Remarque 8. Les tableaux 2 , 4 , 6 , 8 disent respectivement que

$$\left.\frac{\partial \Phi_1(a,b)}{\partial b}\right|_{b=b_1} = \varphi_1'(b_1) < 0, \text{ pour tout } a \in \left(0, \frac{1}{(1+q)^{1/q}}\right)$$

$$\left.\frac{\partial \Phi_5(a,b)}{\partial b}\right|_{b=b_3} = \varphi_5'(b_3) > 0 \text{ et } \left.\frac{\partial \Phi_7(a,b)}{\partial b}\right|_{b=b_5} = \varphi_7'(b_5) < 0$$

pour tout $a \in \left(0, \frac{1}{(1+p)^{1/p}}\right)$ et $\left.\frac{\partial \Phi_9(a,b)}{\partial b}\right|_{b=b_7} = \varphi_9'(b_7) > 0$, pour tout

$a \in \left(0, \left(\frac{1+p}{1+q}\right)^{\frac{1}{q-p}}\right)$, de sorte que les racines b_1 , b_3 , b_5 , b_7 sont des fonctions continues et à dérivées du premier ordre continues par rapport à a dans les intervalles respectifs d'existence.

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"BABES-BOLYAI" UNIVERSITY, Faculty of Mathematics
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Preprint Nr. 1 , 1985, pp. 143 - 153.

ON STRONGLY PROXIMINAL SETS IN BANACH SPACES

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0. Introduction. We have investigated in the recent papers [12] and [13] the existence of strongly proximinal or other similar sets in normed or other spaces . There were obtained also some characteristics of the unit balls in normed spaces which contain bounded respectively compact strongly proximinal sets.

The aim of the present note is to give new results in this direction. We begin with a result concerning the existence of weakly compact convex strongly proximinal sets in normed spaces. Next one obtains as generalization of a result in [12] , that the unit balls of normed spaces containing weakly compact strongly proximinal sets cannot contain exposed points, k - exposed points or vertices. It is also showed that there exists a wide class of Banach spaces which does not contain closed convex and bounded strongly proximinal sets with the Radon-Nikodým property.

1. Definitions, notations, preliminary results. Let X be a normed vector space and M a non-empty subset of X. The metric projection with respect to the set M is the (generally multivalued) application $P_M : X \longrightarrow 2^M$, defined by :

$$P_M(x) = \{ m \in M \ ; \ \| x - m \| = d(x, M) \},$$

where $d(x, M)$ is the distance of x from the set M.

The set M is called proximinal if $P_M(x) \neq \emptyset$, for all $x \in X$. M is said strongly proximinal if for every $x \in X \setminus M$ we have card $P_M(x) \geq 2$. A normed space contains a strongly proximinal set if and only if X is non-strictly convex. It is known [12], that if X is a Banach space and $M \subset X$ is a strongly proximinal set, then card $P_M(x) \geq c$, for every $x \in X \setminus M$ and that in this proposition the completity of the space is essential. Since the proximinal sets are closed, it follows that so are the strongly proximinal sets.

Let X^* be the topological dual of X. A point $m \in M$ is an extremal point if it is'nt the middle of a segment contained in M. A functional $x^* \in X^* \setminus \{0\}$ is called the support functional of M, if x^* attains its supremum (or its infimum) on M. If x^* attains its supremum (or its infimum) on a single point $m_0 \in M$, then x^* is said exposing functional for M and m_0 is called exposed point. Denote by $\mathcal{S}(M)$ the set of support functionals and by $\mathcal{E}(M)$ the set of exposing functionals of M.

Let K be a closed bounded convex subset of a Banach space X, and suppose that $x_0 \in K$. The point $x_0 \in K$ is a strongly exposed point of K if there is an $x^* \in X^*$ which exposes x such that $\{ S(K, x^*, \alpha) : \alpha > 0 \}$ is a neighborhood base for x in K in the norm topology, where

$$S(K, x^*, \alpha) = \{ x \in K : x^*(x) > \sup_{y \in K} (x^*(y) - \alpha) = \\ = x^*(x_0) - \alpha \}.$$

We say that x^* strongly exposes x. We will denote the set of strongly exposing functionals by $\mathcal{SE}(K)$, and the set of strongly exposed points of K by str. exp (K).

The closed convex bounded subset K of the Banach space X has the Radon-Nikodým property whenever given a probability measure space (Ω, Σ, μ) and a countably additive μ continuous $F : \Sigma \longrightarrow X$ for which $\{ F(E) / \mu(E) : E \in \Sigma, \ \mu(E) \neq 0 \} \subset K$, there is a Bochner integrable $f : \Omega \longrightarrow X$ such that

$$F(E) = \int_E f(\omega) \, d\mu(\omega),$$

for each $E \in \Sigma$, [5]. A Banach space has the Radon-Nikodým property whenever every closed convex and bounded subset of X has this property.

It is well known the following theorem of R. R. Phelps [10] and J. Bourgain [3].

THEOREM. *Let K be a closed bounded convex subset of X and assume that K has the Radon-Nikodým property. Then $K = \overline{\text{co}}\,(\text{str. exp}\,(K))$ and $\mathcal{SE}(K)$ is a dense G_δ subset of X.*

3. Main results. In [8], S. V. Konjagin has given examples of Banach spaces containing bounded, respectively compact strongly proximinal sets. In [12] we proved that the unit ball of a normed space is strongly proximinal if and only if it has no extremal points. It follows, for example, that c_0 and $L_1[0, 1]$ contains strongly proximinal closed bounded convex bodies.

The question is if in a Banach space there exist strongly proximinal compact (weakly compact) convex sets. Modifying an example of S. V. Konjagin it can be proved that the set M of c_0 (with the sup-norm), given by

$$M = \{ x = (x_i)_{i=1}^{\infty} \in c_0 : \ x_i \in [0, 1/i], \ i = 1, 2, \ldots \},$$

is a compact convex set and that M is a strongly proximinal set too. We shall prove in the sequel that there exists a large class of normed spaces which do not contain weakly compact and convex strongly proximinal sets. It will be showed indeed a more general result. Firstly, we will introduce some notations.

For a functional $x^* \in X^*$ and for a given set M we consider the following sets

$$M^{x^*} = \{ m \in M : \; x^*(m) = \sup_{m' \in M} x^*(m') \}$$

$$M_{x^*} = \{ m \in M : \; x^*(m) = \inf_{m' \in M} x^*(m') \} .$$

It is clear that $M^{x^*} \cup M_{x^*}$ is a non-void set if and only if x^* is a support functional for M and M^{x^*} or M_{x^*} consists of a single point if and only if x^* is an exposing functional. Denote by $\mathcal{S}^*(M)$ $(\mathcal{E}^*(M))$ the set of support (exposing) functionals which attain their suppremum on M. An analogous notation $\mathcal{S}_*(M)$ $(\mathcal{E}_*(M))$ is used for the set of functionals which attain their infimum on M. We have :

$$\mathcal{S}^*(M) = - \mathcal{S}_*(M), \qquad \mathcal{S}^*(M) \cup \mathcal{S}_*(M) = \mathcal{S}(M) ,$$

and the analogous relations for $\mathcal{E}(M)$. For $x_1^*, x_2^* \in X^*$ denote by $M^{x_1^* x_2^*}$ the set $(M^{x_1^*})^{x_2^*}$. Analogously for $M_{x_1^* x_2^*}$.

LEMMA 1. <u>Let</u> X <u>be a normed space and</u> M <u>a subset of</u> X. <u>If</u> M <u>is strongly proximinal, then for any functionals</u> $x_1^*, \dots , x_n^* \in X^*$ <u>we have</u> :

$$(1) \qquad \mathcal{E}^*(M^{x_1^* \dots x_n^*}) \cap \mathcal{S}_*(U_{x_1^* \dots x_n^*}) = \emptyset ,$$

U <u>being the closed unit ball of</u> X.

<u>Proof</u>. Without loss of generality we can suppose that $\|x_1^*\| = 1$. Let's now suppose that there exists an $x^* \in \mathcal{E}^*(M^{x_1^* \dots x_n^*}) \cap$ $\cap\, \mathcal{S}_*(U_{x_1^* \dots x_n^*})$. Then there exists $m_0 \in M^{x_1^* \dots x_n^*}$ and $x_0 \in$

$\in U_{x_1^* \dots x_n^*}$ such that :

$$(2) \qquad \begin{aligned} x^*(m_0) &> x^*(m) \quad , \text{ for all } m \in M^{x_1^* \dots x_n^*} , \quad m \neq m_0 , \\ x^*(x_0) &\le x^*(x) \quad , \text{ for all } x \in U_{x_1^* \dots x_n^*} . \end{aligned}$$

Since $m_0 \in M^{x_1^*}$ and $x_0 \in U_{x_1^*}$ we have :

$$(3) \qquad \begin{aligned} x_1^*(m_0) &\ge x_1^*(m) \quad , \text{ for all } m \in M , \\ x_1^*(x_0) &\le x_1^*(x) \quad , \text{ for all } x \in U . \end{aligned}$$

From (3) it follows

$$x_1^*(x_0) = \inf_{x \in U} x_1^*(x) = - \sup_{x \in U} x_1^*(x) = - \|x_1^*\| = -1 , \quad \text{and} \quad \|x_0\| = 1.$$

Let $x' = m_0 - x_0$. We have $\|x' - m_0\| = \|- x_0\| = 1$. Let $m' \in M$ be fixed. Then :

$$(*) \qquad \|x' - m'\| \ge |x_1^*(x' - m')| = |x_1^*(m_0 - x_0 - m')| =$$

$$= |x_1^*(m_0 - m') - x_1^*(x_0)| = x_1^*(m_0 - m') + 1 \ge$$

$$\ge 1 = \|x' - m_0\| .$$

Hence $m_0 \in P_M(x')$. Since $x_0 \in U_{x_1^* x_2^*}$ and $m_0 \in M^{x_1^* x_2^*}$ we have:

$$(4) \qquad \begin{aligned} x_2^*(m_0) &\ge x_2^*(m) \quad , \text{ for all } m \in M^{x_1^*} , \\ x_2^*(x_0) &\le x_2^*(x) \quad , \text{ for all } x \in U_{x_1^*} . \end{aligned}$$

From $\|x' - m'\| \ge 1$ it follows $\|x' - m'\| > 1$ (therefore $m' \notin P_M(x')$) or $\|x' - m'\| = 1$. In this last case from $(*)$ it follows

$$x_1^*(m') = x_1^*(m_0) ,$$

and (3) implies $m' \in M^{x_1^*}$, and hence

$$x_1^*(m' - x') = x_1^*(m_o - x') = x_1^*(x_o).$$

From (3) and the last relation we have : $m' - x' \in U_{x_1^*}$. From (4) and the last relation is obtained :

$$x_2^*(m' - x') \geq x_2^*(x_o) = x_2^*(m_o - x') \quad \text{and}$$

$$x_2^*(m') \geq x_2^*(m_o).$$

Because $m' \in M^{x_1^*}$, we have from (4)

$$x_2^*(m') = x_2^*(m_o) \; ; \quad \text{also from (4)}, \quad m' \in M^{x_1^* x_2^*}.$$

Moreower

$$x_2^*(m' - x') = x_2^*(m_o - x') = x_2^*(x_o),$$

and then

$$m' - x' \in U_{x_1^* x_2^*} \; ; \quad x_3^*(m' - x') \geq x_3^*(x_o) = x_3^*(m_o - x') \; ;$$

$$x_3^*(m') \geq x_3^*(m_o),$$

where from $m' \in M^{x_1^* x_2^*}$ it follows

$$x_3^*(m') = x_3^*(m_o)$$

and so one. We have finally :

$$x_n^*(m') = x_n^*(m_o),$$

$$m' \in M^{x_1^* \dots x_{n-1}^*} \; ; \text{ and } \; m' - x' \in U_{x_1^* \dots x_{n-1}^*}.$$

Since $m_o \in M^{x_1^* \dots x_n^*}$; $x_o \in U_{x_1^* \dots x_n^*}$ we see that

(5) $$x_n^*(m_o) \geq x_n^*(m), \quad \text{for all } m \in M^{x_1^* \dots x_{n-1}^*}$$

$$x_n^*(x_o) \leq x_n^*(x), \quad \text{for all } x \in U_{x_1^* \dots x_{n-1}^*}.$$

From (5), $m' \in M^{x_1^* \dots x_{n-1}^*}$ and $x_n^*(m') = x_n^*(m_o)$ it follows

$$m' \in M^{x_1^* \dots x_n^*} \text{ and } x_n^*(m' - x') = x_n^*(m_o - x') = x_n^*(x_o).$$

From (5) and the last relation we have :

$$m' - x' \in U_{x_1^* \dots x_n^*}.$$

But from (2) and $m' \in M^{x_1^* \dots x_n^*}$ we have $m' = m_o$ or

$$x^*(m_o) > x^*(m').$$

In the last case we have :

$$x^*(m' - x') < x^*(m_o - x') = x^*(x_o),$$

which is in contradiction with the relation $m' - x' \in U_{x_1^* \dots x_n^*}$ and with (2). Therefore we have $\|x' - m'\| > 1$ or $m' = m_o$. Then $P_M(x') = \{m_o\}$. This fact contradicts the strong proximinality of M. The lemma is proved.

<u>Remark</u>. If the relation (1) in the lemma is changed in

(1') $$\varphi^*(M^{x_1^* \dots x_n^*}) \cap \mathcal{E}_*(U_{x_1^* \dots x_n^*}) = \emptyset,$$

then with an analogous proof it can be showed that if M is strongly proximinal, it implies that (1') is verified

Therefore, for every strongly proximinal set $M \subset X$ and for any functional $x_1^*, \dots, x_n^* \in X^*$ we have :

(1") $$[\varphi^*(M^{x_1^* \dots x_n^*}) \cap \mathcal{E}_*(U_{x_1^* \dots x_n^*})] \cup$$

$$\cup [\mathcal{E}^*(M^{x_1^* \dots x_n^*}) \cap \varphi_*(U_{x_1^* \dots x_n^*})] = \emptyset.$$

For $n = 0$ the condition (1") becomes :

$$(1^{o}) \qquad [\varphi^{*}(M) \cap \mathcal{E}_{*}(U)] \cup [\mathcal{E}^{*}(M) \cap \varphi_{*}(U)] = \phi .$$

This condition is verified by every strongly proximinal set, according Lemma 1 in [12] .

THEOREM 2. <u>In a Banach space with the Radon-Nikodým property every strongly proximinal set is unbounded.</u>

<u>Proof</u>. Suppose that there exists the bounded set B which is strongly proximinal. Then by R.R. Phelps' and J. Bourgain's theorem, it follows that $\mathcal{E}^{*}(U)$ is a dense G_{δ} set of X^{*}. Then $\mathcal{E}_{*}(U) = -\mathcal{E}^{*}(U)$ is a dense G_{δ} set too. On the other hand $\varphi^{*}(B)$ is a dense G_{δ} set of X^{*} as a consequence of a theorem of R. Huff and P.D. Morris [7] (see [4] pp. 70). Then $\mathcal{E}_{*}(U) \cap \varphi^{*}(B) \neq \emptyset$ as a consequence of the Baire category theorem. This is a contradiction with the condition (1^{o}).

Theorem 2 improved a result obtained by the author in [12] pp. 165. For finite dimensional Banach spaces this sentence is known, (see [8]).

PROPOSITION 3. <u>If</u> X <u>is a normed space</u>, $M \subset X$ <u>is a strongly proximinal weakly compact set, then</u> U <u>does not contain exposed points, k-exposed points and vertices.</u>

The proof was given in [12] in the case when M is compact. With minor changes the proposition 3 can be proved in an analogous way. The notion of k-exposed point was introduced in [12]. A similar notion was introduced in [1]. A vertex m_0 of M is a point in which the set of support functionals of M is total on X^{*}.

THEOREM 4. <u>Let</u> X <u>be a Banach space and</u> K <u>a closed convex and bounded set with the Radon - Nikodým property. If the set</u> K <u>is strongly proximinal, then for every</u> $n \in N$ <u>and any functionals</u> $x_1^{*}, \ldots, x_n^{*} \in X^{*}$, <u>we have that</u> $\varphi_{*}(U_{x_1^{*} \ldots x_n^{*}})$ <u>is a first category set in</u> X^{*}. <u>Particularly</u>, $\varphi(U) = \varphi_{*}(U)$ <u>is a first category set</u>.

<u>Proof</u>. Suppose that there exists the functionals $x_1^{*}, \ldots, x_n^{*} \in X^{*}$ for which $\varphi_{*}(U_{x_1^{*} \ldots x_n^{*}})$ is a second category set. Since $M^{x_1^{*} \ldots x_n^{*}}$ is a closed bounded and convex subset of M, it follows that $M^{x_1^{*} \ldots x_n^{*}}$ has the Radon - Nikodým property. From the Phelps-Bourgain theorem we have that $\mathcal{E}^{*}(M^{x_1^{*} \ldots x_n^{*}})$ is a dense G_{δ} set of X^{*}. From the Baire category theorem we have:

$$\varphi_{*}(U_{x_1^{*} \ldots x_n^{*}}) \cap \mathcal{E}^{*}(M^{x_1^{*} \ldots x_n^{*}}) \neq \emptyset ,$$

which contradicts the relation (1"). The theorem is proved.

COROLLARY 5. <u>Let</u> X <u>be a Banach space and</u> M <u>a strongly proximinal weakly compact and convex subset of</u> X. <u>Then</u>

$$\varphi_{*}(U_{x_1^{*} \ldots x_n^{*}})$$

<u>is a first category set for every choice of the functionals</u> $x_1^{*}, \ldots, x_n^{*} \in X^{*}$.

Indeed, from the J. Lindenstrauss' [9] and S.L. Troyanski's [14] theorem (see also [4], pp. 60 and [2]) we have that M has the Radon - Nikodým property and the corollary follows.

COROLLARY 6. <u>Let</u> X <u>be a Banach space, admitting a closed bounded convex strongly proximinal set with the Radon - Nikodým property. Then the intersection of a support hyperplane of</u> U <u>with</u> U <u>is not weakly compact</u>.

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