,\ G : D \subseteq \mathbb{R}^2 \to \mathbb{R}$.

The result of Szebehely (1974) follows from that of Stavinschi and Mioc (1993) for the particular value $n = 2$. The force function in these cases is a function of x, y, respectively of x_i, $i = 1,\dots, n$, so it does not depend on velocities; it means that the system (2) which contains the terms $-2\dot{y}$ and $2\dot{x}$ due to the Coriolis forces is of another kind, so Drâmbă's result cannot be obtained as a special case of (1) or (3).

In the following we consider a generalized Lagrangian system

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{\mathrm{q}}} = L_{\mathrm{q}}, \qquad (5)$$

where $L : D' \times D \subseteq \mathbb{R}^{2n} \to \mathbb{R}$, $\mathrm{q} = \begin{pmatrix} q_1 \\ \dots \\ q_n \end{pmatrix} \in D$, $\dot{\mathrm{q}} = \begin{pmatrix} \dot{q}_1 \\ \dots \\ \dot{q}_n \end{pmatrix} \in D'$ is given by

$$L(\dot{\mathrm{q}}, \mathrm{q}) = \langle \dot{\mathrm{q}}, \mathrm{G}(\mathrm{q})\dot{\mathrm{q}} \rangle/2 + \langle \dot{\mathrm{q}}, \mathrm{f}(\mathrm{q}) \rangle + U\mathrm{q}. \qquad (6)$$

By $\langle .\ ,\ . \rangle$ we denote the inner product in $\mathbb{R}^n$, $\mathrm{f} = \begin{pmatrix} f_1 \\ \dots \\ f_n \end{pmatrix} : D \to \mathbb{R}^n$ and $\mathrm{G} =$ $= (g_{ij})_1^n : D \to \mathcal{M}_n^{\mathrm{sym}}(\mathbb{R})$, $U, f_i, g_{ij} = g_{ji} \in C^1(D)$, $i = 1,\dots, n$, $j = 1,\dots, n$.

The first term in the expression of L is a "kinetic energy"

$$T(\dot{\mathrm{q}}, \mathrm{q}) = \langle \dot{\mathrm{q}}, \mathrm{G}(\mathrm{q})\dot{\mathrm{q}} \rangle/2,$$

while $V(\dot{\mathrm{q}}, \mathrm{q}) = \langle \dot{\mathrm{q}}, \mathrm{f}(\mathrm{q}) \rangle + U(\mathrm{q})$ represents a "generalized force function", the first part of which contains the effect of the Coriolis forces.

LEMMA 1. *The system* (5) *admits an energy integral*

$$T(\dot{\mathrm{q}}, \mathrm{q}) - U(\mathrm{q}) = h, \qquad (7)$$

i.e. $\langle \dot{\mathrm{q}}, \mathrm{G}(\mathrm{q})\dot{\mathrm{q}} \rangle/2 - U(\mathrm{q}) = h$.

Proof. Multiplying the i^{th} equation in (5) by $\dot{q}_i$ and adding we have

$$\sum_{i=1}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{q}_i} \right) \dot{q}_i = \sum_{i=1}^{n} L_{q_i} \dot{q}_i .$$

Hence

$$\frac{\mathrm{d}}{dt}\left(-L+\sum_{i=1}^{n} L_{\dot q_i}\dot q_i\right) = -\sum_{i=1}^{n} L_{q_i}\dot q_i - \sum_{i=1}^{n} L_{\dot q_i}\ddot q_i + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} L_{\dot q_i}\dot q_i =$$

$$= -\sum_{i=1}^{n}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}L_{\dot q_i}\right)\dot q_i + L_{\dot q_i}\ddot q_i\right] + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} L_{\dot q_i}\dot q_i = 0$$

and $-L+\sum_{i=1}^{n} L_{\dot q_i}\dot q_i = h.$

But

$$L_{\dot q_i} = \sum_{i=1}^{n} g_{ik}(\mathbf{q})\dot q_k + f_i,\ i=1,\dots,n,\ \text{so}\ \sum_{i=1}^{n} L_{\dot q_i}\dot q_i = 2T + \langle \mathbf{f},\ \dot{\mathbf{q}}\rangle$$

and the above relation gives $T(\dot{\mathbf{q}}, \mathbf{q}) - U(\mathbf{q}) = h.$ ■

The system (5) has an equivalent explicit form. In order to obtain it, we define, following Wintner (1941), p. 118, the functions

$$P_{ik},\ \Gamma_{ijk},\ \Gamma^i_{jk},\ P^i_k : D \to \mathbb{R}$$

of the position $\mathbf{q}$ in the configuration space, in terms of the coefficients g_{ik}, f_i in (6) and of their first order partial derivatives:

$$P_{ik} = f_{iq_k} - f_{kq_i}$$

$$\Gamma_{ijk} = (g_{ikq_j} + g_{jkq_i} - g_{ijq_k})/2$$

$$\Gamma^i_{jk} = \sum_{l=1}^{n} g^{il}\Gamma_{jkl}$$

$$P^i_k = \sum_{l=1}^{n} g^{il}P_{lk}\ ,$$

where $(g^{il})^n_1 = \mathbf{G}^{-1}$ is the inverse of the matrix $\mathbf{G}$.

Let us denote

$$\Gamma^i = (\Gamma^i_{jk})^n_1,\ \ \mathbf{P}^i = \begin{pmatrix} P^i_1 \\ \vdots \\ P^i_n \end{pmatrix},\ \ \mathbf{g} = \begin{pmatrix} g^{i1} \\ \vdots \\ g^{in} \end{pmatrix},\quad i=1,\dots,n.$$

LEMMA 2. *The system* (5) *with L given by* (6) *where* $\mathbf{G}$ *is a symmetric matrix with* $\det \mathbf{G} \neq 0$ *has the normal form*

$$\ddot q_i = -\langle \dot{\mathbf{q}}, \Gamma^i(\mathbf{q})\dot{\mathbf{q}}\rangle - \langle \mathbf{P}^i(\mathbf{q}), \dot{\mathbf{q}}\rangle + \langle \mathbf{g}^i(\mathbf{q}), U_{\mathbf{q}}(\mathbf{q})\rangle,\quad i=1,\dots,n. \tag{8}$$

Proof. The system (5) is equivalent to

$$\sum_{k=1}^{n} \ddot{q}_k L_{\dot{q}_i\dot{q}_k} + \sum_{k=1}^{n} \dot{q}_k L_{q_i q_k} - L_{q_i} = 0, \quad i = 1,\dots, n$$

which is in fact

$$\sum_{k=1}^{n} g_{ik}\ddot{q}_k + \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma_{jki}\, \dot{q}_j\dot{q}_k + \sum_{k=1}^{n} P_{ik}\dot{q}_k - U_{q_i} = 0, \quad i = 1,\dots, n.$$

The matrix **G** having the inverse $\mathbf{G}^{-1}$, we have

$$\ddot{q}_i = -\sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma^i_{jk}(\mathbf{q})\dot{q}_j\dot{q}_k - \sum_{k=1}^{n} P^i_k(\mathbf{q})\dot{q}_k + \sum_{k=1}^{n} g^{ik}(\mathbf{q})\, U_{q_k}(\mathbf{q}),$$

which is exactly (8). ■

Let there be given the family of curves

$$F_p(\mathbf{q}) = c_p, \quad p = 1,\dots, n-1 \tag{9}$$

where $\mathbf{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^{n-1}$ is twice differentiable on the open set D and at least one $(n-1) \times (n-1)$ minor of $\mathbf{F_q} = (F_{pq_j})$, $p = 1,\dots, n-1$, $j = 1,\dots, n$ is nonzero. We suppose that the curves of the family (9) are orbits for the system (5).

Differentiating in (9) we obtain the linear system

$$\langle \mathbf{F_q}, \dot{\mathbf{q}} \rangle = 0 \tag{10}$$

which has the solution

$$\dot{\mathbf{q}} = a(\mathbf{q})\mathbf{L}(\mathbf{q}), \tag{11}$$

where $\mathbf{L} = \begin{pmatrix} L_1 \\ \dots \\ L_n \end{pmatrix}$ contains the determinants

$$L_i = (-1)^{i+1} \begin{vmatrix} F_{1q_1} \dots F_{1q_{i-1}} & \dots & F_{1q_{i+1}} \dots F_{1q_n} \\ F_{n-1,q_1} \dots F_{n-1,q_{i-1}} & & F_{n-1,q_{i+1}} \dots F_{n-1,q_n} \end{vmatrix}. \tag{12}$$

LEMMA 3. *Let* **G** *be a symmetric and positive definite matrix; then the system* (10) *to which the energy integral is joined has a unique (up to a sign) solution* $\dot{\mathbf{q}}$.

Proof. Introducing $\dot{\mathbf{q}} = a(\mathbf{q})\mathbf{L}(\mathbf{q})$ in (7) one has

$$a^2(\mathbf{q})\langle \mathbf{L}(\mathbf{q}), \mathbf{G}(\mathbf{q})\mathbf{L}(\mathbf{q})\rangle/2 = h + U(\mathbf{q}),$$

hence we can consider

$$a(\mathbf{q}) = \pm\, [2(h + U(\mathbf{q}))/\langle \mathbf{L}(\mathbf{q}), \mathbf{G}(\mathbf{q})\mathbf{L}(\mathbf{q})\rangle]^{1/2}. \ ■ \tag{13}$$

Now we can state the main theorem.

THEOREM. *Let us suppose that the curves of the family* (9) *are orbits for the system* (5) *with* **G** *a symmetric and positive definite matrix, so that* $U, f_i, g_{ij} \in C^1(D)$, $i = 1,\ldots, n$, $j = 1,\ldots, n$. *Then the function U satisfies the following system of partial differential equations:*

$$a^2\left\langle \mathbf{L},\ \left(F_{p\mathbf{qq}} - \sum_{i=1}^{n} F_{pq_i}\Gamma^i\right)\mathbf{L}\right\rangle - a\sum_{i=1}^{n} F_{pq_i}\langle \mathbf{P}^i,\ \mathbf{L}\rangle + \sum_{i=1}^{n} F_{pq_i}\langle \mathbf{g}^i,\ U_{\mathbf{q}}\rangle = 0,\quad p = 1,\ldots, n-1 \tag{14}$$

where a is given by (13), **L** by (12) *and* Γ^i, $\mathbf{P}^i$, $\mathbf{g}^i$ *depend only on the coefficients* g_{ij}, f_i *in* (6), *all the functions involved having the argument* **q**.

Proof. Let us differentiate in (10). We have

$$\langle \dot{\mathbf{q}},\ \mathbf{F}_{p\mathbf{qq}}\dot{\mathbf{q}}\rangle + \langle \mathbf{F}_{p\mathbf{q}},\ \ddot{\mathbf{q}}\rangle = 0,\quad p = 1,\ldots, n-1$$

where $\mathbf{F}_{p\mathbf{qq}} = (F_{pq_jq_k})_1^n$ is the Hessian of F_p.

Replacing $\ddot{\mathbf{q}}$ by the expression obtained in (8), we have

$$\langle \dot{\mathbf{q}},\ F_{p\mathbf{qq}}\dot{\mathbf{q}}\rangle - \sum_{i=1}^{n} F_{pq_i}[\langle \dot{\mathbf{q}},\ \Gamma^i\dot{\mathbf{q}}\rangle + \langle \mathbf{P}^i,\ \dot{\mathbf{q}}\rangle - \langle \mathbf{g}^i,\ U_{\mathbf{q}}\rangle] = 0,$$

$$\left\langle \dot{\mathbf{q}},\ \left(F_{p\mathbf{qq}} - \sum_{i=1}^{n} F_{pq_i}\Gamma^i\right)\dot{\mathbf{q}}\right\rangle - \sum_{i=1}^{n} F_{pq_i}\langle \mathbf{P}^i,\ \dot{\mathbf{q}}\rangle + \sum_{i=1}^{n} F_{pq_i}\langle \mathbf{g}^i,\ U_{\mathbf{q}}\rangle = 0.$$

Now $\dot{\mathbf{q}}$ is given by (11) and (12) and we get the relations (14). ■

Remark. The functions U and its partial derivatives appear in the function a given by (13) and in the last sum in (14).

The system (3) is a Lagrangian system with **L** given by (6) for $\mathbf{G}(\mathbf{q}) = I_n$, $\mathbf{f} = 0$ and $U = W$. In this case $\Gamma^i = 0$, $\mathbf{P}^i = 0$, $i = 1,\ldots, n$ and (14) becomes

$$a^2\langle \mathbf{L},\ F_{p\mathbf{qq}}\mathbf{L}\rangle + \sum_{i=1}^{n} F_{pq_i}U_{q_i} = 0,$$

where $a = \pm 2\Omega / \sum_{i=1}^{n} L_i^2$ with $\Omega = W + h$, and $U_{q_i} = \Omega_{q_i}$. Here we have $\mathbf{q} = x$.

We have obtained

COROLLARY 1. (Stavinschi, Mioc). *Let us suppose that a particle has a trajectory given by* (9). *If its motion is governed by the system* (3), *the function W satisfies*

$$\left(2(W+h)/\sum_{i=1}^{n} L_i^2\right)\sum_{i=1}^{n}\left(\sum_{k=1}^{n} F_{px_ix_k}L_iL_k + F_{pq_i}W_{x_i}\right) = 0,$$

h being the energy constant and L_i given by (12). ■

The system (2) has $n = 2$, $\mathbf{G} = I_2, f_1(x, y) = -y, f_2 = x$ and the energy integral

$$(\dot{x}^2 + \dot{y}^2)/2 = \Omega(x, y) + h.$$

Consider the system

$$\begin{aligned} \ddot{x} + \lambda(x, y)\dot{y} &= \Omega_x \\ \ddot{y} + \lambda(x, y)\dot{x} &= \Omega_y \end{aligned} \tag{15}$$

where $\lambda(x, y) = u_y(x, y) - v_x(x, y)$, which is a slight generalization of (2) considered by Birkhoff (1917).

This is a Lagrangian system whith $n = 2$, $\mathbf{G} = I_2$, $(f_1, f_2) = (u, v)$ which has the same energy integral as (2). The system being 2-dimensional, the equation of the orbit is given by

$$F(x, y) = c. \tag{16}$$

Hence $L_1 = F_y$, $L_2 = -F_x$ and $a = \pm\, [2(h + \Omega)/(F_x^2 + F_y^2)]^{1/2}$.

We have $\Gamma_{ijk} = 0$, $\Gamma^i_{jk} = 0$ for every $i, j, k \in \{1, 2\}$ and $P^i_k = P_{ik} = f_{iq_k} - f_{kq_i}$, so $P^i_k = 0$ for $i = k$, and $P^1_2 = -P^2_1 = \lambda$.

In this case (14) contains one equality and this is

$$a^2(F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2)\, \mu\, a\lambda(F_x^2 + F_y^2) + F_x\Omega_x + F_y\Omega_y = 0,$$

the argument of all the functions being (x, y).

COROLLARY 2. *Let us suppose that a particle has a trajectory given by* (16) *in the 2-dimensional space. If its motion is governed by system* (15), *with* $\lambda = u_y - v_x$, *then the force function* Ω *satisfies*

$$\begin{aligned} &[2(h + \Omega)/(F_x^2 + F_y^2)](F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2) \mp \\ &\mp [2(h + \Omega)]^{1/2}\lambda(F_x^2 + F_y^2)^{1/2} + F_x\Omega_x + F_y\Omega_y = 0, \end{aligned}$$

h being the energy constant. ■

The planar circular restricted three-body problem (2) is a special case of (15) with $u(x, y) = -y$, $v(x, y) = x$, so $\lambda(x, y) = -2$.

From Corollary 2 we obtain

COROLLARY 3 (Drâmbă). *Let us suppose that a particle has a trajectory in 2-dimensional space given by* (16). *If the motion is governed by the system* (2), *then the function* Ω *satisfies*

$$\begin{aligned} &[2(h + \Omega)/(F_x^2 + F_y^2)](F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2) \pm \\ &\pm 2[2(h + \Omega)]^{1/2}(F_x^2 + F_y^2)^{1/2} + F_x\Omega_x + F_y\Omega_y = 0, \end{aligned} \tag{17}$$

h being the energy constant. ■

The system (1) is obtained from (15) with $\lambda(x, y) \equiv 0$, or from (3) with $n = 2$, hence Corollaries 1 and 2 give

COROLLARY 4 (Szebehely). *Let us suppose that a particle has a trajectory in the 2-dimensional space given by* (16). *If the motion is governed by the system* (1), *then the function V satisfies*

$$[2(h + V)/(F_x^2 + F_y^2)](F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2) + F_xV_x + F_yV_y = 0, \qquad (18)$$

h being the energy constant. ■

In fact, the relation (18) appears in Whittaker (1902a), and (17) in Whittaker (1902b), where they are deduced as a consequence of a variational principle of Maupertuis' type.

Acknowledgements. The author expresses her thanks to Prof. A Pal and Dr. V. Mioc for valuable suggestions and useful discussions, and to Prof. G. Bozis for drawing her attention to the paper of Borghero and Melis (1990) and other ones which contain similar generalizations of two-dimensional inverse problem.

REFERENCES

Birkhoff, G. D.: 1917, *Trans. Amer. Math. Soc.*, **18**, 199.
Borghero, F., Melis, A.: 1990, *Celest. Mech. Dyn. Astron.*, **49**, 273.
Dainelli, U.: 1880, *Giorn. Mat.*, **18**, 271.
Drâmbă, C.: 1963, *St. Cerc. Astron.*, **8**, 7.
Stavinschi, M., Mioc, V.: 1993, *Astron. Nachr.*, **314**, 91.
Szebehely, V.: 1974, in: E. Proverbio (ed.): *Proceedings of the International Meeting on Earth's Rotation by Satellite Observations* Cagliari, 16–18 April 1973, Bologna, 31.
Whittaker, E. T.: 1902a, *Mon. Not. Roy. Astron. Soc.*, **62**, 186.
Whittaker, E. T.: 1902b, *Mon. Not. Roy. Astron. Soc.*, **62**, 346.
Whittaker, E. T.: 1904, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, London and New York.
Wintner, A.: 1941, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, Princeton, New Jersey.

Received on 5 April, 1995