

Approximation Properties of the M_n -Operators

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The purpose of this paper is to study the sequence (M_n) , $(n=1, 2, \dots)$, of linear positive operators which were introduced by Meyer-König and Zeller [10] for the uniform approximation of continuous functions. These operators are formally related to the Pascal distribution of probability theory and the Taylor matrix of summability theory – quite in the same way as the well-known Bernstein operators [5] are connected with the Bernoulli distribution of probability theory and the Euler-Knopp matrix of summability theory. Due to this analogy, from a constructive point of view it is necessary to study which of the properties of Bernstein's operators are also maintained for the Meyer-König and Zeller operators. This study may be motivated by taking into account the close relationships between the two probability distributions and summability methods.

The first paragraph contains some results and specifications which are necessary for our further investigations. In § 2 the order of approximation given by the sequence of M_n -operators is analysed, and at the same time an asymptotic formula is found. Likewise an estimation of the remainder is given by using a second order modulus of continuity. These inequalities are obtained as immediate consequences of some general theorems on the order of approximation by means of certain sequences of linear positive operators. The next paragraph is devoted to point out some monotonicity properties of the sequence of M_n -operators on the space of first order convex functions. Likewise the simple form of the remainder is established. In § 4 the sequence of Meyer-König and Zeller operators is modified obtaining other linear positive operators with the same properties. The convergence of the sequence formed by the higher-order derivatives of the functions $M_n f$ is studied in the last paragraph.

All these results justify the denomination Bernstein power series given to the sequence $(M_n f)$, $n=1, 2, \dots$, this terminology being entirely motivated by the analogy of the properties of these series with those of the Bernstein polynomials.

1. Basic Definitions and Properties

Let $C(K)$, $K=[0, 1]$, denote – as usually – the linear space of complex-valued functions f , defined and continuous on K and normed by the uniform norm $\|f\| = \max_{x \in K} |f(x)|$. The n -th Meyer-König and Zeller operator M_n ($n=1, 2, \dots$) is attaching to a function $f \in C(K)$ the n -th 'Bernstein power series'

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$$(M_n f)(x) = \sum_{k=0}^{\infty} m_{nk}(x) f\left(\frac{k}{n+k}\right), \quad 0 \leq x < 1, \quad (1.1)$$

where

$$m_{nk}(x) := (1-x)^{n+1} x^k \binom{n+k}{k},$$

and thus maps the space $C(K)$ into the space $C(K')$, $K' = [0, 1]$. $(M_n f)(1)$ is defined as (cf. [10])

$$(M_n f)(1) := \lim_{x \uparrow 1} (M_n f)(x) = f(1), \quad (1.1')$$

and thus we can say that M_n is a positive endomorphism of $C(K)$ with the norm

$$\|M_n\| := \sup_{\|f\| \leq 1} \|M_n f\| = 1.$$

It is readily verified that for $e_i(x) := x^i$ and for each $n = 1, 2, \dots$ we have

$$M_n e_i = e_i \quad \text{for } i = 0, 1. \quad (1.2)$$

For $i = 2$ and $n > 1$ we obtain

$$\begin{aligned} (M_n e_2 - e_2)(x) &= x(1-x) \sum_{k=0}^{\infty} \frac{1}{n+k+1} m_{n-1,k}(x) \\ &= -\frac{x(1-x)^2}{n-1} \left[\sum_{k=0}^{\infty} m_{n-2,k}(x) - 2 \sum_{k=0}^{\infty} \frac{1}{n+k+1} m_{n-2,k}(x) \right] = \frac{x(1-x)^2}{n} - R_n(x), \end{aligned}$$

where

$$|R_n(x)| \leq \frac{8}{27 n(n-1)},$$

and thus

$$(M_n e_2)(x) = e_2(x) + \frac{x(1-x)^2}{n} + o\left(\frac{1}{n}\right) \quad (n \rightarrow \infty). \quad (1.3)$$

H. van Iperen (Delft) kindly pointed out to us that he even could give the following estimation [3a]:

$$\sup_{\substack{n=1,2,\dots \\ x \in [0,1]}} n [(M_n e_2 - e_2)(x)] = \frac{4}{27}. \quad (1.3')$$

Useful in connection with Bernstein power series are the functions (cf. [12], no. 2.9 and [14])

$$S_{t,n}(x) := \frac{1}{1-x} \sum_{k=0}^{\infty} \left(k - \frac{nx}{1-x} \right)^t m_{nk}(x) \quad (t = 0, 1, \dots; n = 1, 2, \dots; x \in K'). \quad (1.4)$$

We easily compute

$$S_{0,n}(x) = \frac{1}{1-x},$$

$$S_{1,n}(x) = \frac{x}{(1-x)^2},$$

and in order to compute $S_{t,n}(x)$ ($t \geq 2$) we use the recurrence relation

$$S_{t+1,n}(x) = \frac{ntx}{(1-x)^2} S_{t-1,n}(x) + x \frac{d}{dx} S_{t,n}(x),$$

which follows from (1.4) by differentiation. An induction on t leads to the estimation

$$0 \leq S_{2t,n}(x) \leq \frac{W_n(2t) n^t}{(1-x)^{2t+1}}, \quad x \in K', \quad (1.5)$$

where $W_n(2t)$ is monotonously decreasing in n for a fixed t .

2. The Order of Approximation in Two Different Metrics

a) Uniform metric

We denote by $C^{(p)}(K)$ the linear subspace of those functions $f \in C(K)$ that are p -times continuously differentiable and by Z_β^N ($\beta \in [1, 2]$; N a positive constant) the linear subspace of all functions $f \in C(K)$ satisfying the inequality

$$|f(x+2h) - 2f(x+h) + f(x)| \leq N|h|^\beta, \quad h \in [0, \tfrac{1}{2}], \quad 0 \leq x+2h \leq 1.$$

If $f \in C^{(1)}(K)$ and $f' \in \text{Lip}_M \alpha$, $\alpha \in (0, 1]$, then the mean value theorem yields

$$|f(x+2h) - 2f(x+h) + f(x)| = |f'[x + (\theta_1 + 1)h] - f'(x + \theta_2 h)| \cdot |h|$$

$$\leq M[(1 + \theta_1 + \theta_2)|h|]^\alpha |h| < 3M|h|^{\alpha+1}, \quad \theta_1, \theta_2 \in (0, 1),$$

more precisely $f \in Z_{\alpha+1}^{3M}$.

THEOREM 2.1. *If $f \in C^{(2)}(K)$, then*

$$\|M_n f - f\| \leq \frac{2}{27n} \|f''\| + o\left(\frac{1}{n}\right), \quad (n \rightarrow \infty).$$

Proof. Let $f \in C^{(2)}(K)$. We estimate the norm of the remainder using the following theorem of R. G. Mamedov on the asymptotic value of the approximation by sequences of linear positive operators (see [9], [12] and [18]) (We formulate this theorem in terms of the M_n -operators, $n=1, 2, \dots$): Let $x \in K'$ be a fixed point. If the

M_n -operators have the properties

$$\begin{aligned}(M_n e_0)(x) &= e_0(x) + o\left(\frac{1}{\varphi(n)}\right) \quad (n \rightarrow \infty), \\ (M_n e_i)(x) &= e_i(x) + \frac{Q_i(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right) \quad (n \rightarrow \infty), \quad i = 1, 2\end{aligned}$$

with $\varphi(n) \neq 0$ and $\lim_{n \rightarrow \infty} \varphi(n) = \infty$, and if for a certain integer $m > 1$ holds

$$\lim_{n \rightarrow \infty} \varphi(n) (M_n \Omega_{2m})(x) = 0, \quad (2.1)$$

where

$$\Omega_j(t) := |t - x|^j, \quad j = 0, 1, \dots, \quad (2.2)$$

then we have

$$(M_n f - f)(x) = \frac{2f'(x)Q_1(x) + f''(x)[Q_2(x) - 2xQ_1(x)]}{2\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right). \quad (2.3)$$

On account of (1.2) and (1.3) we have

$$Q_1(x) = 0, \quad Q_2(x) = x(1-x)^2, \quad \varphi(n) = n.$$

Moreover

$$(M_n \Omega_4)(x) = \sum_{k=0}^{\infty} m_{nk}(x) \Omega_4\left(\frac{k}{n+k}\right),$$

and taking into account that

$$\frac{k}{n+k} - x = \frac{1-x}{n+k} \left(k - \frac{nx}{1-x}\right)$$

from (1.5) follows

$$\begin{aligned}\varphi(n) (M_n \Omega_4)(x) &\leq \frac{\varphi(n)(1-x)^4}{n^4} \sum_{k=0}^{\infty} \left(k - \frac{nx}{1-x}\right)^4 m_{nk}(x) \\ &= \varphi(n) \frac{(1-x)^5}{n^4} S_{4,n}(x) \leq \frac{W_n(4)}{n^3},\end{aligned}$$

i.e., the condition (2.1) is satisfied for $m=2$. The equation (2.3) gives now the estimation

$$(M_n f - f)(x) = \frac{x(1-x)^2 f''(x)}{2n} + o\left(\frac{1}{n}\right) \quad (n \rightarrow \infty), \quad (2.4)$$

and from there follows our assertion, since $x(1-x)^2 \leq \frac{4}{27}$ and (2.4) is trivially true for $x=1$.

Our result is sharper than a result due to E. W. Cheney and A. Sharma [1].

Theorem 2.1 shows that if $f'' \neq 0$, then the order of approximation to $f \in C^{(2)}(K)$ by the sequence $(M_n f)$, $n = 1, 2, \dots$, is exactly $\mathcal{O}(n^{-1})$. We can obtain this order even under less restrictive assumptions on the approximated function f . First we establish two general theorems for a general sequence of linear positive operators.

THEOREM 2.2. *Let $L_n: C(K) \rightarrow C(K)$, $n = 1, 2, \dots$, be a sequence of linear positive operators satisfying $L_n e_0 = e_0$. Then we have for any $\delta > 0$*

$$\|L_n f - f\| \leq \inf_{m=1, 2, \dots} \{1 + \delta^{-2m} \|L_n \Omega_{2m}\|\} \omega(f; \delta) \quad (2.5)$$

where $\omega(f; \delta)$ is the modulus of continuity of f and Ω_{2m} is defined by (2.2).

Proof. In the sequel we make use of the following property of the modulus of continuity: if λ is a real positive number, then we have

$$\omega(f; \lambda \delta) \leq (1 + [\lambda]) \omega(f; \delta),$$

where $[\lambda]$ denotes the integral part of λ .

For an arbitrary $x \in K$ we put

$$F(t) := f(t) - f(x), \quad e(t) := t - x, \quad \lambda(t) := \frac{|e(t)|}{\delta}, \quad (\delta > 0)$$

and may write then

$$|L_n f - f| \leq L_n(|F|) \leq \omega(f; \delta) [1 + L_n([\lambda])]. \quad (2.6)$$

But

$$[\lambda] = 0 \quad \text{for } \lambda < 1$$

and therefore, taking into account that L_n is a monotone operator, we have

$$L_n([\lambda]) \leq L_n(\lambda) \leq L_n(\lambda^2) \leq \dots \leq L_n(\lambda^{2m}) \leq L_n(\lambda^{2m}) = \delta^{-2m} L_n(\Omega_{2m}).$$

From (2.6), the above inequality and observing (1.1') we obtain (2.5).

COROLLARY 2.3. *For the sequence of M_n -operators and a function $f \in C(K)$ we have*

$$\|M_n f - f\| \leq \frac{31}{27} \omega\left(f; \frac{1}{\sqrt[n]{n}}\right), \quad n = 1, 2, \dots$$

Proof. On account of (1.2) and (1.3') follows

$$M_n \Omega_2 = M_n e_2 - 2e_1 M_n e_1 + e_2 M_n e_0 = M_n e_2 - e_2$$

and from this

$$\|M_n \Omega_2\| \leq \frac{4}{27n}.$$

If we insert this last inequality into (2.5) and if we take $m=1$ and $\delta=n^{-1/2}$ our proof is complete.

This estimation is similar to Popoviciu's theorem on Bernstein polynomials [5].

THEOREM 2.4. *Let $L_n: C(K) \rightarrow C(K)$, $n=1, 2, \dots$, be a sequence of linear positive operators satisfying $L_n e_i = e_i$, $i=0, 1$. If $f \in C^{(1)}(K)$ and $\omega(f'; \delta)$ is the modulus of continuity of f' , then we have for any $\delta > 0$*

$$\|L_n f - f\| \leq \inf_{m=1, 2, \dots} \{ \|L_n \Omega_2\|^{1/2} + \delta^{-m} \|L_n \Omega_{m+1}\| \} \omega(f'; \delta). \quad (2.7)$$

Proof. We use here the same notation as in the proof of theorem 2.2. Since $f \in C^{(1)}(K)$ we observe that for x fixed in K we have

$$F(t) = e(t) f'(x) + e(t) [f'(T) - f'(x)]$$

where

$$\min(x, t) < T = T(x, t) < \max(x, t).$$

As the operators L_n are linear and positive and on the fact that $L_n e = 0$ it follows immediately the inequality

$$|L_n f - f| \leq L_n(|F_1|)$$

where $F_1(t) := e(t) [f'(T) - f'(x)]$. Further, by applying the above property of the modulus of continuity with

$$\lambda := \frac{|T - x|}{\delta} \quad (\delta > 0),$$

we have

$$L_n(|F_1|) \leq \omega(f'; \delta) [L_n(|e|) + L_n(|e| [\lambda])].$$

Schwarz's inequality yields

$$L_n(|e|) \leq \sqrt{L_n(\Omega_2)},$$

and because of

$$L_n(|e| [\lambda]) \leq L_n(|e| \lambda) \leq \dots \leq L_n(|e| \lambda^m) \leq \delta^{-m} L_n(\Omega_{m+1}),$$

our assertion is proved.

With $m=1$ and $\delta=n^{-1/2}$ inequality (2.7) gives for M_n -operators

COROLLARY 2.5. *On $C^{(1)}(K)$ we have*

$$\|M_n f - f\| \leq \frac{2(2 + 3\sqrt{3})}{27\sqrt{n}} \omega\left(f'; \frac{1}{\sqrt{n}}\right), \quad n = 1, 2, \dots$$

Since this result is similar to a theorem of Lorentz for Bernstein polynomials [5], we call it 'Theorem of Lorentz for M_n -operators'.

COROLLARY 2.6. If $f' \in \text{Lip}_M \alpha$, $0 < \alpha \leq 1$, then

$$\|M_n f - f\| \leq \frac{2(2 + 3\sqrt{3})}{27} n^{-(\alpha+1)/2},$$

i.e. the order of approximation to f by the sequence $(M_n f)$ is at least $O(n^{-(\alpha+1)/2})$.

Let $L: C(K) \rightarrow C(K)$ be a linear operator. G. Freud [2] estimates the uniform norm $\|Lf\|$ with the aid of the second order modulus of continuity of f , that is

$$\omega_2(f; \delta) := \max_{\substack{|h| \leq \delta \\ x \in K}} |f(x+2h) - 2f(x+h) + f(x)|, \quad 0 \leq \delta \leq \frac{1}{2}.$$

If we change the proof of his theorem by dividing the interval K only in j ($j=1, 2, \dots$) double-strips (instead of $2j$ double-strips) we obtain

$$\|Lf\| \leq (\|L\| + j^2 \|L^{(2)}\|) \omega_2\left(f; \frac{1}{j}\right), \quad (j=1, 2, \dots), \quad (2.8)$$

where $\|L\|$ is the norm of the operator L and

$$\|L^{(2)}\| = \sup_{\|f''\| \leq 1} \|Lf\|, \quad f \in C^{(2)}(K)$$

is a seminorm.

For the special sequence $(M_n - I)$, $n=1, 2, \dots$, of positive endomorphisms of $C(K)$ (I being the identical operator) holds

$$\|M_n - I\| = \sup_{\|f\| \leq 1} \|M_n f - f\| \leq 2, \quad (n=1, 2, \dots),$$

and applying theorem 2.1 we have (on the space $C^{(2)}(K)$)

$$\|(M_n - I)^{(2)}\| = \sup_{\|f''\| \leq 1} \|M_n f - f\| \leq \frac{1}{9n}.$$

Inserting the last two inequalities into (2.8) and putting here $j = [\sqrt{n}]$ we get

THEOREM 2.7. If $\omega_2(f; \delta)$ is the second order modulus of continuity of $f \in C(K)$, then

$$\|M_n f - f\| \leq \frac{19}{9} \omega_2\left(f; \frac{1}{[\sqrt{n}]}\right), \quad n=1, 2, \dots$$

COROLLARY 2.8. If $f \in Z_\beta^N$, $\beta \in [1, 2]$, then

$$\|M_n f - f\| \leq \frac{19}{9} N[n]^{-\beta/2},$$

where $[\cdot]$ denotes the integral part. In other words, the order of approximation of a function $f \in Z_\beta^N$ by the sequence $(M_n f)$, $n=1, 2, \dots$, is at least $n^{-\beta/2}$.

Remark. One of us has shown [13] that the saturation class (Favard-class) of the M_n -operators in the space $C(K)$ is

$$\text{Sat. Cl. } [M_n]_{C(K)} = \{f/f' \text{ exists and belongs to Lip } 1\}.$$

The saturation order is $O(n^{-1})$ and the only invariant elements are the linear functions. Independently of us Y. Suzuki (Sendai) obtained similar results which he will publish in a forthcoming paper.

b) *Hausdorff–Pompeiu metric*

Bl. Sendov [19] defines the modulus of non-monotonicity on K of a bounded function f by

$$\mu(f; \delta) := \sup_{|x_1 - x_2| \leq \delta} \left\{ \sup_{x_1 \leq x \leq x_2} [|f(x_1) - f(x)| + |f(x_2) - f(x)|] - |f(x_1) - f(x_2)| \right\}, \quad (x_1, x_2 \in K; \delta > 0). \quad (2.9)$$

Two simple properties of this modulus are:

- (I) $\mu(f; \delta) = 0$ if f is monotone on K ,
- (II) $\mu(f; \delta) \leq 2\omega(f; \delta)$ for any function f (the constant 2 being the best possible).

DEFINITION 2.9. We denote by $H(K)$ the linear space of all bounded real valued functions on K satisfying $\lim_{\delta \rightarrow 0} \mu(f; \delta) = 0$. On account of (II) we have $C(K) \subseteq H(K)$.

If $f \in H(K)$, then there exist $f(0+)$ and $f(1-)$ and both $f(x+0)$ and $f(x-0)$ at interior points of K (see [19]). In the following we put

$$f(0) := f(0+), \quad f(1) := f(1-), \quad (f \in H(K)).$$

DEFINITION 2.10. The point set

$$\mathcal{G}_e(f) := \{(x, y) \in R^2 \mid x \in K, f(x-0) \leq y \leq f(x+0) \text{ or } f(x+0) \leq y \leq f(x-0)\}$$

is called the *extended graph* of the function $f \in H(K)$. (For $f \in C(K)$ the extended graph coincides with the graph in the ordinary sense and will be denoted by $\mathcal{G}(f)$).

Using the extended graphs we are able now to define the Hausdorff–Pompeiu distance $r(f, g)$ between two functions $f, g \in H(K)$:

$$r(f, g) := \max \left\{ \max_{x_1 \in \mathcal{G}_e(f)} \min_{x_2 \in \mathcal{G}_e(g)} \|X_1 - X_2\|_0, \max_{x_1 \in \mathcal{G}_e(g)} \min_{x_2 \in \mathcal{G}_e(f)} \|X_1 - X_2\|_0 \right\},$$

where the distance $\|X_1 - X_2\|_0$ between the two points $X_1(x_1, y_1)$ and $X_2(x_2, y_2)$ is defined by

$$\|X_1 - X_2\|_0 = \max(|x_1 - x_2|, |y_1 - y_2|)$$

(see [3], [15], [19]).

The following estimations hold for a very general class of sequences of linear positive operators $\hat{M}_n: H(K) \rightarrow C(K)$, $n = 1, 2, \dots$, which are defined by

$$(\hat{M}_n f)(x) = \sum_{k \geq 0} p_{nk}(x) f(t_{nk})$$

(where the points $\{t_{nk}\} \subset K$ and the functions $p_{nk}(x) \geq 0$ are such that $\sum_{k \geq 0} p_{nk}(x) = 1$) and which are satisfying

$$\hat{M}_n e_i = e_i, \quad i = 0, 1, \quad (\hat{M}_n f)(0) = f(0), \quad (\hat{M}_n f)(1) = f(1).$$

Evidently the sequence of M_n -operators is belonging to this class.

For every $\delta > 0$ and $n = 1, 2, \dots$ we have

$$\hat{M}_n e_0 \leq \delta^{-2} [(\hat{M}_n e_2)(x) - x^2] =: \delta^{-2} R_{n2}(x), \quad (x \in K). \quad (2.10)$$

LEMMA 2.11. *If $f \in H(K)$, then for any point $X_1 \in \mathcal{G}(\hat{M}_n f)$ there is a point $X_2 \in \mathcal{G}_e(f)$ such that for every $\delta > 0$*

$$\|X_1 - X_2\|_0 \leq \max(\delta, 2H\delta^{-2}R_{n2}),$$

where

$$H := \sup_{x \in K} |f(x)|.$$

Proof. It is sufficient to prove that for x_1 an arbitrary point from K there is a point $x_2 \in \Delta(x_1, \delta) := [x_1 - \delta, x_1 + \delta] \cap [0, 1]$ and a value y_2 such that

$$X_2(x_2, y_2) \in \mathcal{G}_e(f)$$

and

$$|(\hat{M}_n f)(x_1) - y_2| \leq 2H\delta^{-2}R_{n2}.$$

From the continuity supposition the point $X_1 \in \mathcal{G}(\hat{M}_n f)$ is of the form

$$X_1(x_1, (\hat{M}_n f)(x_1)).$$

Now let the function g be defined as

$$g(x) = g(x_1, f; x) = \begin{cases} f(x), & x \in \Delta(x_1, \delta) \\ f(x_1), & x \notin \Delta(x_1, \delta), \end{cases}$$

and let $h := g - f$. With the notation

$$\inf_{x \in \Delta(x_1, \delta)} f(x) =: m_f, \quad \sup_{x \in \Delta(x_1, \delta)} f(x) =: M_f$$

we have for $x \in K$

$$m_f \leq g(x) \leq M_f$$

and

$$\sup |h(x)| \leq 2 \sup |f(x)| \leq 2H.$$

In this case we may write $\hat{M}_n f = \hat{M}_n g - \hat{M}_n h$ with

$$m_f \leq \hat{M}_n g \leq M_f. \quad (2.12)$$

Let $\mathcal{F} = \{(x, y) \in \mathcal{G}_e(f) \mid x \in \Delta(x_1, \delta)\}$; from (2.11) we observe that there are $x_m, x_M \in \Delta(x_1, \delta)$ such that the points $X_m(x_m, m_f)$ and $X_M(x_M, M_f)$ belong to $\mathcal{F}$. From the properties of the set $\mathcal{F}$, for an arbitrary $y_2 \in [m_f, M_f]$ there is a point $x_2 \in [x_m, x_M] \subset \Delta(x_1, \delta)$ such that $X_2(x_2, y_2) \in \mathcal{F} \subset \mathcal{G}_e(f)$. Taking into account (2.12) we can take $y_2 = (\hat{M}_n g)(x_1)$. Further we have

$$|(\hat{M}_n f)(x_1) - y_2| = |(\hat{M}_n h)(x_1)| \leq 2H(\hat{M}_n e_0)(x_1) \quad (2.13)$$

$t_{nk} \in K_1$

where $K_1 := K - \Delta(x_1, \delta)$. On the other hand $t_{nk} \in K_1$ implies $|x_1 - t_{nk}| > \delta$, so that (2.13) gives the inequality

$$|(\hat{M}_n f)(x_1) - y_2| \leq 2H\delta^{-2}R_{n2},$$

which proves this lemma.

In the other direction goes

LEMMA 2.12. *If $f \in H(K)$, then for any point $X_1 \in \mathcal{G}_e(f)$ there is a point $X_2 \in \mathcal{G}(\hat{M}_n f)$ such that for every $\delta > 0$*

$$\|X_1 - X_2\|_0 \leq \max\{\delta, \tfrac{1}{2}\mu(f; 2\delta) + 8H\delta^{-2}R_{n2}\},$$

where $\mu(f; \delta)$ is the modulus of non-monotonicity of f on K and $H := \sup_{x \in K} |f(x)|$.

Proof. As above, we prove that if $X_1(x_1, y_1) \in \mathcal{G}_e(f)$ then exists a point $x_2 \in \Delta(x_1, \delta)$ such that

$$|y_1 - (\hat{M}_n f)(x_2)| \leq \tfrac{1}{2}\mu(f; 2\delta) + 8H\delta^{-2}R_{n2}. \quad (2.14)$$

We consider the two cases:

I. $x_1 \in [\delta, 1 - \delta]$

II. $x_1 \in [0, \delta] \cup [1 - \delta, 1]$,

and give the proof only for the first, because the method of proving the other case is the same (see also [19], p. 151).

If $x_1 \in [\delta, 1 - \delta]$, then according to a lemma of Sendov ([19], Lemma 3)

$$f(x) \geq y_1 - \tfrac{1}{2}\mu(f; 2\delta)$$

holds in one of the intervals

$$A_\lambda = \left[x_1 - \frac{1+\lambda}{2}\delta, x_1 + \frac{1-\lambda}{2}\delta \right] \subset \Delta(x_1, \delta),$$

where $\lambda \in (-1, 1)$. For $z = x_1 - \delta\lambda/2 \in \Delta_\lambda$ we have

$$\begin{aligned} (\hat{M}_n f)(z) &= (\hat{M}_n f)_{t_{nk} \in \Delta_\lambda}(z) + (\hat{M}_n f)_{t_{nk} \in K - \Delta_\lambda}(z) = \\ &= (\hat{M}_n f)_{|t_{nk} - z| \leq \delta/2}(z) + (\hat{M}_n f)_{|t_{nk} - z| > \delta/2}(z) \geq [y_1 - \tfrac{1}{2} \mu(f; 2\delta)] (\hat{M}_n e_0)_{|t_{nk} - z| \leq \delta/2}(z) \\ &\quad - H (\hat{M}_n e_0)_{|t_{nk} - z| > \delta/2}(z) \geq [y_1 - \tfrac{1}{2} \mu(f; 2\delta)] - 4\delta^{-2} R_{n2} [H + y_1 - \tfrac{1}{2} \mu(f; 2\delta)] \geq \\ &\geq [y_1 - \tfrac{1}{2} \mu(f; 2\delta)] - 8H\delta^{-2} R_{n2} \end{aligned}$$

$$\text{or} \quad -\tfrac{1}{2} \mu(f; 2\delta) - 8H\delta^{-2} R_{n2} \leq (\hat{M}_n f)(z) - y_1. \quad (2.15)$$

Let

$$\Delta_v = \left[x_1 - \frac{1+v}{2} \delta, x_1 + \frac{1-v}{2} \delta \right] \subset \Delta(x_1, \delta), \quad v \in \{-1, 1\},$$

be the intervals on which we have

$$f(x) \leq y_1 + \tfrac{1}{2} \mu(f; 2\delta).$$

If $w = x_1 - \delta v/2$ we obtain

$$\begin{aligned} (\hat{M}_n f)(w) &= (\hat{M}_n f)_{t_{nk} \in \Delta_v}(w) + (\hat{M}_n f)_{t_{nk} \in K - \Delta_v}(w) = (\hat{M}_n f)_{t_{nk} \in \Delta_v}(w) + (\hat{M}_n f)_{|t_{nk} - w| > \delta/2}(w) \\ &\leq [y_1 + \tfrac{1}{2} \mu(f; 2\delta)] (\hat{M}_n e_0)_{|t_{nk} - w| \leq \delta/2}(w) + 4H\delta^{-2} R_{n2} \end{aligned}$$

$$\text{or} \quad (\hat{M}_n f)(w) - y_1 \leq \tfrac{1}{2} \mu(f; 2\delta) + 4H\delta^{-2} R_{n2}. \quad (2.16)$$

If $\lambda = v$ from (2.15) and (2.16) we obtain the inequality (2.14) with $x_2 = x_1 - \delta\lambda/2 \in \Delta(x_1, \delta)$. For $\lambda = -v$, by using the continuity of $\hat{M}_n f$, from (2.15) and (2.16) there result that there is a point $x_2 \in [x_1 - \delta/2, x_1 + \delta/2] \subset \Delta(x_1, \delta)$ where (2.14) is true.

As a simple consequence of the two lemmas 2.11 and 2.12 we deduce now an estimation of the order of approximation by the operators $\hat{M}_n$ in the Hausdorff-Pompeiu metric.

THEOREM 2.13. *If $f \in H(K)$, then*

$$r(f, \hat{M}_n f) \leq \max[\delta, 8H\delta^{-2} R_{n2} + \tfrac{1}{2} \mu(f; 2\delta)], \quad n = 1, 2, \dots; \quad \delta > 0.$$

In the special case of M_n -operators we have by (1.3') and (2.2)

$$R_{n2} = M_n(\Omega_2) \leq \frac{4}{27n}, \quad n = 1, 2, \dots$$

Thus Theorem 2.13 gives with $\delta = n^{(-1+\alpha)/2}$

THEOREM 2.14. If $f \in H(K)$, then for $n = 1, 2, \dots$

$$r(f, M_n f) = \begin{cases} \frac{32}{27} \frac{H}{n^\alpha} + \frac{1}{2} \mu(f; N(n)) & \text{if } \mu(f; \delta) \geq \delta \\ \frac{32}{27} \frac{H}{n^\alpha} + n^{(-1+\alpha)/2} & \text{if } \mu(f; \delta) < \delta, \end{cases}$$

where $0 < \alpha < 1$, $N(n) := 2n^{(-1+\alpha)/2}$ and $H := \sup_{x \in K} |f(x)|$.

3. Form of the Remainder. Properties of the Sequence of Meyer-König and Zeller Operators

The aim of this paragraph is to establish the form of the remainder and for this we likewise carry out some properties of the sequence of operators M_n on the subspace of convex functions. We recall some notions which we use below. The symbol

$$[x_1, x_2, \dots, x_{s+2}; f]$$

denotes the $s+1$ -order divided difference of the function f on the points ('knots') $x_1, x_2, \dots, x_{s+2}$. According to T. Popoviciu [16] a real function f is called *convex*, *non-concave*, *polynomial*, *non-convex* respectively *concave* of s -order, on K , if the divided differences of order $s+1$ are positive, non-negative, zero, non-positive respectively negative on any system of $s+2$ knots from K .

DEFINITION 3.1. A linear functional F on $C(K)$ is said to have the *simple form* of s -order if for each $f \in C(K)$ it has the following representation

$$F(f) = A[x_1, x_2, \dots, x_{s+2}; f]$$

where $A \neq 0$ is independent of f and the distinct knots $x_j \in K, j = 1, 2, \dots, s+2$, generally depend on the choice of the function f .

LEMMA 3.2. For $f \in C(K)$ the following relation holds

$$(M_{n+1} - M_n)(f) = - \sum_{k=0}^{\infty} m_{nk} \mu_{kn}(f), \quad (3.1)$$

where

$$\mu_{kn}(f) := \frac{1}{(k+n+1)(k+n+2)} \left[\frac{k}{k+n+1}, \frac{k+1}{k+n+2}, \frac{k+1}{k+n+1}; f \right].$$

The proof of (3.1) follows immediately after some elementary operations. From this lemma follows

THEOREM 3.3. *On the subspaces of $C(K)$ which are formed by convex, non-concave, polynomial, non-convex respectively concave functions of the first order on K , the sequence of the Meyer-König and Zeller operators is decreasing, non-increasing, stationary, non-decreasing respectively increasing.*

Proof. If we denote

$$\Delta M_n := M_n - M_{n+1},$$

then on account of (3.1)

$$\operatorname{sgn} \left[\frac{k}{k+n+1}, \frac{k+1}{k+n+2}, \frac{k+1}{k+n+1}; f \right] \quad \begin{pmatrix} k=0, 1, \dots \\ n=1, 2, \dots \end{pmatrix}$$

implies $\operatorname{sgn} \Delta M_n$ and this completes the proof (see also [1]).

Evidently it also arises the possibility of proving a converse theorem. By using the same proof as E. Popoviciu-Moldovan, establishing in [11] a similar result about Bernstein polynomials, we can easily obtain a converse statement. More precisely, if for $f \in C(K)$ the sequence $(M_n f)$, $n=1, 2, \dots$, is non-increasing respectively non-decreasing then the function f is non-concave, respectively non-convex of the first order on K . With this opportunity we also point out the following interesting property of the Meyer-König and Zeller operator which was established by one of the authors in [7]: the operator M_n ($n=1, 2, \dots$) maps convex functions of $-1, 0, 1, 1 \& 2$ order into functions which are also convex of the same order.

Further we obtain the following

THEOREM 3.4. *Let $f \in C(K)$ and let x_0 be a fixed point on K . Then we have*

$$(f - M_n f)(x_0) = -A_n(x_0) [\zeta_{1n}, \zeta_{2n}, \zeta_{3n}; f] \quad (n \geq 1),$$

where

$$0 \leq A_n(x_0) \leq \frac{4}{27n}$$

and ζ_{in} ($i=1, 2, 3$) are distinct knots on K .

Proof. We are using the following remarkable result [17] which permits to obtain the simple form of the remainder: A functional F on $C(K)$ has the simple form of s -order iff the following conditions are verified

- i) F has the exactness degree s , that is $F(e_j) = 0$, $j=0, 1, \dots, s$, $e_j(x) = x^j$, but $F(e_{s+1}) \neq 0$,
- ii) $F(f) \neq 0$ for each $f \in C(K)$ which is convex of order s .

For the fixed point $x_0 \in K$ we consider on $C(K)$ the functional

$$R_n[f; x_0] := (f - M_n f)(x_0) \quad (n = 1, 2, \dots).$$

Taking into account (1.2) and (1.3) we have

$$R_n[e_0; x_0] = 0, R_n[e_1; x_0] = 0 \quad \text{and} \quad R_n[e_2; x_0] \neq 0.$$

On the other hand, using theorem 3.3, we know that $R_n[f; x_0] < 0$ if f is a convex function of the first order on K . Therefore $R_n[f; x_0]$ admits a simple form of the first order, namely

$$R_n[f; x_0] = -A_n(x_0) [\zeta_{1n}, \zeta_{2n}, \zeta_{3n}; f] \quad (\zeta_{in} \in K).$$

The value of $A_n(x_0)$ – which is independent of f – can be easily estimated with the aid of a suitable choice of f . For instance, with $f = e_2$, using (1.3), we get the upper bound

$$0 \leq A_n(x_0) \leq \frac{4}{27n},$$

and the theorem is proved.

In a similar way, for obtaining other information about the convergence of the sequence (M_n) , $n = 1, 2, \dots$, we shall prove the following statement.

THEOREM 3.5. *If x_0 is an arbitrary point of K and $f \in C(K)$, then*

$$(M_n f - M_{n+1} f)(x_0) = B_n(x_0) [\varepsilon_{1n}, \varepsilon_{2n}, \varepsilon_{3n}; f] \quad (n = 1, 2, \dots),$$

where

$$0 \leq B_n(x_0) \leq \frac{4}{27n(n+1)}$$

and ε_{in} ($i = 1, 2, 3$) are distinct knots on K .

Proof. For each function $f \in C(K)$ which is convex of the first order on K the functionals $\mu_{kn}(f)$ in lemma 3.2 are positive and so $[(\Delta M_n) f](x_0) := (M_n f - M_{n+1} f)(x_0)$ doesn't vanish (see lemma 3.2). On the other hand we have

$$(\Delta M_n) e_j = 0 \quad (j = 0, 1) \quad \text{and} \quad (\Delta M_n) e_2 \neq 0.$$

Thus the functional $[(\Delta M_n) f](x_0)$ has a simple form of first order. We observe that

$$[x_1, x_2, x_3; e_2] = 1 \quad (x_i \text{ arbitrary points on } K),$$

$$0 \leq (M_n e_2 - M_{n+1} e_2)(x_0) \leq \frac{4}{27n(n+1)}$$

– the latter inequality on account of (1.3') and theorem 3.3 – and conclude just in the same way as in the proof of theorem 3.4.

4. Some Modified Bernstein Power Series

We propose to see how, starting from the M_n -operators, other new operators may be obtained possessing similar approximation properties. A first modification consists in considering the integral part of the coefficients, another one – which as a matter of fact imposes itself naturally – changes the interpolation knots into equi-

spaced knots. In any case we do not propose to generalize the Meyer-König and Zeller operator but only to obtain some operators from its initial form.

Let

$$p_{nk}(f) := \left[\binom{n+k}{k} f\left(\frac{k}{n+k}\right) \right]$$

and

$$a_{nk}(x) := (1-x)^{n+1} x^k, \quad \begin{pmatrix} n=1, 2, \dots \\ k=0, 1, \dots \end{pmatrix},$$

where the brackets denote the integral part. We consider then the operators $\tilde{M}_n: C(K) \rightarrow C(K')$, ($n=1, 2, \dots$),

$$\tilde{M}_n f := \sum_{k=0}^{\infty} a_{nk} p_{nk}(f). \quad (4.1)$$

$\tilde{M}_n f$ is a Bernstein power series with integral coefficients.

THEOREM 4.1. *A function $f \in C(K)$ is approximated almost uniformly on K' by the sequence $(\tilde{M}_n f)$, $n=1, 2, \dots$, of its Bernstein power series with integral coefficients.*

Proof. We have to show that on each closed interval $[a, b] \subset K'$ the sequence of functions $\tilde{M}_n f$ converges uniformly to f . With $I=[a, b]$ we have

$$\|\tilde{M}_n f - M_n f\|_{C(I)} = \max_{x \in I} \sum_{k=0}^{\infty} a_{nk}(x) = (1-b)^n \rightarrow 0 \quad (\text{as } n \rightarrow \infty).$$

The assertion follows immediately on account of (see [1], [10])

$$\|M_n f - f\|_{C(I)} \rightarrow 0 \quad (n \rightarrow \infty).$$

Now, if we consider a change of the interpolation knots, we are led to consider the following Bernstein power series (notation as in (1.1))

$$C_n f := \sum_{k=0}^{\infty} m_{n-1,k} f\left(\frac{k}{n}\right), \quad f \in Q[0, a^*] \quad (n=1, 2, \dots), \quad (4.2)$$

where $Q[0, a^*]$, $a^* = a/(1-a)$, $a \in (0, 1)$, is the linear space of all functions which are defined and bounded on the interval $[0, \infty)$ and which are continuous on the interval $[0, a^*]$, continuous to the right at $x=a^*$. In [6] it was established that for $f \in Q[0, a^*]$ the sequence $(C_n f)$, $n=1, 2, \dots$, converges uniformly on $[0, a]$ to $f(x/(1-x))$ and that the operator C_n has the same properties of monotonicity and convexity as the operator M_n .

5. Derivatives of the Bernstein Power Series

We denote by A_R the linear space of all complex-valued functions f , defined on

$K' = [0, 1)$ and satisfying a condition of the type

$$|f(x)| \leq P e^{a/(1-x)} \quad (P > 0, a \geq 0 \text{ real constants}).$$

It is well known that the M_n -image of a function $f \in A_R$ at a fixed point $x_0 \in K'$ exists for $n > 1 - [a/\ln \frac{1}{2} x_0]$ (cf. [10], Hilfssatz 1).

The following theorem extends for Bernstein power series a result on generalized Bernstein power series which is due to A. Jakimovski and D. Leviatan (cf. [4], Theorem 2.3) to the space A_R .

Let

$$D^m = \frac{d^m}{dx^m} \quad \text{and} \quad a_{nk}(x) := (1-x)^{n+1} x^k.$$

LEMMA 5.1. *On the open interval $(0, 1)$ holds*

$$D^m[a_{nk}(x)] = Q(x) a_{n-m, k-m}(x) \quad (m = 0, 1, \dots),$$

where

$$Q(x) = \sum_{2i+j \leq m} (k+n)^i [k(1-x) - nx]^j q_{ij}^{(m)}(x), \quad (i, j > 0),$$

and the $q_{ij}^{(m)}(x)$ are polynomials in x , independent of k and n .

The assertion is trivial if $m=0$. For the proof in the general case we refer to the corresponding proof in [5], p. 26.

THEOREM 5.2. *If $f \in A_R$, then*

$$\lim_{n \rightarrow \infty} D^m(M_n f) = D^m f \quad (m = 1, 2, \dots) \quad (5.1)$$

at each point from $(0, 1)$, where $D^m f$ exists.

Proof. Let us suppose that $D^m f$ exists at $x_0 \in (0, 1)$. Then we may write

$$f(x) = \sum_{i=0}^m \frac{(D^i f)(x_0)}{i!} (x - x_0)^i + g(x) (x - x_0)^m =: P_m(x) + h(x), \quad (5.2)$$

where $g(x) \rightarrow 0$ as $x \rightarrow x_0$. Outside of every neighbourhood of the point x_0 we have by virtue of $f \in A_R$ and (5.2)

$$|g(x)| \leq Q e^{a/(1-x)} \quad (Q > 0 \text{ a real constant, } x \in K'). \quad (5.3)$$

Let $n_0 = 1 - [a/\ln \frac{1}{2} x_0]$. Then $D^m(M_n f)(x_0)$ exists for $n \geq n_0$. Following the first part of Jakimovski and Leviatan's proof we have only to show that

$$\lim_{n \rightarrow \infty} D^m(M_n h)(x_0) = 0 \quad (m = 0, 1, \dots).$$

Given $\varepsilon > 0$, we can find a $\delta = \delta(\varepsilon) > 0$ (where $0 < \delta < 1 - x_0$) so that

$$\left| g\left(\frac{k}{n+k}\right) \right| < \varepsilon \quad \text{for} \quad \left| \frac{k}{n+k} - x_0 \right| < \delta. \quad (5.4)$$

We choose now an index $n_1 \geq n_0$ such that

$$x_0 e^{a/n} < x_0 + \frac{\delta}{2}$$

is still valid. In what follows let $n \geq n_1$. By lemma 5.1 we have

$$D^m(M_n h)(x_0) = \left(\sum_{2i+j \leq m} \overline{q}_{ij}(x_0) \right) (S_1 + S_2), \quad (5.5)$$

where

$$S_t = \sum_{k \in I_t} m_{nk}(x_0) (k+n)^{i+j} \left(\frac{k}{n+k} - x_0 \right)^{m+j} g\left(\frac{k}{n+k}\right), \quad (t=1, 2),$$

$$I_1 = \left\{ k \mid k=0, 1, \dots, \left| \frac{k}{n+k} - x_n \right| < \frac{\delta}{2} \right\}$$

$$I_2 = \left\{ k \mid k=0, 1, \dots, \left| \frac{k}{n+k} - x_n \right| \geq \frac{\delta}{2} \right\} \quad (x_n = x_0 e^{a/n}),$$

and $\overline{q}_{ij}(x_0)$ is $q_{ij}(x_0)$ multiplied with suitable factors x_0^m respectively $(1-x_0)^m$. Let $k \in I_1$. On account of our supposition on n follows

$$\left| \frac{k}{n+k} - x_0 \right| < \delta.$$

Using (5.4) and the fact that $m+j \geq 2(i+j)$ we obtain

$$|S_1| \leq \varepsilon \sum_{k \in I_1} m_{nk}(x_0) (k+n)^{i+j} \left(\frac{k}{n+k} - x_0 \right)^{2(i+j)}$$

$$\leq \frac{\varepsilon(1-x_0)}{n^{i+j}} (1-x_0)^n \sum_{k=0}^{\infty} \left(k - \frac{nx_0}{1-x_0} \right)^{2(i+j)} \left(\frac{n+k}{k} \right) x_0^k$$

$$< \varepsilon n^{-(i+j)} S_{2(i+j), n}(x_0) = \text{const. } \varepsilon.$$

Let $k \in I_2$. Taking into account (5.3), for $s > i+j$ we have

$$|S_2| \leq Q e^a \sum_{k \in I_2} m_{nk}(x_0) (k+n)^{i+j} e^{ka/n} \leq$$

$$\leq Q e^a \left(\frac{2}{\delta} \right)^{2s} (1-x_0)^{n+1} \sum_{k \in I_2} \left(\frac{k}{n+k} - x_n \right)^{2s} \binom{n+k}{k} x_n^k (k+n)^{i+j}$$

$$\leq R_n(x_0) n^{i+j-2s} \left(\frac{1-x_0}{1-x_n} \right)^n S_{2s, n}(x_n),$$

where $R_n(x_0) := Q e^a (2/\delta)^{2s} (1-x_0)(1-x_n)^{2s}$ is bounded. Since the sequence

$$\left\{ \left(\frac{1-x_0}{1-x_n} \right)^n \right\}, \quad n = 1, 2, \dots,$$

is convergent, the relation (1.5) yields

$$|S_2| = \mathcal{O}(n^{i+j-s}) \quad (n \rightarrow \infty).$$

This completes the proof of our theorem.

The result of theorem 5.2 for $m=1$ still holds at the point $x=0$ since

$$D(M_n f)(0) = -(n+1)f(0) + (n+1)f\left(\frac{1}{n+1}\right)$$

leads to

$$\lim_{n \rightarrow \infty} D(M_n f)(0) = (Df)(0).$$

For $m > 1$ however we need sharper suppositions on $D^m f$. For instance, in the case $m=2$, we have (cf. [7], p. 82)

$$(D^2 f)(0) = \frac{2n}{n+2} \left[0, \frac{1}{n+1}, \frac{2}{n+2}; f \right]$$

and thus at the point $x=0$

$$\lim_{n \rightarrow \infty} D^2(M_n f) = D^2 f$$

if $D^2 f$ is continuous at least at the point $x=0$.

Added in proof (January 28, 1970). The paper of Y. Suzuki mentioned in our remark after COROLLARY 2.8 has appeared in the meantime (cf. SUZUKI, Y. and WATANABE, S., *Some Remarks on Saturation Problem in the Local Approximation II*, Tôhoku Math. J. 21, 65–83 (1969)).

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