Seminar of Functional Analysis and
Numerical Methods, Preprint Nr. 1, 1984, pp. 159 - 168.

STRONGLY PROXIMINAL SETS IN ABSTRACT SPACES

by

IOAN ŞERB

Let X be a Hausdorff locally convex space with the topology generated by a family $(p_\alpha)_{\alpha \in A}$ of semi-norms on X and let M be a non-void subset of X. By the <u>metric projection</u> on M with respect to the family $(p_\alpha)_{\alpha \in A}$ we understand the application $P_M : X \longrightarrow 2^M$ defined by

$$(1) \quad P_M(x) = \{m \in M : p_\alpha(x - m) = d_\alpha(x,M) = \inf_{m' \in M} p_\alpha(x - m'), \forall \alpha \in A\}.$$

(see [6]). If X is a normed space then P_M is the usual metric projection on M. In this case $d(x,M) = \inf\{\|x - m'\| : m' \in M\}$. If (X, ρ) is a metric space and M is a non-void subset of X the metric projection $P_M : X \longrightarrow 2^M$ is defined similarly by

$$(2) \qquad P_M(x) = \{m \in M : \rho(x,m) = \inf_{m' \in M} \rho(x,m')\}.$$

In the definitions given below X denotes a locally convex or a metric space, M is a subset of X and P_M is the corresponding metric projection, given by (1) or (2), respectively.

If for every $x \in X \setminus M$, card $P_M(x) \geq 2$, $2 \leq$ card $P_M(x) < \aleph_0$ or card $P_M(x) = \aleph_0$, then the metric projection P_M is called <u>totally multivalued, finitely multivalued or countably multivalued</u>, respectively (see [11]).

The set M is called <u>proximinal</u> if $P_M(x) \neq \emptyset$ for all $x \in X$.

If P_M is totally multivalued then the set M is called <u>strongly proximinal</u>. It is well known that every proximinal set is closed so that every strongly proximinal set is cosed too.

In [11] we proved that if M is a strongly proximinal set of a Banach space X, then $\operatorname{card} P_M(x) \geq c$ for every $x \in X \setminus M$, where c denotes the power of continuum, and that the completeness of X is essential for the validity of this result.

Suppose now that (X, ρ) is a metric linear space, where ρ is a translation invariant metric generating the topology of X. We gave in [11] an example of a finitely dimensional linear metric space (X, ρ) containing a subset M with finitely multivalued metric projection. The unit ball of this space isn't convex.

In the following we study the cardinality of $P_M(x)$ for a strongly proximinal subset M of a complete metrizable locally convex space (a Fréchet space).

Let $(p_n)_{n \in N}$ be a family of semi-norms generating the topology of the Fréchet space X. In the following theorem P_M denotes the metric projection with respect to the family $(p_n)_{n \in N}$.

THEOREM 1. *If M is a strongly proximinal subset of a Fréchet space X, then*

(3) $$\operatorname{card} P_M(x) \geq c$$

for all $x \in X \setminus M$.

Proof. Let M be a strongly proximinal subset of X and let $x \in X \setminus M$. Let $B_n(x,r) = \{ y \in X : p_n(x - y) \leq r \}$ and let $d_n = d_n(x,M) = \inf \{ p_n(x - m) : m \in M \}$, for $n = 1, 2, \ldots$. As a strongly proximinal subset of X the set M is closed. It follows that

$$P_M(x) = M \cap \bigcap_{n=1}^{\infty} B_n(x, d_n) ,$$

is also closed. We shall show that $P_M(x)$ has no isolated points. This will imply that $P_M(x)$ is a perfect subset of the complete metric space X, so that by Theorem 6.65 [5] , $\operatorname{card} P_M(x) \geq c$.

Let's now show that $P_M(x)$ has no isolated points. Suppose on the contrary that $P_M(x)$ contains an isolated point m_0. It follows that there exist a finite subset $\{n_1, n_2, \ldots, n_k\}$ of N and a number $\delta > 0$ such that

(4) $$p_{n_i}(m_0 - m) > \delta$$

for all $m \in P_M(x) \setminus \{m_0\}$ and $i = 1, 2, \ldots, k$. We shall show first that $d_{n_i}(x,M) > 0$ for $i = 1, 2, \ldots, k$. Suppose that $d_{n_i}(x,M) = 0$ for an element i of $\{1, 2, \ldots, k\}$. Then $p_{n_i}(x - m) = 0$, i.e. $x - m \in p_{n_i}^{-1}(0)$ for all $m \in P_M(x)$. Since M is strongly proximinal, there exists $m_1 \in P_M(x) \setminus \{m_0\}$. It follows that

$$m_0 - m_1 = (x - m_1) - (x - m_0) \in p_{n_i}^{-1}(0),$$

in contradiction with (4).

Therefore $d_{n_i}(x,M) > 0$ for $i = 1, 2, \ldots, k$. Let $r = \max \{ d_{n_i} : i = 1, 2, \ldots, k \}$ and $\varepsilon = \delta / r$. The relations (4) become:

(5) $$p_{n_i}(m_0 - m) > \delta = \varepsilon r \geq \varepsilon d_{n_i}(x,M),$$

for all $m \in P_M(x) \setminus \{m_0\}$ and $i = 1, 2, \ldots, k$. Choose $\delta > 0$ such that $\varepsilon < 1$. For $x_0 = (\varepsilon/3)\, x + (1 - (\varepsilon/3))\, m_0$ we have

$$p_n(x - x_0) = p_n(x - (\varepsilon/3)\, x - (1 - (\varepsilon/3))\, m_0) =$$

$$= (1 - (\varepsilon/3))\, p_n(x - m_0) = (1 - (\varepsilon/3))\, d_n(x,M) ,$$

for all $n \in N$. Since $d_{n_i}(x,M) > 0$ for $i = 1, 2, \ldots, k$ it follows that $x_o \in X \setminus M$. On the other hand

$$p_n(x_o - m_o) = p_n((\mathcal{E}/3)\, x + (1 - (\mathcal{E}/3))\, m_o - m_o) =$$

$$= (\mathcal{E}/3)\, p_n(x - m_o) = (\mathcal{E}/3)\, d_n(x,M)\ ,$$

for all $n \in N$, so that $d_n(x_o,M) \leq (\mathcal{E}/3)\, d_n(x,M)$. Let now $m \in M$. If $m \notin P_M(x)$ then there exists $n \in N$ such that

$$p_n(x_o - m) \geq p_n(x - m) - p_n(x_o - x) > d_n(x,M) - p_n(x_o - x) =$$

$$= d_n(x,M) - (1 - (\mathcal{E}/3))\, d_n(x,M) = (\mathcal{E}/3)\, d_n(x,M),$$

which implies that $m \notin P_M(x_o)$. If $m \in P_M(x) \setminus \{m_o\}$ then, by (5)

$$p_{n_i}(x_o - m) \geq p_{n_i}(m_o - m) - p_{n_i}(m_o - x_o) > \mathcal{E}\, d_{n_i}(x,M) -$$

$$- (\mathcal{E}/3)\, d_{n_i}(x,M) = (2\,\mathcal{E}/3)\, d_{n_i}(x,M),$$

for $i = 1,2,\ldots,k$, so that $m \notin P_M(x_o)$.

Therefore, $P_M(x_o) = \{m_o\}$ which contradicts the hypothesis that M is strongly proximinal. The proof is complete.

Let now X be a real Banach space and $U(X)$ its closed unit ball. Let X^* be the dual space of X and $U(X^*)$ its closed unit ball.

A point x_o of a subset M of X is called k - exposed if there exists a linear independent subset $\{x_1^*, x_2^*, \ldots, x_k^*\}$ of $U(X^*)$ such that

$$
(6)\qquad
\begin{aligned}
& x_1^*(x_o) \geq x_1^*(x), \text{ for all } x \in M,\\
& x_2^*(x_o) \geq x_2^*(x), \text{ for all } x \in M \cap H_1\ ,\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
& x_{k-1}^*(x_o) \geq x_{k-1}^*(x) \text{ for all } x \in M \cap H_1 \cap \ldots \cap H_{k-2}\ ,\\
& x_k^*(x_o) > x_k^*(x), \text{ for all } x \in M \cap H_1 \cap \ldots \cap H_{k-1} \setminus \{x_o\}\ ,
\end{aligned}
$$

where H_i is the hyperplane given by the equation $x_i^*(x) = x_i^*(x_o)$, $i = 1,2,\ldots,k-1$, [11] . For $k = 1$ one obtains the usual notion of an exposed point ([3]). Every k-exposed point of M is also n-exposed for all $n \geq k$.

We proved in [11] that if X is a normed space which contains a strongly proximinal compact subset then $U(X)$ has no k-exposed points, $k \in N$, arbitrary. By using this result we will exhibit a large class of spaces which contain no strongly proximinal compact sets. In particular, all strictly convex spaces belong to this class.

PROPOSITION 2. If X is a Banach space such that $U(X)$ admits of at least a support hyperplane H with $H \cap U(X)$ compact, then X contains no strongly proximinal compact sets.

Proof. Recall, that a closed hyperplane H is called a support hyperplane of a set M if $H \cap M \neq \emptyset$ and M is contained in a closed semispace determined by H.

Suppose that H is a support hyperplane of the unit ball $U(X)$ such that $U(X) \cap H$ is compact. Since the set $H \cap U(X)$ is also convex, by the Krein-Milman theorem, it has an extreme point. Moreover the set $H \cap U(X)$ contains an exposed point x_o see [3] Corollary 3.5.7 .

Suppose that H is given by the equation $x^*(x) = x^*(x_o)$.

Let x_1^{**} be an exposing functional for x_o, i.e.

$$(7)\qquad x_1^*(x) < x_1^*(x_o),$$

for all $x \in H \cap U(X) \setminus \{x_o\}$.

Without loosing the generality we can suppose that

$$(8)\qquad x^*(x) \leq x^*(x_o),$$

for all $x \in U(X)$. The relations (7) and (8) show that x_o is a

Ω-exposed point of $U(X)$. By [11], it follows that X contains no strongly proximinal compact sets.

A Banach space X is called of (CR) - type if for every closed hyperplane $H \subset X$ and every $x \in X$ the set $P_H(x)$ is compact (eventually void). The strictly convex and the finite dimensional spaces are of (CR)-type.

The above proposition implies

COROLLARY 3. <u>If</u> X <u>is a Banach space of</u> (CR)-<u>type then</u> X <u>contains no strongly proximinal compact sets</u>.

In the following we strengthen this result, by showing that the spaces of (CR)-type contain no strongly proximinal sets with the Radon-Nikodým property.

A subset K of the Banach space X has the <u>Radon-Nikodým property</u> whenever given a probability measure space (Ω, Σ, μ) and a countably additive μ-continuous $F : \Sigma \longrightarrow X$ for which $\{F(E)/\mu(E) : E \in \Sigma,\ \mu(E) \neq 0\} \subset K$, there is a Bochner integrable $f : \Omega \longrightarrow X$ such that

$$F(E) = \int_E f(\omega)\, d\mu(\omega),$$

for each $E \in \Sigma$, [4].

Let K be a closed bounded convex subset of a Banach space X and suppose that $x_0 \in K$. The point $x_0 \in K$ is a strongly exposed point of K if there is an $x^* \in X^*$ which exposes x such that $\{S(K, x^*, \alpha) : \alpha > 0\}$ is a neighborhood base for x in K in the norm topology, where :

$$S(K, x^*, \alpha) = \{x \in K : x^*(x) > \sup_{y \in K} (x^*(y) - \alpha) = x^*(x_0) - \alpha\}.$$

Then x^* <u>strongly exposes</u> x. We will denote the set of strongly exposed points of K by $\mathrm{str.exp}\,(K)$, and the set of strongly exposing functionals by $SE(K)$.

In the sequel we have need of the following result :

THEOREM A. (R.R. Phelps [9], J. Bourgain [2]). <u>Let</u> K <u>be a closed bounded convex subset of</u> X <u>and assume that</u> K <u>has the Radon-Nikodým property. Then</u> $K = \overline{co}\,(\mathrm{str.exp}\,(K))$. <u>Moreover</u>, $SE(K)$ <u>is a dense</u> G_δ <u>subset of</u> X.

In particular, every compact and convex subset of a Banach space has the Radon-Nikodým property.

THEOREM 4. <u>Let</u> X <u>be a Banach space of</u> (CR)-<u>type</u>. <u>Then</u> X <u>contains no bounded convex strongly proximinal sets with the Radon-Nikodým property</u>.

<u>Proof</u>. Suppose that X contains a bounded convex strongly proximinal set M with the Radon-Nikodým property. Let $x' \in X \setminus M$. The set $P_M(x') = B(x', d(x', M)) \cap M$ is closed bounded convex and contains at least two points. (Here $B(x,r)$ denotes the closed ball of radius r with center x). Since the intersection of $P_M(x')$ with the interior of $B(x', d(x', M))$ is empty, we can apply the Hahn-Banach theorem to find a closed hyperplane H supporting $B(x', d(x', M))$ and separating the sets M and $B(x', d(x', M))$. It follows that $P_M(x') \subset H$. Suppose that H is given by the equation

$$x_1^*(x) = x_1^*(m_0), \tag{9}$$

where m_0 is a fixed element in $P_M(x')$.

We can suppose that $x_1^*(x) \geq x_1^*(m_0)$ for all $x \in B(x', d(x', M))$ and $x_1^*(m) \leq x_1^*(m_0)$ for all $m \in M$. Since X is of (CR)-type it follows that the convex set $S_1 = H \cap B(x', d(x', M)) = H \cap B(x', d(x', H)) = P_H(x')$ is compact. The set $S_2 = H \cap M$ is a closed bounded convex subset of M. Since M has the Radon-Nikodým property, the set S_2 will have too this property. Then by Theorem A there exists an exposing functional $x_2^* \in X^*$ and an element $x_2 \in S_2$ such that

(10) $\qquad x_2^*(x) > x_2^*(x_2)$, for all $x \in S_2 \setminus \{x_2\}$.

By the compactness of the set S_1 there exists an element $x_1 \in S_1$ such that

(11) $\qquad x_2^*(x) \le x_2^*(x_1)$, for all $x \in S_1$.

Define now $x'' \in X \setminus M$ by

(12) $$x'' = x' + x_2 - x_1.$$

Without restricting the generality we can suppose that $\|x_1^*\| = \|x_2^*\| = 1$. The relation $x_1^*(x) \ge x_1^*(m_o)$, for all $x \in B(x',d(x',M))$ implies $x_1^*(x' - m_o) \ge x_1^*(x' - x)$ for all $x \in B(x',d(x',M))$. Therefore

$$x_1^*(x' - m_o)/d(x',M) = x_1^*(x' - m_o)/\|x' - m_o\| =$$

$$= \sup_{x \in B(x',d(x',M))} \{x_1^*(x' - x)/\|x' - m_o\|\} = \|x_1^*\| = 1.$$

so that $x_1^*(x' - m_o) = d(x',M) = \|x' - m_o\|$.

Then

$$\|x'' - m\| \ge |x_1^*(x' + x_2 - x_1 - m)| = |x_1^*(x' - m) + x_1^*(x_2) -$$

$$- x_1^*(x_1)| = |x_1^*(x' - m) + x_1^*(m_o) - x_1^*(m_o)| = |x_1^*(x' - m_o) +$$

$$+ x_1^*(m_o - m)| = \|x' - m_o\| + x_1^*(m_o - m) \ge \|x' - m_o\| = d(x',M),$$

for all $m \in M$, which implies $d(x'',M) \ge d(x',M) > 0$ and $x'' \in X \setminus M$. Since $x_2 \in M$ and $\|x'' - x_2\| = \|x' - x_1\| = d(x',M)$, it follows that $x_2 \in P_M(x'')$ and $d(x'',M) = d(x',M)$.

Let's prove now that $P_M(x'') = \{x_2\}$. Let $m' \in P_M(x'')$, $m' \ne x_2$. Since $x_1 \in S_1 = H \cap B(x',d(x',M))$ we have

$$x_1^*(x'' - x_2) = x_1^*(x' + x_2 - x_1 - x_2) = x_1^*(x' - x_1) =$$

$$= \|x' - x_1\| = d(x',M) = \|x'' - x_2\|.$$

which shows that the hyperplane H supports the ball $B(x'',d(x'',M))$ at the point $x_2 \in M$. Since H is also a supporting hyperplane for M it results that $x_1^*(m - x_2) \le 0$, for all $m \in M$, so that $H \cap B(x'',d(x'',M)) \cap M = P_M(x'') \subset S_2$.

Since $m' \in P_M(x'') \setminus \{x_2\} \subset S_2 \setminus \{x_2\}$, taking into account the relation (10), one obtains :

(13) $$x_2^*(m') > x_2^*(m_2).$$

If $H_o = H - x_1$ is the maximal subspace of X paralel to H, then :

(14) $$H \cap B(x'',d(x'',M)) = P_H(x'') = P_H(x' + x_2 - x_1) =$$

$$= P_{H_o}(x' + x_2) = P_{H_o}(x' + x_2 - x_1 + x_1) =$$

$$= P_{H_o}(x' + x_1) + x_2 - x_1 = P_H(x') + x_2 - x_1 =$$

$$= S_1 + x_2 - x_1.$$

By (11) and (14) one obtains

$$m' = m_1 + x_2 - x_1$$

where $m_1 \in S_1$ and

(15) $$x_2^*(m') = x_2^*(m_1 + x_2 - x_1) \le x_2^*(x_1 + x_2 - x_1) = x_2^*(x_2).$$

But, then the contradictory relations (13) and (15) imply that $P_M(x'') = \{x_2\}$, and since $x'' \in X \setminus M$, these will imply that the set M is not strongly proximinal. The theorem is proved.

R E F E R E N C E S

[1] Asplund, E., A k-extreme point is the limit of k-exposed points, Israel J. Math. 1,3, 161-162 (1963).

[2] Bourgain, J., On dentability and the Bishop-Phelps property, Israel J. Math. 28, 265-271 (1977).

[3] Bourgin, R.D., Geometric Aspects of Convex Sets with the Radon-Nikodým Property, Lectures Noted in Math., Springer-Verlag Berlin Heidelberg New York Tokyo, 1983.

[4] Diestel, J., Uhl, J.J.Jr., Vector Measures, Math. Surveis 15, Amer. Math. Soc., Providence, R.I., 1977.

[5] Hewitt, E., Stromberg, K., Real and Abstract Analysis, Springer-Verlag Berlin Heidelberg, 1968.

[6] Hirschfeld, R.A., On Approximation Theory, p. 184, vol. ed. by P.L. Butzer and J. Korevaar, Birkhauser Verlag, Basel und Stuttgart, 1964.

[7] Holmes, R.B., Geometric Functional Analysis and its Applications, Springer-Verlag New York Heidelberg Berlin, 1975.

[8] Konjagin, S.V., Approximation properties of arbitrary sets in Banach spaces, (Russian) Dokl. Acad. Nauk. SSSR, 239, 2, 261-264 (1978).

[9] Phelps, R.R., Dentability and extreme points in Banach spaces, J. Functional Analysis (1) 17, 78-90 (1974).

[10] Singer, I., Cea mai bună aproximare în spaţii vectoriale normate prin elemente din subspaţii vectoriale, Bucureşti, 1967.

[11] Şerb, I., On the multi-valued metric projection in normed vector spaces, I, II, Anal. Numer. Teor. Approx. 10, 1 101-111 (1981) & 11, 1-2, 155-166 (1982).