„BABEŞ—BOLYAI" UNIVERSITY
FACULTY OF MATHEMATICS AND PHYSICS
RESEARCH SEMINARS

SEMINAR ON

FUNCTIONAL ANALYSIS AND NUMERICAL METHODS

Preprint Nr.1 , 1989

CLUJ-NAPOCA
ROMANIA

CONTENTS

$$\lim_{n \to \infty} \| P(x_n) \| = 0$$

c'est-à-dire

$$P(\bar{x}) = \theta ,$$

égalité qui exprime le fait que $\bar{x}$ est une solution de l'équation (1).

De l'identité

$$P(\bar{x}) - P(x_n) = [\bar{x} , x_n ; P] (\bar{x} - x_n)$$

il résulte que

$$\| \bar{x} - x_n \| \leq B \| P(x_n) \| \leq B \cdot \rho^{1/(1-q)} \varepsilon_0^{q^n} .$$

Le théorème est donc démontré.

B I B L I O G R A P H I E

1 PAVALOIU, I , Asupra operatorilor iterativi, Studii şi Cercetări Matematice , 23 (1971) 10, 1537 - 1544.

2 PAVALOIU, I,. Introducere în teoria aproximării soluţiilor ecuaţiilor , Ed. Dacia, 1976 .

3 UL'M, S., Ob oboboscennyh razdelennîh raznostiak I , Izv. Akad. Nauk Estonskoi SSR 16 (1967) 1 , 13 - 36 .

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This paper is in final form and no version of it is or will be submitted for publication elsewhere.

"BABES-BOLYAI" University
Faculty of Mathematics and Physics
Research Seminars
Seminar on Functional Analysis and Numerical Methods
Preprint Nr. 1 1989, pp. 111 - 120

BANACH SPACES CONTAINING FINITE DIMENSIONAL STRONGLY PROXIMINAL SUBSPACES

Ioan Şerb

Denote by X a real Banach space and by X^* his dual. Let $U(X)$ be the closed unit ball of X and for a non-void proper subset M of X denote by P_M the <u>metric projection</u> on M defined by :

$$P_M(x) = \{ m \in M : \| x - m \| \leq \| x - m' \| , \forall m' \in M \} , \forall x \in X.$$

The subset M is said to be <u>proximinal</u> if $P_M(x) \neq \emptyset$ for all $x \in X$ and <u>strongly proximinal</u> (see [6]) if card $P_M(x) \geqslant 2$ for all $x \in X \setminus M$.

In this paper we consider the problem of the existance of finite dimensional strongly proximinal subspaces in some Banach spaces. The case of $C(K)$ with K a Hausdorff compact space was considered by M.L. Ivenov [3], [4] and independently by A.L. Garkavi and S.V. Konyagin [2] who also disscused the existence of finite dimensional strongly proximinal subspaces of $L_\infty(S, \Sigma , \mu)$.

This models are used in order to obtain some more general results.

Let X be, firstly, an n-dimensional Banach space. Suppose there exist the functionals $x_1^*, \dots, x_n^* \in X^*$, such that

$$\|x_i^*\| = 1, \quad i = 1,2, \dots ,n \quad \text{and}$$

$$U(x) = \{x \in X : \max_{i=\overline{1,n}} |x_i^*(x)| \le 1\}.$$

Let Y be an m-dimensional $(1 \le m < n)$ subspace of X. Then we have

PROPOSITION 1. *If for every $y \in Y \setminus \{0\}$ there exist at least two functionals x_i^* and x_j^*, $i \ne j$, $i, j = 1,2, \dots ,n$ such that $x_i^*(y) \ne 0$ and $x_j^*(y) \ne 0$, then Y is not a strongly proximinal subspace of X.*

The proof is the same as in [3] where it was given for the case $C(K)$ where K is a Hausdorff compact with card $K = n$.

This result will be used in the sequel in order to obtain analogous results for infinite dimensional Banach spaces. Let now X be a real infinite dimensional Banach space. The set $Z = \{x_i^*\}_{i \in I}$, $x_i^* \in X^*$, $\forall i \in I$ is said to be *total over* X if from $x_i^*(x) = 0$, for all $i \in I$, it follows $x = 0$.

PROPOSITION 2. *Let X be a (real infinite dimensional) Banach space, and $Z = \{x_i^*\}_{i \in I}$ a total subset over X of continuous functionals, $Z \subset X^*$. Let Y be an n-dimensional subspace of X such that for every $y \in Y \setminus \{0\}$ there exist at least two functionals $x^*, y^* \in Z$ such that $x^*(y) \ne 0$ and $y^*(y) \ne 0$. Then there exists a finite subset $Z_0 = \{x_1^*, \dots ,x_m^*\} \subseteq Z$, such that for every $y \in Y \setminus \{0\}$, there exist at least two functionals $x_i^*, x_j^* \in Z_0$ with $x_i^*(y) \ne 0$ and $x_j^*(y) \ne 0$.*

Proof. Consider the following situations:

1) For every $y \in Y \setminus \{0\}$, the set $Z(y) = \{x^* \in Z : x^*(y) \ne 0\}$ is finite. Then the finite set will be $Z' = \bigcup_{i=1}^{n} Z(y_i)$, where $\{y_i\}_{i=1}^{n}$ is a base for Y.

Indeed, let $y \in Y$, $y = \sum_{i=1}^{n} \lambda_i y_i$. Then for $x^* \in Z \setminus Z'$ we have

$x^*(y) = \sum_{i=1}^{n} \lambda_i x^*(y_i) = 0$ and the set Z' is the one we need.

2) For every $y \in Y \setminus \{0\}$, $Z(y)$ is an infinite set. Let

$$Z_1 = \{x^* \in \mathrm{sp}\,(Z) : x^*(y) = 0, \ \forall y \in Y\}.$$

Then Z_1 is a subspace of $\mathrm{sp}\,(Z)$, $Z_1 \ne \mathrm{sp}\,(Z)$. Let Z_2 be an algebraic complement of Z_1 in $\mathrm{sp}\,(Z)$. It follows that for every $x^* \in Z_2 \setminus \{0\}$ there exists at least an $y \in Y$ such that $x^*(y) \ne 0$. Observe that the pair (Y, Z_2') where

$$Z_2' = \{z^*|_Y : z^* \in Z_2\}$$

is a dual pair with respect to the canonical application $f : Y \times Z_2' \to R$, defined by $f(y, z^*) = z^*(y)$, $\forall y \in Y$, $\forall z^* \in Z_2'$. Indeed, let $y \in Y$ and suppose $z^*(y) = 0$, $\forall z^* \in Z_2'$. Since $y^*|_Y(y) = 0$, $\forall y^* \in Z_1$ and because for every $x^* \in \mathrm{sp}\,(Z)$ we have $x^* = y^* + z^*$, $y^* \in Z_1$ and $z^* \in Z_2$, it implies

$$x^*|_Y(y) = y^*|_Y(y) + z^*|_Y(y) = 0.$$

Z being a total set over X we have $y = 0$.

Suppose that for fixed $z^* \in Z_2'$, we have $z^*(y) = 0$, $\forall y \in Y$. Then, from the construction of Z_2 we have $z^* = 0$. But the pair (Y, Z_2') is in duality with respect to the canonical duality biliniar form on $Y \times Y^*$, so Z_2' is $\sigma(Y^*, Y)$ dense in Y^*, i.e. $Y^* = Z_2'$. (Y being finite dimensional). Then there exist the independent functionals: $z_1'^*, \dots ,z_n'^* \in Z_2'$. From Theorem 1,1 p. 124 [5], there exist $x_1, \dots ,x_n \in Y$ such that $z_i'^*(x_j) = \delta_{ij}$, $i,j = 1,\dots,n$, δ_{ij} being the symbol of Kronecker. We can suppose that the $z_i'^*$ are restrictions to Y of the projections z_i^* of functionals $x_i^*\in Z$, $i=1,\dots,n$. But from

$$x_i^*(x_j)=y_i^*(x_j)+z_i^*(x_j)\ , \quad i,j=1,\dots,n\ ,$$

it follows that $x_i^*(x_j)=\delta_{ij}$, $i,j=1,\dots,n$ and $\{x_i\}_{i=1}^n$ is a base for Y. Because for x_1 there exist an infinite set $Z(x_1)$ of functionals in Z such that $x^*(x_1)\neq 0$, $\forall x^*\in Z(x_1)$ (and by analogy for $x_2,\dots,x_n$) it results that there exist the functionals $y_1^*,\dots,y_n^*\in Z$ with $\{y_1^*,\dots,y_n^*\}\cap\{x_1^*,\dots,x_n^*\}=\emptyset$ such that $y_i^*(x_i)\neq 0$, $i=1,\dots,n$. Let $y=\sum_{i=1}^n\lambda_i x_i$ an arbitrary element of $Y\setminus\{0\}$. a) If a single coefficient λ_1 is different from 0, then $y=\lambda_1 x_1$ and $y_1^*(x_1)\neq 0$, $x_1^*(x_1)=1\neq 0$. b) If there exist at least two coefficients $\lambda_1\neq 0$ and $\lambda_2\neq 0$, then

$$x_1^*\Big(\sum_{i=1}^n\lambda_i x_i\Big)=\lambda_1 x_1^*(x_1)\neq 0\ ,$$

$$x_2^*\Big(\sum_{i=1}^n\lambda_i x_i\Big)=\lambda_2 x_2^*(x_2)\neq 0.$$

From a) and b) it results that the set $\{x_1^*,\dots,x_n^*,y_1^*,\dots,y_n^*\}$ verifies the conditions of the theorem.

3) There exists an element $y\in Y$ such that $Z(y)=\{x^*\in Z: x^*(y)\neq 0\}$ consists of a finite number of elements, and there exists $z\in Y$ such that $Z(z)=\{x^*\in Z: x^*(z)\neq 0\}$ is an infinite set. The set

$$\overline{\{y\in Y: \operatorname{card} Z(y)<\infty\}\cup\{0\}}\ ,$$

is a proper subspace of Y of dimension m, $1\le m<n$. Suppose that $\{x_1,\dots,x_m\}$ is a base for this space. Let $Z_1'=\bigcup_{i=1}^m Z(x_i)$. Then, for every $y\in sp\{x_1,\dots,x_m\}\setminus\{0\}$, Z_1' verifies the conditions of the theorem.

Also we can construct a finite set of functionals $Z_2''\subset Z\setminus Z_1'$

such that for every $y\in sp\{x_{m+1},\dots,x_n\}\setminus\{0\}$ to have card $Z(y)\ge 2$. This is possible because one can apply the result from the point 2) for the total set of restrictions of functionals in $Z\setminus Z_1'$ to the complement of $sp\{x_1,\dots,x_m\}$ in X.

Let now $Z'=Z_1'\cup Z_2''$. Then Z' verifies the conditions of the theorem. For $y\in Y\setminus\{0\}$, $y=y'+y''$, with $y'\in sp\{x_1,\dots,x_m\}$ and $y''\in sp\{x_{m+1},\dots,x_n\}$. Suppose $y'\neq 0$ or $y''\neq 0$. If $y''\neq 0$ then from $x^*(y')=0$, for all $x^*\in Z_2''$, there exist two functionals $x_1^*, x_2^*\in Z_2''$ such that $x_i^*(y)=x_i^*(y'+y'')\neq 0$, $i=1,2$. If $y=y'\neq 0$, then there exi. $x_3^*, x_4^*\in Z_1'$ such that $x_i^*(y)\neq 0$, $i=3,4$.

PROPOSITION 3. Let X be a (real infinite dimensional) Banach space and let $Z=(x_i^*)_{i\in I}$ a family of independent functionals. Suppose that

1) $\|x_i^*\|=1$, $\forall i\in I$;

2) $U(X)=\{x\in X: \sup_{i\in I}|x_i^*(x)|\le 1\}$.

Let Y be an n-dimensional subspace of X. such that for every $y\in Y\setminus\{0\}$ there exist at least two functionals $x_i^*, x_j^*\in Z$ such that $x_i^*(y)\neq 0$ and $x_j^*(y)\neq 0$, $i\neq j$. Then Y is not a strongly proximinal subspace of X.

Proof. By Proposition 2 there exist the functionals $x_1^*,\dots,x_r^*\in Z$ such that for every $y\in Y$, $y\neq 0$ there exist $i,j\in\{1,\dots,r\}$, $i\neq j$ such that $x_i^*(y)\neq 0$, $x_j^*(y)\neq 0$.

Observe first that $r>n$. Indeed, if we have $r\le n$, then there exists a subspace $Y_r\subseteq Y$ such that

$$(Y_r,\ sp\{x_i^*|_{Y_r}\}_{i=1}^r)$$

to be a dual pair with respect to $f(y,x^*)=x^*(y)$, $\forall y\in Y_r$

and $x^* \in sp\{x_i^*|_{Y_r}\}_{i=1}^r$. Then there exist $y_1, \dots, y_r$ such that

$$x_i^*(y_j) = \delta_{ij}, \quad i, j = 1, \dots, r,$$

in contradiction with the choice of $\{x_1^*, \dots, x_r^*\}$. Then we have $r > n$. Let X_1 be the subspace of codimension r given by :

$$X_1 = \{x \in X : x_i^*(x) = 0, \ \forall i = 1, \dots, r\},$$

and let X_2 be a topological complement of X_1. Then $X = X_1 \oplus X_2$. It is clear that $\dim X_2 = r$. The pair

$$(X_2, sp\{x_i^*|_{X_2}\}_{i=1}^r)$$

is a dual pair with respect to the canonic bilinear form. We consider the space X_2 of dimension r and the subspace $P_2(Y)$ where P_2 is the projection $P_2 : X \longrightarrow X_2$. Then $\forall y \in Y$, $y \neq 0$, $y = y' + y''$, $y' \in X_1$, $y'' \in P_2(Y)$ and $x_i^*(y) = x_i^*(y') + x_i^*(y'')$.

Hence for every $y'' \in P_2(Y)$, $y'' \neq 0$, there exist x_i^*, x_j^*, $i \neq j$, $i, j \in \{1, \dots, r\}$ such that

$$x_i^*(y'') \neq 0, \quad x_j^*(y'') \neq 0,$$

i.e. $\quad x_i^*|_{X_2}(y'') \neq 0, \quad x_j^*|_{X_2}(y'') \neq 0.$

From Proposition 1 it results that $P_2(Y)$ is not strongly proximinal in X_2 if X_2 is endowed with the norm

$$\|x''\|_{X_2} = \sup_{i=\overline{1,r}} |x_i^*(x'')|.$$

Because $Z = (x_i^*)_{i\in I}$ is a total set over X we can interpret every element $x \in X$ as a continuous linear functional on $sp\{x_i^*\}_{i\in I}$ with the norm :

$$\|x\|_X = \sup_{i\in I} |x_i^*(x)| = \sup_{i\in I} |x(x_i^*)|.$$

Then the subspace X_2 will be the subspace of restrictions to $sp\{x_i^*\}_{i=1}^r = Z_o$ of the functionals on $sp\{x_i^*\}_{i\in I}$. The induced norm is :

$$\|x''\| = \sup_{i\in I} |x''|_{Z_o}(x_i^*)| = \max_{i=\overline{1,r}} |x''|_{Z_o}(x_i^*)|.$$

We prove now that Y is not strongly proximinal. From Proposition 1 there exists $x'' \in X_2 \setminus P_2(Y)$ such that $P_{P_2(Y)}(x'') = 0$. From the Hahn - Banach theorem there exists a norm preserving extension x of x'' to $sp(Z)$. We have $x \in X$ and we prove that $P_Y(x) = 0$.

Suppose that there exists $y \in Y \setminus \{0\}$ such that $y \in P_Y(x)$. Then for $y'' = P_2(Y)$ we have :

$$\|x'' - y''\| = \max_{i=\overline{1,r}} |x''(x_i^*) - y''(x_i^*)| =$$

$$= \max_{i=\overline{1,r}} |x|_{Z_o}(x_i^*) - y|_{Z_o}(x_i^*)| \leq \sup_{i\in I} |x(x_i^*) - y(x_i^*)| \leq$$

$$\leq \sup_{i\in I} |x(x_i^*)| = \max_{i=\overline{1,r}} |x|_{Z_o}(x_i^*)| = \max_{i=\overline{1,r}} |x''(x_i^*)| = \|x''\|,$$

in contradiction with $P_{P_2(Y)}(x'') = 0$. Then $P_Y(x) = 0$, and the proposition is proved.

COROLLARY 4. <u>Let X be a (real infinite dimensional) Banach space and $Z = (x_i^*)_{i\in I}$ a family of independent functionals. Suppose</u>

1) $\|x_i^*\| = 1, \ \forall i \in I$;

2) $U(X) = \{x \in X : \sup_{i\in I} |x_i^*(x)| \leq 1\}$.

<u>Then X has a finite dimensional strongly proximinal subspace Y if and only if there exist $i \in I$ and $z \in X$ such that:</u>

$$(*) \qquad x_i^*(z) = 1 \quad \text{and} \quad x_j^*(z) = 0, \quad \forall j \neq i.$$

COROLLARY 5. Let X be a (real infinite dimensional) Banach space and $Z = (x_i^*)_{i\in I}$ a family of independent functionals such that:

1) $\|x_i^*\| = 1$, $\forall i \in I$;

2) $U(X) = \{x \in X : \sup_{i\in I} |x_i^*(x)| \leq 1\}$.

Then Y is a finite dimensional strongly proximinal subspace of X , if there exist $i \in I$ and $y \in Y$ such that:

$$x_i^*(y) = 1 \quad , \quad x_j^*(y) = 0 \ , \quad \forall j \neq i \ .$$

Proof of the Corollary 4. If the condition (*) is not verified then by Proposition 3 each finite dimensional subspace Y is not strongly proximinal.

If there exists $i \in I$ and $z \in X$ such that $x_i^*(z) = 1$, $x_j^*(z) = 0$, $\forall j \neq i$, then let Y be a finite dimensional subspace containing z. Let $x \in X \setminus Y$, and $y \in P_Y(x)$. Denote by $d = \|x - y\| > 0$ and let $\alpha \in R$, $|\alpha| < d$. Then for $y_\alpha \in Y$

$$y_\alpha = y - (x_i^*(x) - x_i^*(y) + \alpha)\, z \in Y \ ,$$

we have:

$$\|x - y_\alpha\| = \sup_j |x_j^*(x - y_\alpha)| =$$

$$= \max\left(\sup_{j\neq i} |x_j^*(x - y_\alpha)| \ , |x_i^*(x - y_\alpha)|\right) =$$

$$= \max\left(\sup_{j\neq i} |x_j^*(x - y) - (x_i^*(x) - x_i^*(y) + \alpha)\, x_j^*(z)| ,\right.$$

$$\left. , |x_i^*(x - y) - x_i^*(z)\,(x_i^*(x) - x_i^*(y) + \alpha)| \right) =$$

$$= \max\left(\sup_{j\neq i} |x_j^*(x - y)|, |x_i^*(x) - x_i^*(y) - x_i^*(x) + x_i^*(y) + \alpha| \right) =$$

$$= \max\left(\sup_{j\neq i} |x_j^*(x - y)|, |\alpha|\right) \leq \max\ (d, |\alpha|) = d \ ,$$

Corollary 5 follows from Proposition 3 and Corollary 4.

Remark. For the sufficience part of Corollary 4 the independence of the functionals is not used.

Particular cases. If $X = C(K)$, where K is a compact Hausdorff space and if we put for the family of functionals $(x_i^*)_{i\in K}$ the evaluation functionals:

$$x_i^*(x) = x(i) \quad , \ \forall x \in X \ , \ \forall i \in K \ ,$$

then by Corollary 4 we obtain that $C(K)$ contains finite dimensional strongly proximinal spaces if and only if K has at least an isolated point. ([4], [2]).

Let now H be a Hilbert space and on product space $H \times R$ define the norm:

$$\|(x, t)\| = \max\ (\|x\|, |t|) \quad , \quad (x, t) \in H \times R.$$

With this norm $H \times R$ is a Banach space. Consider the following family of continuous functionals on $H \times R$:

$$x_h^*(x,t) = (x|h) \quad , \quad \forall h \in U(H) \ ; \quad \|h\| = 1 \ ;$$

$$x_0^*(x,t) = t.$$

We have :

$$\max\left\{ \sup_{h\in U(H)} |x_h^*(x,t)| \ , |x_0^*(x,t)| \right\} =$$

$$= \max\left\{ \sup_{h\in U(H)} |(x|h)| \ , |t| \right\} = \max\left\{ \|x\| , |t| \right\} = \|(x, t)\| \ .$$

Finally, if $(0, t) \in H \times R$ and $t \neq 0$, then :

$$0 = x_h^*(0,t) \ , \ \forall h \in U(H), \ \|h\| = 1 \quad ; \quad x_0^*(0,t) = t \neq 0.$$

From Corollary 4, $(H \times R, \|\cdot\|)$ contains finite dimensional strongly proximinal subspaces.

REFERENCES

[1] Holmes, R.B., Geometric Functional Analysis and its Applications, Springer Verlag New York Heidelberg Berlin, 1975.

[2] Garkavi, A.L., Konyagin, S.V., Antichebyshevian subspaces in C(Γ), (Russian), Teoriya priblizh. funkts., Trudy Mezhdunar. Konf. Priblizh. Funkts., Kiew, 31 may - 5 june 1983 ; 1987 p. 113.

[3] Ivanov, M.L., On the uniqueness of the best uniform approximation of a function defined on a finite set, Primen. Funkts. Anal. Teor. Priblizh. Kalinin, 56 - 66, 1983. (Russian)

[4] Ivanov, M.L., Structure of non-Chebyshev systems of functions, Vestnik Moskowskogo Universiteța, Seriya 15,Vychisli-tel'naya Matematika i Kibernetika 3 (1985) 27 - 30.

[5] Schaefer, H.H., Topological Vector Spaces, The MacMillan Company, New York 1966.

[6] Șerb, I., On the multi-valued metric projection in normed vector spaces I, II, Anal. Numer. Teor. Approx. (Cluj-Napoca) 10, 1 (1981) 101-111 ; 11, 1-2 (1982) 155 - 166.

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This paper is in final form and no version of it is or will be submitted for publication elsewhere.