ROMANIAN ACADEMY

ROMANIAN ASTRONOMICAL JOURNAL

Vol. 5, No. 1

1995

EDITURA ACADEMIEI ROMÂNE

ROMANIAN ASTRONOMICAL JOURNAL

Vol. 5, No. 1, 1995

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ISSN 1220–5168

Rom. Astron. J., Vol. 5, No. 1, p. 1-100, Bucharest, 1995

Quasar ProImpex Ltd.
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Lei 2000

ON THE PHOTOGRAVITATIONAL MODEL OF C. POPOVICI

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Abstract. For the system governed by a central force depending on $r = \left(x^2 + y^2\right)^{1/2}$ and $\dot{r}$ considered by C. Popovici

$$\ddot{x} = -\left(k + 1\dot{r}\right) r^{-3} x$$
$$\ddot{y} = -\left(k + 1\dot{r}\right) r^{-3} y$$

one studies the existence and linear stability of the equilibrium points.

Key words: Radiation Pressure – Equilibrium Points

The influence of the solar radiation pressure on small particles (from the comet tails, for example) was considered long time ago by Johannes Kepler (*Harmonices mundi*, 1619). Outstanding scientists like Newton, Huygens, Coulomb, Maxwell dedicated parts of their work to this study. After the experiments of P.N. Lebedev during 1899-1907, which permitted to measure the force of radiative pressure, K.E. Tsiolkovski in 1921 and F.A. Tsander in 1924 imagined cosmic spacecrafts with large mirrors being driven by the incoming solar radiation. H. Oberth described in 1923 a cosmic reflecting mirror and in 1954 he considered the propulsion by means of light pressure. Since the 1950s there appeared studies on trajectories of solar sails, followed by projects of construction and precise calculations in order to realize a rendezvous mission to comet Halley. Unfortunately, no solar sail has been constructed, but one may hope that sailing using the pressure of sunlight will soon be possible. In fact, light pressure was already used for the orientation and stabilization of Mariner 10, launched in 1973, when it approached the planet Mercury. Since 1964, the spacecrafts Mariner 3 and Mariner 4 had four solar plates disposed in X-shape with sails at the ends. Historical surveys of the efforts in understanding and using the effects of solar radiation pressure were made by E.N. Polyakhova (1986) and C. McInnes (1991).

In 1923, C. Popovici considered a modified Newton-Coulomb law which is applicable for a "radial attractive or repulsive (either gravitational, electrical or light repulsion)" force. Later on, in 1940, he described explicitly the problem of a body moving in the field of gravitational attraction and luminous repulsion of a star. These pioneering ideas of C. Popovici are analysed by A. Pal (1992) and M.-C. Anisiu (1993). The luminous attraction was also studied by G. Armellini (1937) and J. Chazy (1939), while V. Nadolschi and I. Plăcințeanu (1940) applied the law of Popovici-Armellini to the motion of electrons around the atomic nucleus.

The law introduced by C. Popovici is obtained considering the light acting according to a generalized inverse square law.

The Newtonian attraction force is given by

$$F_1 = -Ar^{-2}, \tag{1}$$

A being the attraction of the luminous body (Sun) at the unit of distance. The force due to the light of the central body is

$$F_2 = Rr^{-2}, \tag{2}$$

where R is the light repulsion at the unit of distance. The force of luminous attraction will be given by

$$F = -kr^{-2}, \tag{3}$$

where $k = A - R$. This force will be attractive for bodies on which the luminous effect has a small influence (for example, the planets), in which case it is of Newtonian type with a "reduced" constant, but it will decrease becoming a repulsive force for a solar sail or a particle in a comet's tail.

Forces as (3) are considered in many recent papers referring to the problem of two bodies or to the restricted three body problem.

The modification operated by C. Popovici on the law (3) consists in adding the term $-R\dot{r}\left(cr^2\right)^{-1}$, representing the force due to the finite light speed. Here, c denotes the light speed, R the repulsion of the light at the unit of distance, and $\dot{r}$ the component of the speed of the attracted body on the radius vector. The force considered by C. Popovici is then

$$\varphi = -Ar^{-2} + Rr^{-2} - R\dot{r}\left(cr^2\right)^{-1}, \tag{4}$$

or, in a more condensed form, given also by C. Popovici

$$\varphi = -k\left(1 + \varepsilon\dot{r}\right)r^{-2}, \tag{5}$$

where $k = A - R$, and $\varepsilon = Rc^{-1}(A-R)^{-1}$, $A - R \neq 0$.. The disadvantage of this formulation is that for $k = 0$ it is no more equivalent with (4). For this reason we shall consider

$$\varphi = -(k + 1\dot{r})r^{-2}, \tag{5'}$$

where $k = A - R$ and $1 = Rc^{-1} > 0$. If $k \neq 0$, the form (5) is obtained from (5') putting $\varepsilon = 1k^{-1}$.

The motion of a body attracted by a luminous source by a modified Newton law (5') will be governed by the system

$$\begin{aligned} \ddot{x} &= -(k + 1\dot{r})r^{-3}x \\ \ddot{y} &= -(k + 1\dot{r})r^{-3}y, \end{aligned} \tag{6}$$

where $r = \left(x^2 + y^2\right)^{1/2}$.

The system is more complex than in the usual Newtonian case, but C. Popovici showed that the trajectory still has a simple form

$$r^{-1} = p^{-1} + \eta_0 p^{-1} e^{-\alpha\theta} \sin\left(\left(1 - \alpha^2\right)^{1/2} \theta - \omega\right), \tag{7}$$

where $p = C^2 k^{-1}$, $\alpha = \varepsilon C(2p)^{-1} = \varepsilon k(2C)^{-1}$, C being the angular momentum constant.

More precisely, the result is the following

THEOREM 1. *The system (6) has the plane solution given by*

$$r^{-1} = q + u_0,$$

where $q = kC^{-2}$, $\alpha = 1(2C)^{-1}$ *and*

$$u_0 = \begin{cases} e^{-\alpha\theta}\left(C_1 e^{\left(\alpha^2-1\right)^{1/2}\theta} + C_2 e^{-\left(\alpha^2-1\right)^{1/2}\theta}\right), & \text{if } \alpha > 1 \\ (C_1 + C_2\theta)e^{-\theta}, & \text{if } \alpha = 1 \\ e^{-\alpha\theta}\left(C_1 \sin\left(1-\alpha^2\right)^{1/2}\theta + C_2 \cos\left(1-\alpha^2\right)^{1/2}\theta\right), & \text{if } 0 < \alpha < 1, \end{cases}$$

$$\dot{\theta} = Cr^{-2},$$

in the case $C \neq 0$; for $C = 0$ *the attracted body moves on a straight line passing through the attractive point, the distance between the bodies being given by*

$$\ddot{r} = -(k + 1\dot{r})r^{-2}. \tag{8}$$

Proof. In polar coordinates the system (6) becomes

$$\ddot{r}-r\dot{\theta}^{2}=-(k+1\dot{r})r^{-2} \qquad (9)$$
$$2\dot{r}\dot{\theta}+r\ddot{\theta}=0\,.$$

The second equation gives $\dot{\theta}=Cr^{-2}$, C being the angular momentum constant. Introducing the value of $\dot{\theta}$ in the first equation we obtain

$$\ddot{r}=-(k+1\dot{r})r^{-2}+C^{2}r^{-3}\,. \qquad (10)$$

In the case $C\neq 0$, considering r as a function of θ, we have

$$\dot{r}=Cr'r^{-2},\ddot{r}=C^{2}\left(r''r-2r'^{2}\right)r^{-5},$$

where $\dot{r}$ denotes the derivative with respect to t, and r' the derivative with respect to θ. The equation (10) becomes

$$C^{2}r''r-2C^{2}r'^{2}-C^{2}r^{2}=-\left(kr^{3}+1Crr'\right).$$

Substituting $r=u^{-1}$ we obtain the linear equation with constant coefficients

$$u''+2\alpha u'+u=q, \qquad (11)$$

where $q=kC^{-1}$, $\alpha=1(2C)^{-1}$.

The general solution of (11) gives for r^{-1} the expression from the theorem, θ being then determined as a function of t from the equation $\dot{\theta}=Cr^{-2}$.

In the case $C=0$, $\theta=$ const and the motion will take place on a line having as origin the attractor body. The position of the attracted body will be determined on that line by equation (8), which is a consequence of (9) with $\dot{\theta}=0$.

In the case considered by C. Popovici, $k\neq 0$, $\varepsilon=1k^{-1}$ and $\alpha<1$. Then, Theorem 1 gives

$$r^{-1}=q+e^{-\alpha\theta}\left(C_1\sin\left(1-\alpha^{2}\right)^{1/2}\theta+C_2\cos\left(1-\alpha^{2}\right)^{1/2}\theta\right)$$

and the expression given by C. Popovici is obtained considering $q=p^{-1}$ and $C_1=\eta_0 p^{-1}\cos\omega$, $C_2=-\eta_0 p^{-1}\sin\omega$.

We study now the existence of the equilibrium points for the equations (8) and (10).

In the case of the motion on a line, which corresponds to the equation (8), there are equilibrium points if and only if $k=0$; in this situation (gravitation and repulsive forces being balanced), any $r_0>0$ is an equilibrium point.

Let now $C\neq 0$, the trajectory being a plane curve. The equations in (9) decouple and we shall study the equilibrium points for the equation (10). We obtain

$$-kr_0^{-2}+C^{2}r_0^{-3}=0,$$

and $r_0 = C^2 k^{-1} = q^{-1}$ for $k > 0$; for $k \le 0$ there are no equilibrium points. These considerations lead us to

THEOREM 2. *In the case of the motion on a line* ($C = 0$), *the equation* (8) *admits an infinity of equilibrium points* $r_0 > 0$ *for* $k = 0$, *and no such point for* $k \ne 0$; *in the case of the planar motion* $(C \ne 0)$ *there is a unique equilibrium point of the equation* (10)

$$r_0 = C^2 k^{-1} = q^{-1} \tag{12}$$

for $k > 0$ *and none for* $k \le 0$.

So, in the case when the light pressure R exceeds the gravitational attraction A, there are no equilibrium points; when the two forces are equal, there is an infinity of equilibrium points for the motion on a straight line through the attractive body; when the light pressure is smaller than the gravitational attraction, there is a unique equilibrium point for the equation (10).

Regarding the stability of the equilibrium points, the following result holds:

THEOREM 3. *In the case of the motion on a line* ($C = 0$), *each equilibrium point of the equation* (8) *with* $k = 0$ *is linear stable; in the case of the planar motion* $(C \ne 0)$, *the unique equilibrium point* r_0 *given by* (12) *for the equation* (10) *with* $k > 0$ *is linear stable.*

Proof. The characteristic equation for (8) with $k = 0$ is

$$\lambda\left(\lambda + 1 r_0^{-2}\right) = 0$$

with $\lambda_1 = 0$, $\lambda_2 = -1 r_0^{-2} < 0$, so each equilibrium point r_0 is linear stable.

For equation (10) with $k > 0$ and r_0 given by (12), the characteristic equation is

$$\lambda^2 + 1 C^{-4} k^2 \lambda + C^{-6} k^4 = 0 .$$

If the solutions are real, they are both negative; if they are complex, their real part is $-1 C^{-4} k^2 / 2 < 0$, so the equilibrium point is linear stable (in fact, in this case it is also stable).

If we consider now a sail in the gravitational field of the Sun, the constants which appear in Popovici's law will have the following values:

$A = G(M_0 + m) \cong G M_0$, where $G = 6.67 \cdot 10^{-11}\ N \cdot m \cdot kg^{-1}$ is the universal gravitation constant and $M_0 = 2 \cdot 10^{30}$ kg is the mass of Sun; $R = L_0 / (2\pi c \sigma)$, where $L_0 = 3.86 \cdot 10^{26}$ W is the solar luminosity, $c = 3 \cdot 10^8\ \mathrm{m \cdot s^{-1}}$ the light velocity and $\sigma = m / a$ the sail loading (the ratio of its mass and area).

So, to have a "levitated" sail, i.e. an equilibrium point for equation (8), the condition $k = 0$ is fulfilled by a sail having $\sigma = L_0 (2\pi c G M_0)^{-1} = 1.53 \cdot 10^{-3}\ \mathrm{kg \cdot m^{-2}}$. If we want the sail to have a circular orbit at a distance r_0 given by (12), the sail must have the loading such that $\sigma > 1.53 \cdot 10^{-3}\ \mathrm{kg \cdot m^{-2}}$. For instance, a sail in a heliostationary orbit following the 25-days solar equatorial rotation at $r_0 = 0.02$ AU must have $k = 13.16 \cdot 10^{19}$, hence $\sigma = 0.114\ \mathrm{kg \cdot m^{-2}}$.

Acknowledgement. The author wishes to express her gratitude to Prof. A. Pal, University of Cluj-Napoca, and to Dr. T. Oproiu, Astronomical Observatory of Cluj-Napoca for their valuable suggestions on this topic.

REFERENCES

Anisiu, M.C. : 1993, *Proc. Nat. Conf. Astron. Dedicated to the Anniversary of 80 Years since the Foundation of the Astronomic Observatory of Iași*, 24-28 Sept. (to appear).

Armellini, G.: 1937, *Rend. Accad.Lincei*, **26**, 209.

Chazy, J.: 1939, *C.R. Acad. Sci. Paris*, **209**, 133.

McInnes, C.R.: 1991, *Vistas Astron.*, **34**, 369.

Nadolschi, V., Plăcințeanu, I.: 1940, *Ann. Scient. Univ. Jassy*, **26**, 397.

Pal, A.: 1992, *Rom. Astron. J.*, **2**, 205.

Polyakhova, Y.N.: 1986, *Space Flight Using a Solar Sail: the Problems and the Prospects, edited by V.A. Yegorov, Ser: Space Flight Mechanics* 19, Nauka, Moscow (in Russian).

Popovici, C.: 1923, *Bull. Astron., Mémoires et variétés* **3**, 257.

Popovici, C.: 1939, *C.R. Acad. Sci.* Paris, **209**, 2052.

Popovici, C.: 1940, *C.R. Acad. Sci.* Paris, **210**, 39.

Received August 30, 1994