2. D. RIPEANU , Sur les racines des solutions de l'équation $y'' + p_1(x)\, y' + p_2(x)\, y = 0$, Mathematica - Revue d'analyse numérique et de théorie de l'approximation, Tome 19(42), No.1, (1977), pp. 95-111.

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ON THE MODULUS OF CONVEXITY OF L_p SPACES

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The modulus of convexity of ℓ_p or $L_p[0,1]$ was obtained first by O. Hanner [3] (see also [4]). In this paper we are proposing to obtain this modulus of convexity for $L_p(\mu, H)$, H being a Hilbert space, using a measure theoretical lemma. This lemma, inspired from another lemma of T. Figiel [2], gives relations between a Banach space X and the space $L_p(\mu, X)$. The well known inequalities of Clarkson and Beurling are derived from this construction in a natural way.

Denote by (S, Σ, μ) the triplet consisting of a set S, an algebra Σ of subsets of S, and a finitely additive function μ defined on Σ and with values in the extended nonnegative semiaxis. A set $E \in \Sigma$, with $\mu(E) \neq 0$ is said to be an atom for μ if $\mu(F) = 0$ or $\mu(F) = \mu(E)$ for every set $F \subset E$, $F \in \Sigma$. The function μ is said to be purely atomic if every set $E \in \Sigma$ with $\mu(E) \neq 0$ contains an atom for μ , and μ is said to be non-atomic if there are no atoms for μ . In the following, if $E \in \Sigma$ is an atom, then automatically we suppose that $\mu(E) < \infty$. Let

$L_p(S, \Sigma, \mu, X) = L_p(\mu, X)$, $1 < p < \infty$ be the normed space of classes of p integrable functions defined on S and having the values in the Banach space X. As it is well-known the norm on $L_p(S, \Sigma, \mu, X)$ is given by

$$\|f\|_p = \left(\int_S \|f(s)\|^p \mu(ds)\right)^{1/p} .$$

If for instance S is the set of natural numbers, Σ is the family of all subsets of S and μ is the counting measure, then we obtain the well-known space $\ell_p(X)$. The vector space of all classes of μ-simple integrable functions defined on S with values in X will be denoted by $S(X)$.

It is known that $S(X)$ is a dense subspace of $L_p(S, \Sigma, \mu, X)$, $1 < p < \infty$, ([1]) and as a subspace of $L_p(\mu, X)$, $S(X)$ (denoted $S_p(X)$) will be a normed space with the L_p - norm. For the set $E \in \Sigma$, denote by χ_E the characteristic function with respect to E.

In the sequel, for a normed space $(Y, \|\cdot\|)$ and for a fixed $p > 1$ we shall use the following notation :

$$A^p_Y = \left\{ (\|x - y\|^p , \|x + y\|^p) : x, y \in Y, \ \|x\| = \|y\| = 1 \right\} ,$$

$$C^p_Y = \left\{ (\|x - y\|^p , \|x + y\|^p) : x, y \in Y, \ \|x\|^p + \|y\|^p = 2 \right\} .$$

In [2] Lemma 13, T. Figiel has proved that

$$A^p_{\ell_p(X)} = C^p_{\ell_p(X)} = \text{cl conv } C^p_X ,$$

for every Banach space X. Here cl conv C^p_X means the closed convex hull of C^p_X .

In the following we need a variant of this lemma in which we consider instead of $\ell_p(X)$ the dense subspace $S_p(X)$ of $L_p(S, \Sigma, \mu, X)$ with certain restrictions on (S, Σ, μ).

LEMMA 1. Let (S, Σ, μ) be a triplet as above. Suppose there exist at least four pairwise disjoint sets F_1 , F_2 , F_3 , F_4 $\in \Sigma$ such that $0 < \mu(F_i) < \infty$, for $i = 1, 2, 3, 4$. Then:

$$(1) \qquad A^p_{S_p(X)} = C^p_{S_p(X)} = \text{conv } C^p_X .$$

Proof. Firstly, it is clear that if there exists an $F \in \Sigma$ with $0 < \mu(F) < \infty$ and without atoms, then the hypothesis of the lemma is verified. In fact this hypothesis is not verified if μ is purely atomic and S can be written as a reunion of at most 3 atoms.

In order to obtain (1) it is sufficient to prove the following implications:

a) $\text{conv } C^p_X \subseteq A^p_{S_p(X)}$;

b) $C^p_{S_p(X)} \subseteq \text{conv } C^p_X$.

Indeed, from the definitions we have $A^p_Y \subseteq C^p_Y$, for every normed space Y and from this fact and a) and b) :

$$\text{conv } C^p_X \subseteq A^p_{S_p(X)} \subseteq C^p_{S_p(X)} \subseteq \text{conv } C^p_X .$$

a) Let $x_1, y_1, z_1, u_1 \in X$ be such that

$$\|x_1\|^p + \|y_1\|^p = 2$$

$$\|z_1\|^p + \|u_1\|^p = 2 ,$$

and consider the element in conv C_X^p of the form:

$$\lambda(\|x_1-y_1\|^p,\ \|x_1+y_1\|^p) + (1-\lambda)(\|z_1-u_1\|^p,\ \|z_1+u_1\|^p),$$

with $\lambda \in [0, 1]$. Define the elements $\xi, \eta \in S_p(X)$ by:

$$\xi = \left(\frac{\lambda}{2}\right)^{1/p}\mu(F_1)^{-1/p}x_1\chi_{F_1} + \left(\frac{\lambda}{2}\right)^{1/p}\mu(F_2)^{-1/p}y_1\chi_{F_2} +$$
$$+\left(\frac{1-\lambda}{2}\right)^{1/p}\mu(F_3)^{-1/p}z_1\chi_{F_3} + \left(\frac{1-\lambda}{2}\right)^{1/p}\mu(F_4)^{-1/p}u_1\chi_{F_4};$$

$$\eta = \left(\frac{\lambda}{2}\right)^{1/p}\mu(F_1)^{-1/p}y_1\chi_{F_1} + \left(\frac{\lambda}{2}\right)^{1/p}\mu(F_2)^{-1/p}x_1\chi_{F_2} +$$
$$+\left(\frac{1-\lambda}{2}\right)^{1/p}\mu(F_3)^{-1/p}u_1\chi_{F_3} + \left(\frac{1-\lambda}{2}\right)^{1/p}\mu(F_4)^{-1/p}z_1\chi_{F_4},$$

where F_1, F_2, F_3, F_4 are the sets considered in the statement of the lemma. Then:

$$\|\xi\|_p^p = \int_{F_1}\frac{\lambda}{2}\mu(F_1)^{-1}\|x_1\|^p\mu(ds) + \int_{F_2}\frac{\lambda}{2}\mu(F_2)^{-1}\|y_1\|^p\mu(ds) +$$
$$+\int_{F_3}\frac{1-\lambda}{2}\mu(F_3)^{-1}\|z_1\|^p\mu(ds) + \int_{F_4}\frac{1-\lambda}{2}\mu(F_4)^{-1}\|u_1\|^p\mu(ds) =$$
$$= \frac{\lambda}{2}(\|x_1\|^p+\|y_1\|^p) + \frac{1-\lambda}{2}(\|z_1\|^p+\|u_1\|^p) = 1,$$

and analogously $\|\eta\|_p^p = 1$.

Moreover

$$\|\xi-\eta\|_p^p = \int_{F_1}\frac{\lambda}{2}\mu(F_1)^{-1}\|x_1-y_1\|^p\mu(ds) + \int_{F_2}\frac{\lambda}{2}\mu(F_2)^{-1}\|x_1-y_1\|^p\mu(ds) +$$
$$+\int_{F_3}\frac{1-\lambda}{2}\mu(F_3)^{-1}\|z_1-u_1\|^p\mu(ds) + \int_{F_4}\frac{1-\lambda}{2}\mu(F_4)^{-1}\|z_1-u_1\|^p\mu(ds) =$$
$$= \frac{\lambda}{2}(\|x_1-y_1\|^p+\|x_1-y_1\|^p) + \frac{1-\lambda}{2}(\|z_1-u_1\|^p+\|z_1-u_1\|^p) =$$

$$= \lambda\|x_1-y_1\|^p + (1-\lambda)\|z_1-u_1\|^p.$$

Also $\|\xi+\eta\|_p^p = \lambda\|x_1+y_1\|^p + (1-\lambda)\|z_1+u_1\|^p$

and a) is proved.

b) Let $x, y \in S_p(X)$. Then x, y are in fact the classes of equivalence of the functions:

$$x = \sum_{i=1}^n \chi_{E_i}x_i, \qquad y = \sum_{i=1}^n \chi_{E_i}y_i,$$

where $(E_i)_{i=1}^n$ are pairwise disjoint subsets of Σ with $0 < \mu(E_i) < \infty$, $i = 1, 2, \dots, n$ and $x_i, y_i \in X$ for every $i = 1, 2, \dots, n$.

Suppose that:

$$\|x\|_p^p + \|y\|_p^p = \sum_{i=1}^n \mu(E_i)\|x_i\|^p + \sum_{i=1}^n \mu(E_i)\|y_i\|^p = 2.$$

We have

$$\|x-y\|_p^p = \sum_{i=1}^n \|x_i-y_i\|^p\mu(E_i) \quad \text{and}$$

$$\|x+y\|_p^p = \sum_{i=1}^n \|x_i+y_i\|^p\mu(E_i).$$

Let $\xi_i, \eta_i \in X$, $i = 1, 2, \dots, n$ given by:

$$\xi_i = \left[\tfrac{1}{2}(\|x_i\|^p + \|y_i\|^p)\right]^{-1/p} x_i$$

$$\eta_i = \left[\tfrac{1}{2}(\|x_i\|^p + \|y_i\|^p)\right]^{-1/p} y_i$$

and $\lambda_i = \frac{1}{2}(\|x_i\|^p + \|y_i\|^p)\mu(E_i) = r_i\mu(E_i)$, $i = 1, \dots, n$.

We have $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$. Also

$$\|\xi_i\|^p + \|\eta_i\|^p = 2\,(\|x_i\|^p + \|y_i\|^p)^{-1}\,(\|x_i\|^p + \|y_i\|^p) = 2\,,$$

and

$$\sum_{i=1}^{m} \lambda_i \left(\|\xi_i-\eta_i\|^p, \|\xi_i+\eta_i\|^p\right) = \sum_{i=1}^{m} \lambda_i \left(k_i^{-1}\|x_i-y_i\|^p, k_i^{-1}\|x_i+y_i\|^p\right) =$$

$$= \sum_{i=1}^{m} \mu(E_i) \left(\|x_i-y_i\|^p, \|x_i+y_i\|^p\right) = \left(\|x-y\|_p^p, \|x+y\|_p^p\right).$$

This implies $C^p_{S_p(X)} \subseteq \operatorname{conv} C^p_X$. The lemma is proved.

Remark. If μ is purely atomic and contains exactly an atom then (1) is not true. Indeed, in this case $x \in S_p(X)$ means that $x = x_1 \chi_S$, with $x_1 \in X$; $\|x\|_p = \|x_1\| (\mu(S))^{1/p}$ and $A^p_{S_p(X)} = A^p_X$. But generally A^p_X is not convex for every Banach space X and $p > 1$ fixed. (For example if $X = R$ and $p > 1$ is arbitrary).

In the particular case $X = R$, it is useful to obtain the set C^p_R. We have

$$C^p_R = \{(|x-y|^p, |x+y|^p) : x, y \in R,\ |x|^p + |y|^p = 2\} =$$

$$= \{(||x|-|y||^p, (|x|+|y|)^p) \vee ((|x|+|y|)^p, ||x|-|y||^p) : x, y \in R,\ |x|^p + |y|^p = 2\},$$

and if we put $u = ||x|-|y||^p$, $v = (|x|+|y|)^p$ then:

$$C^p_R = \{(u,v) : u, v \geq 0,\ (u^{1/p} + v^{1/p})^p + |u^{1/p} - v^{1/p}|^p = 2^{p+1}\}$$

Now, C^p_R is a curve in the plane uOv contained in the first quadrant. As a function $v = v(u)$ it is decreasing. Indeed for $u > v$:

$$v' = -\frac{u^{1/p-1}}{v^{1/p-1}} \cdot \frac{(u^{1/p}+v^{1/p})^{p-1} + (u^{1/p}-v^{1/p})^{p-1}}{(u^{1/p}+v^{1/p})^{p-1} - (u^{1/p}-v^{1/p})^{p-1}} < 0, \quad \forall p > 1.$$

Also

$$v'' = \frac{(p-1)\,u^{1/p-2}}{v}\,(v-uv')^2\,\frac{(u^{1/p}+v^{1/p})^{p-2} - (u^{1/p}-v^{1/p})^{p-2}}{(u^{1/p}+v^{1/p})^{p-1} - (u^{1/p}-v^{1/p})^{p-1}},$$

and the analogous relations for $v > u$. Then we have $v''(u) > 0$ for $p > 2$ and $v''(u) < 0$ for $p \in (1, 2)$. For $p = 2$:

$$C^2_R = \{(u, v) : u, v \geq 0,\ u + v = 4\}.$$

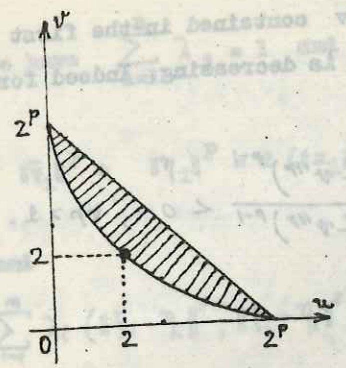
Then

$$\operatorname{conv} C^p_R = \{(u,v) : u, v \geq 0,\ (u^{1/p} + v^{1/p})^p + |u^{1/p} - v^{1/p}|^p \geq 2^{p+1},\ u + v \leq 2^p\},$$

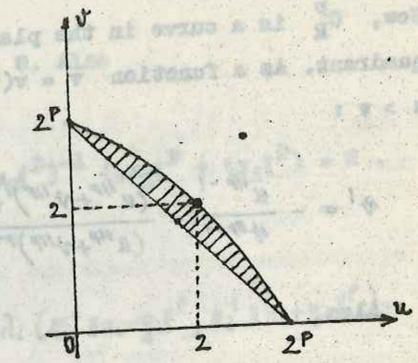
for $p > 2$ and

$$\operatorname{conv} C^p_R = \{(u,v) : u, v \geq 0,\ (u^{1/p} + v^{1/p})^p + |u^{1/p} - v^{1/p}|^p \leq 2^{p+1},\ u + v \geq 2^p\},$$

for $p \in (1, 2)$. Clearly $C^2_R = \operatorname{conv} C^2_R$.



conv C_R^p , $p > 2$



conv C_R^p , $p \in (1, 2)$

Let $(X, \|\cdot\|)$ be a real normed space with $\dim X > 2$ and let $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere. The modulus of convexity of X is defined by

$$\delta_X(\varepsilon) = \inf \{1 - \|x + y\|/2 : x, y \in S_X , \|x - y\| = \varepsilon\},$$

for every $\varepsilon \in [0, 2]$.

O. Hanner [3] has obtained the modulus of convexity for ℓ_p and $L_p[0, 1]$ using two inequalities one of Clarkson and the other of Beurling. This modulus of convexity depends only of p and is the same for ℓ_p and $L_p[0, 1]$. In the following we also obtain this modulus of convexity by using Lemma 1, in the case of $L_p(S, \Sigma, \mu, R) = L_p(\mu)$. Suppose the conditions in Lemma 1 are satisfied. Then from the density of $S_p(R)$ in $L_p(\mu)$ and from Lemma 1 we have:

$$\delta_{L_p(\mu)}(\varepsilon) = \inf \{1 - \|x + y\|/2 : x, y \in S_X , \|x - y\| = \varepsilon\} =$$

$$= \inf \{1 - \tfrac{r}{2} : (\varepsilon^p, r^p) \in A^p_{L_p(\mu)}\} = \inf \{1 - \tfrac{r}{2} : (\varepsilon^p, r^p) \in A^p_{S_p(R)}\}$$

$$= \inf \{1 - \tfrac{r}{2} : (\varepsilon^p , r^p) \in \text{conv } C_R^p\} =$$

$$= \inf \{\delta : (\varepsilon^p , 2^p(1 - \delta)^p) \in \text{conv } C_R^p\}.$$

If $p \in (1, 2)$, from the above determined form of conv C_R^p we have:

$$\delta_{L_p(\mu)}(\varepsilon) = \inf \{\delta : (\varepsilon + 2 - 2\delta)^p + |\varepsilon - 2 + 2\delta|^p \le 2^{p+1},$$
$$\varepsilon^p + 2^p (1 - \delta)^p \ge 2^p \}$$
$$= \inf \{\delta : (1 - \delta + \tfrac{\varepsilon}{2})^p + |1 - \delta - \tfrac{\varepsilon}{2}|^p \le 2 ,$$
$$\delta \le 1 - (1 - (\tfrac{\varepsilon}{2})^p)^{1/p} \}.$$

Since for fixed ε the function:

$$f(\delta) = (1 - \delta + \tfrac{\varepsilon}{2})^p + |1 - \delta - \tfrac{\varepsilon}{2}|^p,$$

is decreasing we have $f(\delta) \le 2$ if and only if $\delta \ge f^{-1}(2)$, and

$$\delta_{L_p(\mu)}(\varepsilon) = \inf \{ \delta : f^{-1}(2) \le \delta \le 1 - (1 - (\tfrac{\varepsilon}{2})^p)^{1/p} \} =$$
$$= f^{-1}(2) ,$$

i.e. $\delta = \delta_{L_p(\mu)}(\varepsilon)$ is the solution of the equation:

$$(1 - \delta + \tfrac{\varepsilon}{2})^p + |1 - \delta - \tfrac{\varepsilon}{2}|^p = 2 , \quad \varepsilon \in [0,2] .$$

If $p > 2$, we obtain in an analogous way:

$$\delta_{L_p(\mu)}(\varepsilon) = \inf \{ \delta : 1 - (1 - (\tfrac{\varepsilon}{2})^p)^{1/p} \le \delta \le f^{-1}(2) \} =$$
$$= 1 - (1 - (\tfrac{\varepsilon}{2})^p)^{1/p} , \quad \varepsilon \in [0,2].$$

Let $(H, \|\cdot\|)$ be a real Hilbert space of dimension at least 2. Then we have:

$$C_H^p = \{ (\|x - y\|^p, \|x + y\|^p) : \|x\|^p + \|y\|^p = 2 \} =$$

$$= \{ ((\|x-y\|^2)^{p/2}, (\|x+y\|^2)^{p/2}) : \|x\| = t;\ \|y\| = (2-t^p)^{1/p},$$

$$t \in [0, 2^{1/p}],\ x, y \in H \} =$$

$$= \{ (u, v) : u = (t^2 + (2 - t^p)^{2/p} - 2t(2 - t^p)^{1/p} \cos\alpha)^{p/2},$$

$$v = (t^2 + (2 - t^p)^{2/p} + 2t(2 - t^p)^{1/p} \cos\alpha)^{p/2} ;$$

$$t \in [0, 2^{1/p}] \},$$

α being the angle between the vectors x and y.

Then the domain C_H^p will be bounded by the curves:

$$u = |t - (2 - t^p)^{1/p}|^p \qquad u = (t + (2 - t^p)^{1/p})^p$$

and

$$v = (t + (2 - t^p)^{1/p})^p \qquad v = |t - (2 - t^p)^{1/p}|^p$$

with $t \in [0, 2^{1/p}]$ and for $t = 1$,

$$u = (2 - 2\cos\alpha)^{p/2} = (4 \sin^2 \tfrac{\alpha}{2})^{p/2}$$

$$v = (2 + 2\cos\alpha)^{p/2} = (4 \cos^2 \tfrac{\alpha}{2})^{p/2},$$

i.e. C_H^p will be given by:

$$(u^{1/p} + v^{1/p})^p + |u^{1/p} - v^{1/p}|^p \leq 2^{p+1}$$

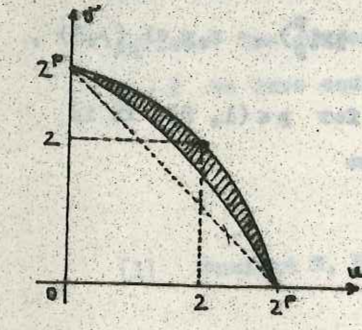
$$u^{2/p} + v^{2/p} \geq 4,$$

for $p \in (1, 2)$ and by

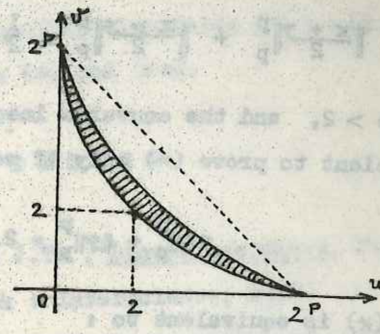
$$(u^{1/p} + v^{1/p})^p + |u^{1/p} - v^{1/p}|^p \geq 2^{p+1}$$

$$u^{2/p} + v^{2/p} \leq 4,$$

for $p > 2$. Also: $C_H^2 = \{(u, v) : u, v \geq 0,\ u + v = 4\}$.



C_H^p, $p \in (1,2)$



C_H^p, $p > 2$

Now it is clear that $\operatorname{conv} C_H^p = \operatorname{conv} C_R^p$ for every $p > 1$, and every Hilbert space H. So from the density of $S_p(H)$ in $L_p(\mu, H)$ and from the Lemma 1 we have:

$$\delta_{L_p(\mu,H)}(\varepsilon) = \inf \{ 1 - \tfrac{r}{2} : (\varepsilon^p, r^p) \in A^p_{L_p(\mu,H)} \} =$$

$$= \inf \{ 1 - \tfrac{r}{2} : (\varepsilon^p, r^p) \in A^p_{S_p(\mu,H)} \} =$$

$$= \inf \{ 1 - \tfrac{r}{2} : (\varepsilon^p, r^p) \in \operatorname{conv} C_H^p \} =$$

$$= \inf \{ 1 - \tfrac{r}{2} : (\varepsilon^p, r^p) \in \operatorname{conv} C_R^p \} = \delta_{L_p(\mu)}(\varepsilon),$$

for every $\varepsilon \in [0, 2]$.

Then the modulus of convexity of $L_p(S, \Sigma, \mu, H)$ is given by the formules of O. Hanner for (S, Σ, μ) in the Lemma and for an arbitrary Hilbert space H.

From Lemma 1 and the construction of C_H^p we can also prove other known inequalities. In the following by $L_p(S, \Sigma, \mu, X)$ we understand an L_p space in which (S, Σ, μ) satisfies the condition of Lemma 1.

The Clarkson's inequalities for $L_p(S, \Sigma, \mu, H)$ are :

$$(*) \qquad \left\|\frac{x+y}{2}\right\|_p^p + \left\|\frac{x-y}{2}\right\|_p^p \le \frac{1}{2}\left(\|x\|_p^p + \|y\|_p^p\right), \quad x, y \in L_p(\mu, H),$$

for $p > 2$, and the converse inequality for $p \in (1, 2)$. It is sufficient to prove $(*)$ only if we suppose

$$\|x\|_p^p + \|y\|_p^p = 2.$$

Then $(*)$ is equivalent to :

$$u + v \le 2^p$$

for $(u, v) \in C^p_{S_p(\mu,H)} = \operatorname{conv} C^p_H$, if $p > 2$, and this is obvious.

Then $(*)$ is proved and in the same manner the converse inequality for $p \in (1, 2)$.

The Beurling inequality for $p > 2$ is

$$(**) \qquad (\|x'\|_p + \|y'\|_p)^p + \left|\, \|x'\|_p - \|y'\|_p \right|^p \ge \|x' + y'\|_p^p + \|x' - y'\|_p^p$$

where $x', y' \in L_p(\mu, H)$. If we put $x' + y' = x$ and $x' - y' = y$ then $(**)$ is equivalent to:

$$(\|x+y\|_p + \|x-y\|_p)^p + \left|\, \|x+y\|_p - \|x-y\|_p \right|^p \ge$$

$$\ge 2^p\left(\|x\|_p^p + \|y\|_p^p\right)$$

and (from the homogeneity of $(**)$) it is sufficient to $(**)$ be proved only for

$$\|x\|_p^p + \|y\|_p^p = 2.$$

Then $(**)$ is equivalent to :

$$(u^{1/p} + v^{1/p})^p + \left|u^{1/p} - v^{1/p}\right|^p \ge 2^{p+1},$$

for every $(u, v) \in C^p_{S_p(\mu,H)} = \operatorname{conv} C^p_H$, if $p > 2$, which is obvious. The converse inequality also holds if $p \in (1, 2)$. For $p = 2$ we have equality in $(*)$ and $(**)$.

R E F E R E N C E S

[1] Dunford N, Schwartz J.T. , Linear operators, Part I, New York : Interscience, 1958.

[2] Figiel T. , On the moduli of convexity and smoothness, Studia Math. 56, 121-155 (1976).

[3] Hanner O. , On the uniform convexity of L^p and ℓ^p , Ark. Mat. 3, 239-244 (1956).

[4] Kadeç M.I. , Unconditional convergence of series in uniformly convex spaces, Uspehi Mat. Nauk 11, 5 (71), 185-190 (1956).