

Non-Injective Collineations on Some Sets in Desarguesian Projective Planes and Extension of Non-Commutative Valuations

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§ 1. Introduction

J. Aczél and W. Benz have established in [2] (see also [1], [3]) the following theorem:

Let $PG(2, K)$ be the projective plane over the division ring K ; S_1, S_2, S_3, S_4 , four lines in general position in $PG(2, K)$, $\mathcal{C}$ the union of the point sets S_i , $i=1, 2, 3, 4$ and f a collineation of $\mathcal{C}$ into $PG(2, K)$ one-one at $S_1 \cap S_2$, $S_2 \cap S_3$, $S_3 \cap S_1$ and such that $f(\mathcal{C})$ is not in a line. Then f is given by the map

$$(x_1, x_2, x_3) \rightarrow (e(x_1), e(x_2), e(x_3)) \quad (1.1)$$

composed with convenient linear transforms, where e is an arbitrary non-constant endomorphism of K (consequently an isomorphism onto a sub-division-ring of K).

As an immediate consequence, f can be embedded in an injective collineation of $PG(2, K)$.

V. Havel, J. F. Rigby and the author of the present paper have generalized independently from each other the above theorem. V. Havel and J. F. Rigby have obtained results for non-Desarguesian projective planes [5], [6], [11]. J. F. Rigby considers also non-injective collineations f of a quadrilateral lying in a plane π into a quadrilateral in an other plane π' and shows that, if π is a Moufang plane and π' satisfies some transitivity property, then f can be embedded in a collineation from π onto a subplane of π' . The author has obtained the following extensions [10]: 1) Replacing the quadrilateral by the union of a triangle and a point non incident with any side of the triangle, the analytic representation (1.1) remains valid, 2) Maintaining the quadrilateral $\mathcal{C}$ and considering non-injective collineations of $\mathcal{C}$ into $PG(2, K)$, one obtains instead of (1.1) an analytic representation of f in the form

$$(x_1, x_2, x_3) \rightarrow (e(qx_1), e(qx_2), e(qx_3)), \quad (1.2)$$

where $e: O \rightarrow K$ is a 'partial endomorphism' of K (O is a subring of K such that for every $x \in K$ we have either $x \in O$ or $x^{-1} \in O$; $e(x+y) = e(x) + e(y)$, $e(xy) = e(x)e(y)$ $\forall x, y \in O$; $e(x) = 0$ if $x^{-1} \notin O$ and $e(O)$ contains at least one element different from 0) and $q \in K$ is selected in a way that $e(qx_i)$ should be defined for $i=1, 2, 3$ and not all of them equal 0, what is always possible.

We shall consider here sets $\mathcal{C}$ in $PG(2, K)$ containing four points A_0, A_1, A_2, A_3 in general position and all points incident with the lines $A_0 A_1$, $A_0 A_2$, $A_1 A_2$, and

collineations $f: \mathcal{C} \rightarrow PG(2, K')$ such that $f(A_i)$, $i=0, 1, 2, 3$ have general position in $PG(2, K')$; no assumption about injectivity will be made. We shall show that the representation (1.2) can be extended for this case. When K is a field, e will become a valuation homomorphism, while in the general case e will be a certain generalization of this concept.

Now we recall some basic facts on valuations of division rings [12]. Let G be a fully (simply) ordered (not necessarily commutative) group with the neutral element e , K a division ring and K^* its multiplicative group. For the subsequent discussion it is convenient to introduce *the completed division ring* $\bar{K} = K \cup \{\infty\}$ by

$$\begin{aligned} x + \infty &= \infty + x = \infty && \text{for every } x \in K \\ x \cdot \infty &= \infty \cdot x = \infty && \text{for every } x \in K^* \cup \{\infty\} \end{aligned}$$

(the symbols $\infty + \infty$, $\infty \cdot 0$, $0 \cdot \infty$ remain undefined).

DEFINITION I. A map $v: K^* \rightarrow G$ is called a *valuation* if

- 1) $v(xy) = v(x) + v(y)$, $\forall x, y \in K^*$
- 2) $v(x+y) \geq \min[v(x), v(y)]$, if $x, y \in K^*$ and $y \neq -x$.

DEFINITION II. A subring O of K is called a *valuation ring* if

- 1) $x \in K^*$ implies $x \in O$ or $x^{-1} \in O$
- 2) O is an invariant subset of K , i.e. we have $x^{-1}Ox = O$ for $x \in K^*$.

DEFINITION III. A map $\varphi: \bar{K} \rightarrow \bar{K}'$, where $\bar{K}$ and $\bar{K}'$ are completed division rings, is called a *valuation homomorphism* if

- 1) $\varphi(\bar{K}) = \bar{K}'$
- 2) $\varphi(x+y) = \varphi(x) + \varphi(y)$, provided the two sums are defined
- 3) $\varphi(xy) = \varphi(x) \varphi(y)$, provided the two products are defined
- 4) $\varphi^{-1}(K')$ is an invariant subset of K .

THEOREM A. If v is a valuation of the division ring K , then $O_v = \{x \mid x \in K^*, v(x) \geq e \text{ or } x=0\}$ is a valuation ring, which is termed as the valuation ring of v . If O is a valuation ring in K , there exists a valuation v , and without regard to isomorphisms, only one such that $O_v = O$.

THEOREM B. Let v be a valuation of the division ring K and O_v the valuation ring of it. Then the set $P = \{x \mid x \in O_v, v(x) > e \text{ or } x=0\}$ is a prime ideal of the ring O_v , whose residue class ring O_v/P is a division ring. Denote by φ_v the extension on $\bar{K}$ of the canonical ring homomorphism $O_v \rightarrow O_v/P$ by putting $\varphi_v(x) = \infty$ for $x \in \bar{K} \setminus O_v$. The map $\varphi_v: \bar{K} \rightarrow \bar{O_v/P}$ is a valuation homomorphism called the canonical valuation homomorphism

of both v and O_v . Conversely, if $\varphi: \bar{K} \rightarrow \bar{K}'$ is a valuation homomorphism, then $\varphi^{-1}(K')$ is a valuation ring and, apart from isomorphism, φ coincides with the canonical valuation homomorphism of $\varphi^{-1}(K')$.

For the trivial valuation, defined by $v(x)=0$ for every $x \in K^*$, we get $O_v = K$ and φ_v is the identical map.

The non-injective collineations will lead us to mappings more general than the valuation homomorphisms considered in Definition III; preserving hypotheses 1)–3) they may not satisfy hypothesis 4) of Definition III.

We want to generalize correspondently also the concepts of valuation and valuation ring in such a way that Theorems A and B could be extended. This can be done in obvious manner for the valuation ring; results and most of the proofs can be carried over even from the commutative case (see e.g. [4], where also a list of commutative valuation theory references is to be found). The generalization of the concept of valuation is still not so immediate. We wish to emphasize that no other structure will be needed in the image than the full ordering. In his fundamental paper [7] W. Krull was introducing the concept of valuation of fields as in Definition I by giving up the field structure of the image set, which had been supposed in the previous papers. He was keeping only the group structure and ordering. Now we are giving up the group structure too preserving only the ordering.

We shall construct an example for the generalized valuation, which does not reduce to a valuation. Thereby we shall have a proof for the independence of postulate 4) from the others in definition III.

Extension of valuation in another direction in which commutativity is preserved is given in [8].

§ 2. Extension of noncommutative valuations

DEFINITION 1. A subring O of the division ring K is said to be *total* in K , if either $x \in O$ or $x^{-1} \in O$ for every $x \in K^*$.

DEFINITION 2. Let $\bar{K}$ and $\bar{K}'$ be completed division rings. A map $\psi: \bar{K} \rightarrow \bar{K}'$ is called a *homomorphism of the completed division ring $\bar{K}$* if

$$\psi(x + y) = \psi(x) + \psi(y), \text{ provided the two sums are defined} \quad (2.1)$$

$$\psi(xy) = \psi(x)\psi(y), \text{ provided the products are defined} \quad (2.2)$$

$$\psi(K) \text{ contains at least 3 different elements.} \quad (2.3)$$

DEFINITION 3. Let O be a total subring in the division ring K and K' a division

ring again. A map $\varphi: O \rightarrow K'$ is called a *partial homomorphism* of K if

$$\varphi(x+y) = \varphi(x) + \varphi(y), \forall x, y \in O \quad (2.4)$$

$$\varphi(xy) = \varphi(x) \varphi(y), \forall x, y \in O \quad (2.5)$$

$$\varphi(x) = 0 \quad \text{when} \quad x \in O \cap K^* \quad \text{and} \quad x^{-1} \notin O \quad (2.6)$$

$$\varphi(O) \text{ contains at least one element different from } 0. \quad (2.7)$$

LEMMA 1. If $\psi: \bar{K} \rightarrow \bar{K}'$ is a homomorphism of the completed division ring $\bar{K}$, then

$$\psi(0) = 0, \quad \psi(\infty) = \infty, \quad \psi(1) = 1.$$

Proof. Choose $a \in K$ such that $\psi(a) = b \notin \{0, \infty\}$. Then $\psi(a+0) = \psi(a) + \psi(0)$, because the two sides are certainly defined. Thus $\psi(0) = 0$.

We have $a \neq \infty$, for $a = \infty$ would lead us to $\psi(x+\infty) = \psi(x) + \psi(\infty)$ for each $x \neq \infty$, from which we would deduce $b = \psi(x) + b$, that is $\psi(x) = 0$, contradicting (2.3). Then $\psi(a+\infty) = \psi(a) + \psi(\infty)$ holds and we get $\psi(\infty) = \infty$.

Finally write relation (2.2) for $x=a, y=1$.

LEMMA 2. If $\psi: \bar{K} \rightarrow \bar{K}'$ is a homomorphism of the completed division ring $\bar{K}$ and $x \in K^*$, then

$$\psi(x) = 0 \Leftrightarrow \psi(x^{-1}) = \infty$$

Proof. Assume $\psi(x) = 0, \psi(x^{-1}) \neq \infty$; then xx^{-1} and $\psi(x)\psi(x^{-1})$ are defined, consequently

$$\psi(xx^{-1}) = \psi(x) \cdot \psi(x^{-1}).$$

Hence $\psi(1) = 0$ in contradiction to Lemma 1. Similarly the assumption $\psi(x) \neq 0, \psi(x^{-1}) = \infty$ is also contradictory.

THEOREM 1. If $\psi: \bar{K} \rightarrow \bar{K}'$ is a homomorphism of the completed division ring $\bar{K}$, then $O_\psi = \{x \mid \psi(x) \in K'\}$ is a total subring in K . The restriction $\tilde{\psi} = \psi \mid O_\psi$ is a partial homomorphism of K .

Proof. $\psi(x) \in K', \psi(y) \in K'$ imply $\psi(x+y) = \psi(x) + \psi(y) \in K', \psi(xy) = \psi(x)\psi(y) \in K'$; and we can observe that $\psi(-x) = -\psi(x) \in K'$. Therefore O_ψ is a ring, and in view of Lemma 2 it is total in K . $\tilde{\psi}$ is clearly a partial homomorphism of K .

It is easily seen:

LEMMA 3. $\psi(\bar{K})$ is a completed division ring.

THEOREM 2. *Every partial homomorphism of the division ring K can be extended to a homomorphism of the corresponding completed division ring $\bar{K}$.*

Proof. First we establish two properties of a partial homomorphism $\varphi: O \rightarrow K'$, where O is a total subring in the division ring K and K' a division ring.

$$i) \quad 1 \in O, \quad \varphi(1) = 1, \quad \varphi(0) = 0.$$

For $1 \notin O$ implies $1^{-1} = 1 \in O$; therefore we have $1 \in O$. Choosing $a \in O$ such that $\varphi(a) \neq 0$, from $\varphi(a \cdot 1) = \varphi(a) \varphi(1)$ we conclude that $\varphi(1) = 1$. Since $\varphi(1 + 0) = \varphi(1) + \varphi(0)$, we get $\varphi(0) = 0$.

We denote

$$U = \{x \mid x \in O, x^{-1} \in O\} \\ N = \{x \mid x \in O, x^{-1} \notin O \text{ or } x = 0\};$$

the set U of all units of O is clearly a subgroup of the multiplicative group K^* ; the elements of N are called the non-units of O .

$$ii) \quad \begin{aligned} x \in N &\Rightarrow \varphi(x) = 0 \\ x \in U &\Rightarrow \varphi(x) \neq 0. \end{aligned}$$

The first part repeats condition (2.6) of Definition 3 and $\varphi(0) = 0$. In order to show the second part assume $x \in U$ and $\varphi(x) = 0$. Then $1 = \varphi(1) = \varphi(x \cdot x^{-1}) = \varphi(x) \varphi(x^{-1}) = 0 \cdot \varphi(x^{-1}) = 0$. The contradiction shows property ii).

Now we set

$$\psi(x) = \begin{cases} \varphi(x) & \text{for } x \in O \\ \infty & \text{for } x \in \bar{K} \setminus O \end{cases}$$

and wish to show that ψ is a homomorphism of the completed division ring $\bar{K}$ into the completed division ring $\bar{K}'$.

In order to prove relation (2.1) we distinguish the following cases: 1) $x \in O, y \in O$; then relation (2.1) holds by (2.4). 2) $x \in O, y \in \bar{K} \setminus O$; then $x + y \in \bar{K} \setminus O$, because otherwise, it would follow $y = (x + y) + (-x) \in O + O = O$. Hence the two sides of (2.1) are ∞ . 3) $x \in \bar{K} \setminus O, y \in O$ is similar to the above case. 4) $x \in \bar{K} \setminus O, y \in \bar{K} \setminus O$; the right side of (2.1) is undefined.

Next we prove relation (2.2). We distinguish some cases again: 1) $x, y \in O$; relation (2.2) is valid by (2.5). 2) $x \in N, y \in \bar{K} \setminus O$; then $\psi(x) \psi(y)$ is undefined. 3) $x \in K \setminus N, y \in \bar{K} \setminus O$; then we have $xy \notin O$, for otherwise we should find $y = x^{-1}(xy) \in O \cdot O = O$; the two sides of (2.2) are ∞ . 4) $x \in \bar{K} \setminus O, y \in K$; analogous to cases 2) and 3). 5) $x = \infty, y = \infty$; (2.2) is obvious.

Since $0, 1, \infty \in \psi(K)$, the proof is completed.

LEMMA 4. *If O is a total subring of the division ring K , then the set N of all non-units in O is a maximal ideal in O .*

Proof. Denoting as above by U the set of all units of O , we have

$$\begin{aligned} x \in U, \quad y \in U &\Rightarrow xy \in U \\ x \in N, \quad y \in O &\Rightarrow xy \in N \\ x \in O, \quad y \in N &\Rightarrow xy \in N. \end{aligned}$$

The first implication is clear. We have only to prove the second. We divide our discussion into two cases: 1) $x \in N, y \in U$; then $xy = z \in U$ would imply $x = zy^{-1} \in U$, contradicting the assumption $x \in N$. 2) $x \in N, y \in N$; it is no restriction to suppose $x \neq 0, y \neq 0$; then $xy \in U$ would imply $v = y^{-1}x^{-1} \in U$, that is, $y^{-1} = vx \in O$, contradicting the assumption $y \in N$.

Next we show that $x \in N, y \in N \Rightarrow x + y \in N$. For we have either $x^{-1}y \in O$ or $y^{-1}x \in O$. Supposing the first instance we get

$$x + y = x(1 + x^{-1}y) \in N \cdot (1 + O) = N \cdot O = N.$$

Finally suppose that A is a right ideal different from N with $N \subset A \subset O$. Then there is an element $u \in U \cap A$. Consequently $uu^{-1} = 1 \in A$ and thus $A = O$. We argue similarly, when A is supposed to be a left ideal.

COROLLARY. *The residue class ring O/N is a division ring.*

THEOREM 3. *If O is a total subring of the division ring K , then there exists a homomorphism ψ of the completed division ring $\bar{K}$ such that $O_\psi = O$. Up to isomorphisms, ψ is uniquely determined.*

Proof. The canonical map $O \rightarrow O/N$ is a partial homomorphism of K , which can be extended by Theorem 2 to a homomorphism ψ of the completed division ring $\bar{K}$. Clearly $O_\psi = O$.

Now let $\psi': \bar{K} \rightarrow \bar{K}'$ be a surjective homomorphism of $\bar{K}$ such that $O_{\psi'} = O$. We get $\psi'(x) = \psi'(y) \Leftrightarrow \psi(x - y) = 0 \Leftrightarrow (x - y) \in N$, if we use property ii) given in the proof of Theorem 2. Thus $\psi'^{-1}(\xi)$ is a coset of the ideal N in O for all $\xi \in K'$ and the obvious relations $\psi'^{-1}(\xi + \eta) = \psi'^{-1}(\xi) + \psi'^{-1}(\eta)$, $\psi'^{-1}(\xi\eta) = \psi'^{-1}(\xi) \cdot \psi'^{-1}(\eta)$ complete the proof.

Now we wish to extend the concept of valuation.

DEFINITION 4. Let K be a division ring and S a fully ordered set containing two elements at least. A map $v: K \rightarrow S$ is called a *generalized valuation* of K if

$$x, y, z \in K, v(x) \leq v(y) \Rightarrow v(xz) \leq v(yz) \quad (2.8)$$

$$v(x + y) \geq \min(v(x), v(y)), \quad \forall x, y \in K \quad (2.9)$$

$$v(K) = S. \quad (2.10)$$

Remarks. a) Let a be a fixed element in S . We have either $a \leq v(0)$ or $v(0) \leq a$. Assuming the first case, let b be any element of S ; we have $a = v(x)$, $b = v(y)$ for some $x, y \in K$; the inequality $v(x) \leq v(0)$ implies by (2.8) $v(xx^{-1}y) \leq v(0)$, that is, $b \leq v(0)$. Consequently there exists in this case a greatest element in S equal to $v(0)$. In a similar manner, we may show that in the other case there exists a smallest element in S equal to $v(0)$. In view of (2.9) we have the first instance and denote $v(0) = \infty$.

b) We have $v(x) \neq \infty$ for $x \neq 0$. For assume that there exists $a \in K^*$ such that $v(a) = \infty$; then from $v(0) \leq v(a)$ we deduce $v(0) \leq v(aa^{-1}b)$, that is, $v(0) \leq v(b)$ for all $b \in K$. Thus $v(b) = \infty$, contradicting the fact that S contains at least two elements.

c) Any valuation v of K is a generalized valuation of it. For $v(x) \leq v(y)$ implies $v(xz) = v(x)v(z) \leq v(y)v(z) = v(yz)$.

d) If $v: K \rightarrow S$ satisfies conditions (2.8), (2.9), (2.10) and

$$v(x) \leq v(y) \Rightarrow v(zx) \leq v(zy), \quad (2.8)$$

then v is a valuation of K . For it is easily seen that

$$v(x) = v(x'), v(y) = v(y') \Rightarrow v(xy) = v(x'y');$$

for this reason we may define a multiplication in S by the formula

$$v(x) \cdot v(y) = v(xy);$$

thereby $S \setminus \{\infty\}$ becomes a fully ordered group with the neutral element $v(1)$.

e) $v(x) = v(-x)$. For we have either $v(x) \leq v(-x)$ or $v(-x) \leq v(x)$. In the first instance by using condition (2.8) we get $v(-x) \leq v(x)$, consequently $v(x) = v(-x)$. In the second instance we obtain the same result.

f) If $v(x) \neq v(y)$, the relation (2.9) holds with equality sign. In fact, supposing without loss of generality $v(x) > v(y)$, if we had $v(x+y) > \min(v(x), v(y))$, then in view of $v(x+y) > v(y)$ we should get the contradiction $v(y) = v(x+y-x) \geq \min[v(x+y), v(-x)] > v(y)$.

THEOREM 4. *If v is a generalized valuation of the division ring K , then the set $O_v = \{x \mid x \in K, v(x) \geq v(1)\}$ is a total subring of K . Conversely, if O is a total subring of K , then there exists a generalized valuation v of K such that $O_v = O$. Moreover, v is uniquely determined by it up to order preserving bijections.*

Proof. If $x \in O_v$, $y \in O_v$, then $v(x+y) \geq \min(v(x), v(y)) \geq v(1)$, that is, $x+y \in O_v$. Furthermore, from $v(x) \geq v(1)$ we get by (2.8) $v(xy) \geq v(y) \geq v(1)$, hence $xy \in O_v$. By remark e) we have $-1 \in O_v$, and thus $x-y \in O_v$. Consequently O_v is a ring. In order to show that O_v is total in K , we take $x \in K \setminus O_v$. Then $v(x) < v(1)$, which implies by (2.8) $v(xx^{-1}) \leq v(x^{-1})$. Thus $v(x^{-1}) \geq v(1)$, that is, $x^{-1} \in O_v$.

Now let O be a total subring of K . We denote by U the set of all units of O

$$U = \{x \mid x \in O, \quad x^{-1} \in O\}.$$

We have $U \cdot U = U$ and $U^{-1} = U$, that is, U is a (not necessarily invariant) subgroup of the multiplicative group K^* . Let S^* be the set of the right cosets of U in K^* and $S = S^* \cup \{\infty\}$. Define $v: K \rightarrow S$ by $v(x) = Ux$ when $x \in K^*$ and $v(0) = \infty$, and put

$$v(x) \leq v(y) \text{ if and only if } yx^{-1} \in O \quad (2.11)$$

This definition is correct, for by a change of the representatives of the cosets Ux , Uy we get from $yx^{-1} \in O$ the following:

$$\begin{aligned} v(x) &= v(x'), v(y) = v(y'), x' = ux, y' = u_1y, u, u_1 \in U \\ y'x'^{-1} &= u_1yx^{-1}u^{-1} \in UOU = O. \end{aligned}$$

We have the following properties:

- i) $v(x) \leq v(0)$ for all $v(x) \in S^*$.
- ii) $v(x) \leq v(y)$, $v(y) \leq v(x)$ imply $v(x) = v(y)$.

In fact, since $yx^{-1} \in O$ and $xy^{-1} \in O$, we obtain $yx^{-1} \in U$. It follows that $y \in Ux = v(x)$, hence $v(y) = v(x)$.

iii) At least one of the relations $v(x) \leq v(y)$, $v(y) \leq v(x)$ always holds. For either $yx^{-1} \in O$ or $(yx^{-1})^{-1} = xy^{-1} \in O$.

- iv) $v(x) \leq v(y)$, $v(y) \leq v(x)$ imply $v(z) \leq v(z)$.

For, since $yx^{-1} \in O$, $zy^{-1} \in O$, we have $zx^{-1} = zy^{-1}yx^{-1} \in O$.

Thus (2.11) defines a full ordering on S .

v) $v(x) \leq v(y)$ implies $v(xz) \leq v(yz)$. In fact, from $yx^{-1} \in O$ we deduce $(yz) \times (xz)^{-1} \in O$.

vi) $v(x+y) \geq \min(v(x), v(y))$. In fact, it is no restriction to assume $v(x) \leq v(y)$; since $yx^{-1} \in O$, $1 \in O$, we get $1+yx^{-1} \in O$ and therefore $(x+y)x^{-1} \in O$. Consequently we can write by (2.11) $v(x) \leq v(x+y)$.

Thus v is a generalized valuation of K and clearly $O_v = O$.

Finally consider an arbitrary generalized valuation $w: K \rightarrow S'$ satisfying the condition $O_w = O$. It is easily seen that $w(x) = w(1) \Leftrightarrow x \in U$. Therefore we obtain by (2.8) $w(x) = w(y) \Leftrightarrow xy^{-1} \in U$. For this reason we get a bijective map $\chi: S' \rightarrow S$ if we set $\chi(w(x)) = v(x)$. Since this map preserves the ordering, our proof is completed.

§ 3. An example

Now we are going to construct an example for a non-invariant total subring in a division ring. This induces by Theorem 4 a generalized valuation which, in view of Theorem A, does not reduce to a valuation in ordinary sense.

We shall use formal power series [9]. Let G be a non-commutative fully ordered group and k a division ring. If $\varphi: G \rightarrow k$ is a map, we write $D(\varphi) = \{x \mid \varphi(x) \neq 0\}$. We consider the following set of maps

$K(G, k) = \{\varphi: G \rightarrow k \mid D(\varphi) \text{ is a well-ordered set in the full ordering of } G\}$
and define addition and multiplication in $K(G, k)$ according to the formulae

$$\begin{aligned}(\varphi + \psi)(x) &= \varphi(x) + \psi(x) \\ (\varphi\psi)(x) &= \sum_{\xi\eta=x} \varphi(\xi)\psi(\eta)\end{aligned}$$

(it is proved in [9] that there is only a finite number of non-zero terms under the summation sign). Thereby $K(G, k)$ becomes a division ring, and

$$O(G, k) = \{\varphi \mid \varphi \in K(G, k), g \geq e \text{ for all } g \in D(\varphi)\}$$

is a valuation ring of it; the corresponding valuation is given by

$$v(\varphi) = \min D(\varphi).$$

Now we shall choose G in a particular manner. Let F be an ordered nonarchimedean field, e.g. the field $\mathbb{R}(t)$ of the rational functions over the field of the real numbers with the ordering given by

$$\frac{a_0 t^n + a_1 t^{n-1} + \dots + a_n}{b_0 t^m + b_1 t^{m-1} + \dots + b_m} > 0 \Leftrightarrow a_0 b_0 > 0.$$

We denote by Γ the group of the linear transformations of F

$$\forall x \in F, \quad x \rightarrow ax + b \quad (a, b \in F, a > 0),$$

ordered lexicographically. An element of Γ can be written in the form (a, b) , where $a, b \in F, a > 0$. The group operation is given by

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, a_2 b_1 + b_2),$$

the neutral element is $e = (1, 0)$, and

$$(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow \text{either } a_1 < a_2 \text{ or } a_1 = a_2, \quad b_1 \leq b_2.$$

It is easily verified that $x_1 \leq x_2 \Rightarrow x x_1 \leq x x_2, x_1 x \leq x_2 x$ for elements x, x_1, x_2 of Γ .

We have in the division ring $K(\Gamma, k)$ the invariant total subring $O(\Gamma, k)$. Denote

$$\Delta = \{x \mid x \in \Gamma, \exists r \in \mathbf{R}, x \geq (1, r)\}.$$

If $x_1, x_2 \in \Delta$, $x_1 x_2 \geq (1, r_1) \cdot (1, r_2) = (1, r_1 + r_2)$, hence $\Delta \cdot \Delta \subset \Delta$. On the other hand Δ is not an invariant set in Γ . For, if $(a, b) \in \Gamma$, then $(a, b)^{-1} = (a^{-1}, -a^{-1}b)$ and

$$(a, b)^{-1}(1, -1)(a, b) = (1, -a).$$

Selecting an element $a \in F$ such that $r < a$ for all $r \in \mathbf{R}$ (in the case of $F = \mathbf{R}(t)$ we may take $a = t$), we obtain a transform of the element $(1, -1) \in \Delta$ which does not belong to Δ . We consider now the following subset of $K(\Gamma, k)$

$$O_1 = \{\varphi \mid \varphi \in K(\Gamma, k), D(\varphi) \subset \Delta\}.$$

Obviously $O_1 + O_1 \subset O_1$ and $\varphi \in O_1 \Rightarrow -\varphi \in O_1$. Since $\Delta \cdot \Delta \subset \Delta$, we get $O_1 \cdot O_1 \subset O_1$. Hence O_1 is a subring in $K(\Gamma, k)$. But O_1 contains $O(\Gamma, k)$, thus O_1 is a total subring in $K(\Gamma, k)$. We assign to each $x \in \Gamma$ the function $\varphi_x \in K(\Gamma, k)$ defined in this way:

$$\begin{cases} \varphi_x(x) = 1 \\ \varphi_x(y) = 0, \quad \text{when } y \in \Gamma \setminus \{x\}. \end{cases}$$

Selecting $a \in F$ as above, we have $\varphi_{(1, -1)} \in O_1$ and

$$\varphi_{(a, 0)}^{-1} \varphi_{(1, -1)} \varphi_{(a, 0)} = \varphi_{(1, -a)} \notin O_1;$$

therefore the total subring O_1 is not invariant.

§ 4. Collineations on some sets in Desarguesian projective planes

We shall need the following lemmas.

LEMMA 5. *In a completed division ring $\bar{K}$, the associative laws and the distributive law hold, provided all expressions occurring in the respective law are defined. Moreover, if we set*

$$0^{-1} = \infty, \quad \infty^{-1} = 0 \quad (4.1)$$

we have in the case where xy is defined

$$(xy)^{-1} = y^{-1}x^{-1}. \quad (4.2)$$

In fact, we have to verify the distributive law

$$(x + y)z = xz + yz \quad (4.3)$$

only in the following cases: 1) $x, y, z \in K$, 2) $x = \infty, y, z \in K, z \neq 0$, 3) $y = \infty, x, z \in K, z \neq 0$, for in all other cases an undefined expression occurs. But in these three cases (4.3) is obviously true. Similarly one verifies the other laws.

LEMMA 6. *Let O be a total subring of the division ring K , U the set of the units in O and $a_i \in K^*$, $i=1, 2, \dots, n$. Then we can choose $q \in K^*$ so that $qa_i \in O$ for every $i=1, 2, \dots, n$ and $qa_j \in U$ for at least one j . If q_0 is such a q , then the set of all q of this kind is given by*

$$q = \tau q_0, \quad \tau \in U. \quad (4.4)$$

Proof. Without loss of generality we can assume that $a_j \in U$ for some j (for if it is necessary we multiply the a_i by a_1^{-1}). We denote

$$I = \{i \mid a_i \in O\}$$

$$I' = \{i \mid a_i \notin O\}.$$

If I' is empty, the proof is achieved. If I' is non empty, we choose $k \in I'$ and multiply all of the a_i by a_k^{-1} . Since $a_k^{-1} \in O$, we have $a_k^{-1}a_i \in O$ for $i \in I \cup \{k\}$. Among $a_k^{-1}a_1, \dots, a_k^{-1}a_n$ there is a larger number of elements lying in O than the number of elements of I , and we have also $a_k^{-1}a_k = 1 \in U$. Repeating this process we finish by proving the first part of Lemma 6. If $\tau \in U$, then $q_0 a_i \in O$ implies $\tau q_0 \cdot a_i \in O$ and $q_0 a_j \in U$ implies $\tau q_0 a_j \in U$. Conversely, let q satisfy the requirements of the first part of Lemma 6. Then, by selecting j such that $q_0 a_j \in U$, we get from $qa_j \in O$

$$\tau = q q_0^{-1} = q a_j (q_0 a_j)^{-1} \in O.$$

If we had $\tau \in O \setminus U = N$, it would follow by Lemma 4 that $\tau(q_0 a_i) \in N \cdot O = N$, contradicting the fact there exists k such that $\tau q_0 a_k = q a_k \in U$. Therefore we have $\tau \in U$ and Lemma 6 is proved.

In general, we say that a map $F: A \rightarrow B$ is *injective at a point* $a \in A$ if $F(x) = F(a)$ implies $x = a$.

LEMMA 7. *If the partial homomorphism $\varphi: O \rightarrow K'$ of the division ring K is injective at a single point $a \in O$, then $O = K$ and φ is an isomorphism of K onto a sub-division-ring of K' .*

Proof. Suppose $x \neq 0$ and $\varphi(x) = 0$. It follows:

$$\varphi(a + x) = \varphi(a) + \varphi(x) = \varphi(a),$$

whence a contradiction with the injectivity at a . Consequently $N = \{x \mid \varphi(x) = 0\} = \{0\}$. If $x \in K \setminus O$ then $x^{-1} \in O \setminus \{0\}$ and (2.6) yields $\varphi(x^{-1}) = 0$, a contradiction. Thus $K = O$. It is easily seen that φ is injective on K .

Now we state our main theorem.

THEOREM 5. *Let K, K' be division rings, $\mathcal{C}$ a set in the projective plane $PG(2, K)$ containing four points A_0, A_1, A_2, A_3 in general position and all points incident with the*

lines A_0A_1 , A_0A_2 , A_1A_2 , and f a collineation from $\mathcal{C}$ into the projective plane $PG(2, K')$ such that the points $f(A_i)$ have general position. Choosing A_0, A_1, A_2, A_3 ; $f(A_0)$, $f(A_1)$, $f(A_2)$, $f(A_3)$ as reference frames for projective coordinate systems in $PG(2, K)$ and $PG(2, K')$, the collineation f is given by

$$K(x_1, x_2, x_3) \rightarrow K'(\varphi(\varrho x_1), \varphi(\varrho x_2), \varphi(\varrho x_3)), \quad (4.5)$$

where $\varphi: O \rightarrow K'$ is a partial homomorphism of K and $\varrho \in K$ is determined according to Lemma 6 such that $\varrho x_i \in O$, $i=1, 2, 3$ and $\varrho x_i \in U$ for at least one i (U being the set of units of O).

Proof. We represent a point P lying on the line A_0A_1 in the form $(x, 0, 1)$ even when P coincides with A_1 setting in this case $x = \infty$. Similarly $y \rightarrow (0, y, 1)$ and $z \rightarrow (1, -z, 0)$ are bijections from the completed division ring $\bar{K}$ onto the lines A_0A_2 , A_1A_2 respectively.

In order that the points $P(x, 0, 1)$, $Q(0, y, 1)$, $R(1, -z, 0)$ be collinear it is necessary and sufficient that

$$\text{either } y = xz \quad \text{or} \quad xz \text{ undefined.} \quad (4.6)$$

As a matter of fact, in the case when $x, y, z \in K$, a simple computation shows the statement. In the case $x = \infty$, we have $P = A_1$ and the collinearity of P, Q, R means $Q = A_2$ or $R = A_1$, that is, $y = \infty$ or $z = 0$, what is equivalent to (4.6). In the other cases we conclude similarly.

Let us denote the images of the points $P(x, 0, 1)$, $Q(0, y, 1)$, $R(1, -z, 0)$ under the collineation f by $(\psi_1(x), 0, 1)$, $(0, \psi_2(y), 1)$, $(1, -\psi(z), 0)$ respectively. We have

$$\psi(0) = 0, \quad \psi(\infty) = \infty \quad (4.7)$$

and

$$y = xz \Rightarrow \begin{cases} \text{either } \psi_2(y) = \psi_1(x) \cdot \psi(z) \\ \text{or } \psi_1(x) \cdot \psi(z) \text{ undefined.} \end{cases} \quad (4.8)$$

Since the point $A_0A_1 \cap A_2A_3$ is mapped in $f(A_0)f(A_1) \cap f(A_2)f(A_3)$, we deduce $(1, 0, 1) \xrightarrow{f} (1, 0, 1)$. We obtain in a similar manner $(0, 1, 1) \xrightarrow{f} (0, 1, 1)$ and further $(1, -1, 0) \xrightarrow{f} (1, -1, 0)$. Thus

$$\psi_1(1) = 1, \quad \psi_2(1) = 1, \quad \psi(1) = 1. \quad (4.9)$$

Substituting in (4.8) $x=1$ and then $z=1$, we obtain

$$\psi_1 = \psi_2 = \psi.$$

It follows from (4.8)

$$\psi(xy) = \psi(x) \cdot \psi(y) \text{ provided } \psi(x) \psi(y) \text{ is defined.} \quad (4.10)$$

(It is not necessary to specify that xy is defined, for if it is not, then $\psi(x)\psi(y)$ is surely undefined too).

Setting in (4.10) $y=x^{-1}$, we have either $\psi(x)\psi(x^{-1})=1$ or $\psi(x)\psi(x^{-1})$ is undefined, whence $\psi(x^{-1})=[\psi(x)]^{-1}$ even in the case $\psi(x)=0$ or ∞ , for otherwise, $\psi(x)\psi(x^{-1})$ would be defined and clearly different from 1. Thus

$$\psi(x^{-1})=[\psi(x)]^{-1} \quad \text{for } x \in \bar{K}. \quad (4.11)$$

The points $P(x, 0, 1)$, $Q(0, y, 1)$, $A_3(1, 1, 1)$ are collinear if and only if

$$\text{either } x^{-1} + y^{-1} = 1 \quad \text{or } x^{-1} + y^{-1} \text{ is undefined.} \quad (4.12)$$

When P differs from A_0 and A_1 we may easily verify this statement. When $P=A_0$, then collinearity of P, Q, A_3 means $P=Q$ and condition (4.12) reduces to its second part, that is $y=0$. When $P=A_1$ collinearity of P, Q, A_3 says $Q=K(0, 1, 1)$, that is $y=1$ and (4.12) becomes the same.

By assumption collinear points $P(x, 0, 1)$, $Q(0, y, 1)$, $A_3(1, 1, 1)$ are mapped in collinear points, consequently

$$x^{-1} + y^{-1} = 1 \Rightarrow \begin{cases} \text{either } [\psi(x)]^{-1} + [\psi(y)]^{-1} = 1 \\ \text{or } [\psi(x)]^{-1} + [\psi(y)]^{-1} \text{ is undefined} \end{cases}$$

or, in view of (4.11)

$$x^{-1} + y^{-1} = 1 \Rightarrow \psi(x^{-1}) + \psi(y^{-1}) = 1 \text{ or undefined.}$$

Now we can easily see

$$\psi(1-x) = 1 - \psi(x) \quad \text{for } x \in \bar{K}.$$

A new change in notation and use of (4.10) yields

$$\psi(1+x) = 1 + \psi(x) \quad \text{for } x \in \bar{K}. \quad (4.13)$$

Now we are able to show

$$\psi(x+y) = \psi(x) + \psi(y), \text{ provided the right hand side is defined.} \quad (4.14)$$

Indeed, if $\psi(x)=\psi(y)=\infty$ we have nothing to show. Suppose, say, $\psi(x) \neq \infty$. We can also suppose without loss of generality that $\psi(x) \neq 0$. Notice $x \neq 0, \infty$. Using Lemma 5 and relations (4.10), (4.11), (4.13), we get

$$\begin{aligned} \psi(x+y) &= \psi(x(1+x^{-1}y)) = \psi(x)\psi(1+x^{-1}y) = \\ &= \psi(x)(1+\psi(x^{-1}y)) = \psi(x)(1+(\psi(x))^{-1}\psi(y)) = \\ &= \psi(x) + \psi(x)(\psi(x))^{-1}\psi(y) = \psi(x) + \psi(y). \end{aligned}$$

Thus, it is proved, in view of (4.7), (4.9), (4.10), (4.14), that $\psi: \bar{K} \rightarrow \bar{K}$ is a homomorphism of the completed division ring $\bar{K}$. Denote by O the corresponding total

subring in K , by φ the restriction $\psi|_O$ and consider the map (4.5), say Φ , which can be regarded as defined on the whole plane $PG(2, K)$. Observe that the image of a point under this map does not depend on the choice of ϱ among its possible values.

Φ is a collineation. For consider a point $(\varrho x_1, \varrho x_2, \varrho x_3)$ lying on a line $(u_1 \varrho', u_2 \varrho', u_3 \varrho')$; then

$$\sum_{i=1}^3 (\varrho x_i) (u_i \varrho') = 0.$$

Choose ϱ, ϱ' according to Lemma 6 (which is evidently valid for right multipliers too); then we have

$$\sum_{i=1}^3 \varphi(\varrho x_i) \varphi(u_i \varrho') = 0,$$

where not all $\varphi(u_i \varrho')$ are 0. That is, points on a line are mapped in a line.

It is easily seen that on the points incident with the sides of the triangle $A_0 A_1 A_2$ and on A_3 we have

$$f = \Phi. \quad (4.15)$$

But then this equality holds on each point P of $\mathcal{C}$. In fact, it is sufficient to consider two lines through P which intersect $A_0 A_1$ in points mapped by f in different images and to observe that these lines have the same images by both f and Φ . It follows that $f(P) = \Phi(P)$.

COROLLARIES. 1) *A collineation $f: \mathcal{C} \rightarrow PG(2, K')$ as in the statement of Theorem 5 can be uniquely imbedded in a collineation from $PG(2, K)$ into $PG(2, K')$.*

2) *If $f: \mathcal{C} \rightarrow PG(2, K')$ satisfies the hypotheses of Theorem 5 and is injective at a single point of $A_0 A_1 \cup A_0 A_2 \cup A_1 A_2$, then we have (4.5) with an isomorphism φ from K into K' .*

Indeed, we can apply Lemma 7 to φ .

The fact that the injectivity of f at a single point implies the injectivity everywhere is proved in [11] under more general assumptions.

§ 5. A property characterizing collineations on a Desarguesian projective plane

In this paragraph we give a consequence to the above theorem.

Collineations on a projective plane are usually defined as maps which transform the points on an *arbitrary* line in points on a line again. Is it actually necessary to require this property on each line? or is there such a family of lines that, if the points of a line belonging to this family are taken in collinear points, then the same will happen to all of the lines? The following theorem yields a positive answer to the second question for Desarguesian planes.

THEOREM 6. *Let $\mathcal{L}$ be a family of lines in a plane $PG(2, K)$ consisting of three pencils with non collinear centers O_1, O_2, O_3 and a line l more. Further let f be a map from the point set of $PG(2, K)$ into the point set of the plane $PG(2, K')$, which transforms three points on a line belonging to $\mathcal{L}$ in three collinear points and also suppose that $f(O_i), i=1, 2, 3$ are non collinear and the points of l are mapped in points of a line not incident with any $f(O_i)$. Then f maps every three collinear points in collinear points again, whence f is a (not necessarily injective) collineation defined on the plane $PG(2, K)$.*

Proof. Consider dual transforms of both planes. Then we have a map of all lines of one plane in the lines of another and a point set $\tilde{\mathcal{L}}$ with the property that lines going through a point of $\tilde{\mathcal{L}}$ are taken into concurrent lines. $\tilde{\mathcal{L}}$ is composed of the points incident with the sides of a triangle and a point more. Our map induces a map of the points of $\tilde{\mathcal{L}}$, which is a collineation satisfying the hypotheses of Theorem 5 with $\mathcal{C} = \tilde{\mathcal{L}}$. The above Corollary 1 yields Theorem 6.

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