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ON A LINEAR SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEM

by

C.I. Gheorghiu

In this work we consider the following one-dimensional convection dominated singularly perturbed convection diffusion problem

$$(P_\varepsilon)\quad \left|\begin{array}{l} -\varepsilon\cdot u'' + a\cdot u' = f, \quad x \in (0,1) \equiv I \\ \quad u(0) = 0,\ u(1) = 0 \end{array}\right.$$

$a \in R$, $a > 0$, fixed, $0 < \varepsilon < 1$, ε being the small parameter with $\varepsilon << a$.

We shall restrict our attention, without loss of generality, to the homogeneous boundary conditions because the non-homogeneous case is easy to be reduced to the first one. (see for example Tikhonov, Vasil'eva and Sveshnikov [8]).

The case when the constant "a" is a function of x will also be in our attention in the future works.

Functional context : the injection from $H^1(0,1)$ to $C^0[0,1]$ is compact, and one can also prove that

$$(1)\qquad \|v\|_{C^0[0,1]} \left(= \max_{x\in[0,1]} |v(x)|\right) \le \sqrt{2}\,\|v\|_{H^1(I)}, \quad \forall\, v \in H^1(I).$$

The standard variational formulation for (P_ε) is (see for example Glowinski [3] or Petrila and Gheorghiu [7])

(PV) find $u \in H^1_0(I)$ such that

$$\varepsilon\int_0^1 u'v'dx + a\cdot\int_0^1 u'v\,dx = \int_0^1 fv\,dx, \quad \forall v \in H^1_0(I).$$

As usually we shall note $a(u,v) = \varepsilon\int_0^1 u'v'dx + a\cdot\int_0^1 u'v\,dx$, $a(\cdot,\cdot): H^1_0(I)\times H^1_0(I) \longrightarrow R$ and $L(v) = \int_0^1 fv\,dx$, $L(\cdot): H^1_0(I)\longrightarrow R$.

The functional $a(\cdot,\cdot)$ is also V-elliptic (we shall note for the sake of the brevity $H^1_0(I)\equiv V$) because

$$\int_0^1 u'v\,dx = -\int_0^1 uv'\,dx, \quad \forall u,v\in V$$

and then

$$a(v,v) = \varepsilon\int_0^1 v'^2dx + a\cdot\int_0^1 v'v\,dx = \varepsilon\int_0^1 v'^2dx = \varepsilon|v|_V$$

where $|\cdot|_V$ is the norm of V space.

Unfortunatelly, the constant of ellipticity is the small parameter ε, so the numerical methods for solving (P_ε) starting from variational formulation (PV) produce severely oscillating solutions (as the usual Galerkin method). However, the fact that the problem (PV) has a unique solution is classic.

Formally, it is natural to believe that u_ε, the solution of the (P_ε) problem, which is also the same for (PV) problem, converges in a suitable topology towards u the solution of

(2) $$a\cdot u' = f \quad , \quad x \in I .$$

Following the strategy of J.-L.Lions [6] (chapter V) we shall try to study the behaviour of the solution of (P_ε) problem when the small parameter $\varepsilon \longrightarrow 0$.

Actually, this is the aim of our study.

The boundary conditions of (P_ε) problem cannot be both satisfied by the solution of (2). The first question is to see whether u_ε actually converges towards a solution of (2) and to determine which one.

Out of technical reasons, which will be shown in the following, we shall attach to equation (2) the boundary condition

(3) $$u(0) = 0 .$$

Thus the problem (2),(3) will be called the <u>reduced problem</u> (PR) or terminal problem. It is obvious that this problem has also a unique solution.

The main result of this work is :

THEOREM 1. <u>If u_ε and u are the solutions of (P_ε) and respectively (PR) problems, then</u>

(4) $$u_\varepsilon \longrightarrow u \quad \text{weakly in } H^1(I).$$

The proof of this theorem will be done taking into consideration two apriori estimates given by the following lemmas.

LEMMA 1. <u>The statement of the above theorem implies</u>

(5) $$\varepsilon\|u_\varepsilon\|^2_{H^1(I)} + c_1\|u\|^2_{L^2(I)} \leqslant C\|f\|^2_{L^2(I)}$$

where the constants c and c_1 are independent of ε.

<u>Proof</u>. Taking the scalar product of all terms of equation of (P_ε) problem by $e^{-x}u_\varepsilon$ and using the integration by parts we have

$$\varepsilon\int_0^1 e^{-x}(u'_\varepsilon)^2dx + \frac{a-\varepsilon}{2}\int_0^1 e^{-x}u_\varepsilon^2\,dx = \int_0^1 f\,e^{-x}u_\varepsilon\,dx$$

or

$$\varepsilon\left[\int_0^1 e^{-x}u_\varepsilon^2\,dx + \int_0^1 e^{-x}(u'_\varepsilon)^2dx\right] + \frac{a-3\varepsilon}{2}\int_0^1 e^{-x}u_\varepsilon^2\,dx = \int_0^1 f\,e^{-x}u_\varepsilon\,dx.$$

Using in the right hand side the Schwarz inequality in $L^2(I)$ and in all integrals the first mean theorem, it is easy to obtain inequality (5) for $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(a)$.

Remarks 1. Using (1), the first term from the left hand side of (5) may be replaced by $\| u_\varepsilon \|^2_{C^0[0,1]}$

2. The inequality (5) implies

$$\| u_\varepsilon \|_{L^2(I)} \leqslant c \| f \| \tag{6}$$

the constant C being every where positive and independent of ε. This estimation is in accordance with that of Kreiss [5].

3. In inequality (5) the constant c_1 is of "a" order and C is of $1/a$ order.

4. In the functional framework of $C^m(I)$ spaces, Dorr, Parter and Shampine [1] consider more complicated (nonlinear) cases analogous to (P_ε).

LEMMA 2. Let φ be a function from $C^1[0,1]$ such that

(7) $\varphi = 0$ in a neighbourhood of 1, $\varphi(o) > o$ and $0 \leqslant \varphi(x) < 1$, $x \in I$

Then $\varphi u'_\varepsilon$ is uniformly bounded in $L^2(I)$, $o < \varepsilon < \varepsilon_0$.

Proof. Taking the scalar product of all terms of equation of (P_ε) problem by $\varphi u'_\varepsilon$, we have

$$-\frac{\varepsilon}{2} \int_0^1 \left[(u'_\varepsilon)^2 \right]' \varphi \, dx + a \cdot \int_0^1 \varphi u'^2_\varepsilon \, dx = \int_0^1 f \varphi u'_\varepsilon \, dx. \tag{8}$$

For the first left hand side term, using (7), follows

$$\begin{aligned} -\frac{\varepsilon}{2} \int_0^1 \varphi \left[(u'_\varepsilon)^2 \right]' dx &= -\frac{\varepsilon}{2} \varphi (u'_\varepsilon)^2 \Big|_0^1 + \frac{\varepsilon}{2} \int_0^1 (u'_\varepsilon)^2 \varphi' dx = \\ &= \frac{\varepsilon}{2} \left[\varphi (u'_\varepsilon)^2 \right](o) + \frac{\varepsilon}{2} \int_0^1 (u'_\varepsilon)^2 \varphi' dx \end{aligned} \tag{9}$$

Due to (5) the last integral from (9) is $O(1/\varepsilon)$. Using (9), it follows from (8) that

$$\left| \int_0^1 a \varphi u'^2_\varepsilon \, dx \right| \leqslant \left| \int_0^1 f \varphi u'_\varepsilon \, dx \right|$$

or, with Schwarz inequality

$$\| \sqrt{a\varphi}\, u'_\varepsilon \|^2_{L^2(I)} \leqslant \frac{1}{2} \| f \|^2_{L^2(I)} + \frac{1}{2} \| \varphi u'_\varepsilon \|^2 .$$

Since $0 \leqslant \varphi(x) < 1$, $x \in I$, it follows from this last inequality that

$$\| \sqrt{a\varphi}\, u'_\varepsilon \|^2_{L^2(I)} \leqslant \| f \|^2_{L^2(I)} \tag{10}$$

But $\sqrt{a\varphi}$ has the same properties as φ, so (10) proves the lemma.

Remarks. 5. Relation (9) justifies the choice of boundary condition (3) in (PR) problem.

6. For regular f one can prove analoguous estimations with (10) for $\varphi u''_\varepsilon$.

Proof of the theorem. The inequalities (5) and (10) show that for $0 < \varepsilon < \varepsilon_0$, countable, from u_ε we can find a subsequence, also noted u, such that

$$\left| \begin{array}{ll} u_\varepsilon \longrightarrow w & \text{weakly in } L^2(I) \\ \varphi u' \longrightarrow \varphi w' & \text{weakly in } L^2(I) \end{array} \right. \tag{11}$$

for φ from a countable family which satisfies (7).

The statement (11) shows that $w \in H^1(I)$ and $u(o) \longrightarrow w(o)$. So, $w(o) = 0$ and passing to limit in the equation of (P_ε) we obtain $a \cdot w' = f$, hence $w = u$.

Because we "loose" a boundary condition passing from (P_ε) to (PR), that in $x = 1$, we must have a boundary layer (mathematically not hydrodinamically !).

How can it be found ?

Moreover, can one construct an approximate solution of

(P_ε) if the solution of (PR) is known ?

The works of Ekhaus and de Jager [2] and Wasow [9] are outstanding in the mathematical acceptance of boundary layer concept.

To illustrate this, we will consider following Wasow[9], the problem

$$(12)\qquad \begin{aligned} -\varepsilon\cdot u''_\varepsilon + u'_\varepsilon = 0 \ , \quad x \in I \ , \quad 0<\varepsilon<1 \\ u(0) = \alpha \ , \quad u(1) = \beta \ ; \ \alpha > 0, \ \beta > 0. \end{aligned}$$

Its reduced problem is

$$(13)\qquad \begin{aligned} u' = 0 \quad , \quad x \in I \\ u(0) = \alpha \quad . \end{aligned}$$

The exact solution of (12) is

$$(14)\qquad u_\varepsilon(x) = (1-e^{-1/\varepsilon - 1}) \left\{ (\alpha-\beta)e^{-x/\varepsilon} + \beta - \alpha e^{-1/\varepsilon} \right\}$$

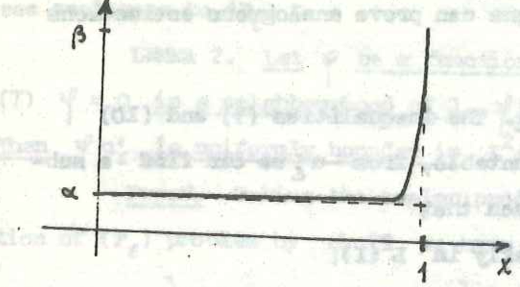

Fig.1.

In Fig.1, the solutions of (12) and (13) are comparatively presented, when $\alpha<\beta$.

As a consequence of the loss of a boundary condition the convergence of $u_\varepsilon(x)$ as $\varepsilon \longrightarrow 0$ cannot be uniform unless $\alpha = \beta$. It is clear that any solution of a two point boundary value problem of the singular perturbation type which converges to a solution of reduced problem in the whole interior of the interval, as $\varepsilon \longrightarrow 0$, must exhibit such a nonuniform behaviour near one, or both boundary points, since some boundary conditions are necesarily lost, as the order of differential equation drops in the limit. These narrow intervals are frequently called boundary layers (in mathematical acception) because the physical

so - called boundary layers that occur in flow of viscous fluids are mathematically described by just such a singular perturbation problem.

The work of Ekhaus and de Jager [2] emphasizes the actual construction of approximation solutions (not numerical!) to problem (P_ε).

In our future works we will try to answer to some of the questions in connection with these two point boundary value problems of the singular perturbation type by using numerical methods which combine formal high accuracy with good stability properties. One of these methods could be the so - called streamline diffusion method, presented in the work of Johnson [4], which in addition has important localization properties. These properties show that the presence of a boundary layer in the exact solution only affects the accuracy of the discrete solution close to the layer. This is in contrast to the usual Galerkin method, where in general the presence of a boundary layer strongly degrades the accuracy in the whole domain.

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AN APPLICATION OF A THEOREM OF Mc SHANE

by
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Let (X,d) be a metric space and let Y be a nonvoid subset of X. A function $f: Y \to R$ is called Lipschitz on Y if there exists a number $L \geq 0$ such that

(1) $$|f(x) - f(y)| \leq L\, d(x,y) \quad ,$$

for all $x,y \in Y$. A number L satisfying (1) is called a Lipschitz constant for f on Y. Denote by $\mathrm{Lip}\, Y$ the set of all real-valued Lipschitz function on Y and by $K_Y(f)$ the set of all Lipschitz constant for the function f on Y. It is easy to check that $K_Y(f) = [L_f, \infty)$, where

(2) $$L_f = \sup \{ |f(x) - f(y)| / d(x,y) \ : \ x,y \in Y \ , \ x \neq y \} .$$

If $Y = X$ then $\mathrm{Lip}\, X$ denotes the set of all real-valued Lipschitz functions on X.

Mc Shane [6] proved the following extension theorem for Lipschitz functions :

THEOREM 1. Let (X,d) be a metric space and let Y be a nonvoid subset of X. If $f \in \mathrm{Lip}\, Y$ and $L \geq 0$ is a Lipschitz constant for f on Y then there exists $F \in \mathrm{Lip}\, X$ such that $F|_Y = f$ and $|F(x) - F(y)| \leq L\, d(x,y)$, for all $x,y \in X$.