

ON THE APPROXIMATION BY LINEAR POSITIVE OPERATORS

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1. INTRODUCTION

This paper is concerned with some properties of linear positive operators which are in analogy with the well-known Bernstein operator. Let $\mathfrak{F}_j(I_j)$, $j = 1, 2$, denote two linear spaces of real functions defined on the sets I_j , $j = 1, 2$, of points of the real axis. Let $L_n: \mathfrak{F}_1(I_1) \rightarrow \mathfrak{F}_2(I_2)$, $n = 1, 2, \dots$, be a sequence of linear positive operators which associate to each function $f \in \mathfrak{F}_1(I_1)$ the functions $L_n f \in \mathfrak{F}_2(I_2)$, $n = 1, 2, \dots$. There has been a considerable amount of research concerned with the convergence of sequences of the form $\{L_n f\}$, $n = 1, 2, \dots$, to the function f (see [1] - [5] and [8], [12]). Often, from the practical view of the theory of approximation it is desirable to construct sequences of linear positive operators such that L_n to preserve some properties of the functions from the space $\mathfrak{F}_1(I_1)$. In this direction, in 1934 T. POPOVICIU [17] published the first result about the convexity-preserving property of the Bernstein operator B_n , defined on $C[0, 1]$ by

$$(B_n f)(x) = \sum_{k=0}^n p_{nk}(x) f\left(\frac{k}{n}\right) \quad x \in [0, 1]$$

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

In [22] W. WEGMÜLLER used in statistics this property of the Bernstein operator. Also, I.J. SCHOENBERG [20-21] established the fact that B_n is a variation diminishing operator. More recently, L. LUPAŞ [12] showed that the Bernstein operator preserves the starshapedness. We further call attention to M.J. MARSDEN [7] which dealt with the convexity preserving property of a spline approximation operator. The fact that some approximation operators preserve certain qualitative properties of the

functions may be of interest in statistics. Indeed, a natural collection of linear positive operators $L_n: \mathfrak{F}_1(I_1) \rightarrow \mathfrak{F}_2(I_2)$ are those of the form

$$(L_n f)(x) = \sum_{k=0}^{m(n)} a_{nk}(x) f(x_{kn})$$

where $m(n)$ may be finite or not, and moreover for $k=0, 1, \dots$ and $n=1, 2, \dots$ we have:

$$\begin{aligned} \text{i)} \quad & 0 \leq a_{nk}(x) \leq 1 \\ \text{ii)} \quad & x_{kn} \in I_1 \\ \text{iii)} \quad & \sum_{k=0}^{m(n)} a_{nk}(x) = 1. \end{aligned}$$

Thus $(L_n f)(x)$ is the mathematical expectation of a certain discrete-one-dimensional variable and the fact that the mathematical expectation preserves some graphic behavior of the data will probably be of importance to the statisticians.

In this paper we investigate other linear positive operators which preserve the convexity and also which have the variation diminishing property. Some results were established by the author in [4]-[5], (see also [2], [8] - [12]).

For the formulation of our results we recall the following definitions.

DEFINITION 1: A real function f is called *convex, non-concave, polynomial, non-convex, respectively concave of k -order, on $[a, b]$* iff

$$[x_1, x_2, \dots, x_{k+2}; f] > 0, \geq 0, = 0, \leq 0 \text{ resp. } < 0,$$

for any system of $k+2$ knots from $[a, b]$.

In the above definition by $[x_1, x_2, \dots, x_{k+2}; f]$ we mean the $k+1$ -order divided difference of the function f on the knots $x_1, x_2, \dots, x_{k+2}$, that is the coefficient of t^{k+1} in the unique polynomial of degree $k+1$ which interpolates f at the knots x_i , $i=1, 2, \dots, k+2$.

DEFINITION 2: An operator $T: \mathfrak{F}_1(I_1) \rightarrow \mathfrak{F}_2(I_2)$ preserves the convexity of k -order (on I_2) if for any convex function f of k -order (on I_1) the function Tf is also convex of k -order (on I_2).

If in this definition we shall replace respectively the term of convex function by non-concave function, polynomial function, non-convex function of k -order, then we say that the operator T preserves the non-concavity, the polynomiality, the non-convexity respectively the concavity of k -order.

DEFINITION 3: Let $V(f; I_1)$ the variation of a function $f \in \mathfrak{B}_1(I_1)$ be defined as follows. For

$$Q: x_0 < x_1 < x_2 < \dots < x_n < \dots, \quad x_i \in I_1,$$

let $V(f; Q)$ denote the number of sign changes for the sequence $\{f(x_k)\}$, $k = 0, 1, 2, \dots$. Then $V(f; I_1) = \sup_Q V(f; Q)$.

DEFINITION 4: An operator $T: \mathfrak{B}_1(I_1) \rightarrow \mathfrak{B}_2(I_2)$ is said to be variation-diminishing (for f) if

$$V(Tf; I_2) \leq V(f; I_1).$$

DEFINITION 5: Let $V_r(f; I_1)$ denote the (possible infinite) variation of order r of f defined on I_1 , being

$$V_r(f; I_1) = \sup_Q \lim_{h \rightarrow 0} V(D_h^r f_s; Q)$$

where $Q: x_0 < x_1 < \dots < x_s < \dots$ is a sequence from I_1 , h is such that $x_s + kh \in I_1$, $s = 0, 1, \dots$, $k = 0, 1, \dots, r$, and $V(D_h^r f_s; Q)$ denote the number of sign changes for the sequence $D_h^r f_s$, $s = 0, 1, \dots$, where $D_h^r f_s = [x_s, x_s + h, \dots, x_s + rh; f]$.

DEFINITION 6: An operator $T: \mathfrak{B}_1(I_1) \rightarrow \mathfrak{B}_2(I_2)$ is said to be variation-diminishing of order r , if for every $f \in \mathfrak{B}_1(I_1)$

$$V_r(Tf; I_2) \leq V_r(f; I_1).$$

The variation of order r is a slight extension of the concept of "number of reversals of order r " which was introduced by B. WOOD [23] and which also remark that if f has a continuous r^{th} -derivative, then $V_r(f; I_1) = V(f^{(r)}; I_1)$.

2. GRAPHIC BEHAVIOR OF THE SZASZ-MIRAKYAN OPERATOR

The set of functions which are defined on $[0, \infty)$ and such that $|f(t)| \leq e^{At}$, for $t \geq 0$ and some finite A is denoted by W . Let S_n the Szász-Mirakyan operator be defined on the space W by

$$(1) \quad (S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

We specify that the series

$$\sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

is uniformly and absolutely convergent on $[0, a]$ for any $a > 0$ (see [3] and [12]) if $f \in W$. In [3] A. JAKIMOVSKI and D. LEVIATAN was established that if $f \in C[0, a] \cap W$ then the sequence $\{S_n f\}$, $n = 1, 2, \dots$, converges uniformly on $[0, a]$ to the function f .

LEMMA 1: *If for a function $f \in W$, of real t variable, we denote*

$$D_k^f(t) = [t, t+1/n, \dots, t+k/n; f],$$

then the k -derivative of $S_n f$ has the following form

$$(2) \quad (S_n f)^{(k)} = k! S_n D_k^f, \quad k = 0, 1, \dots$$

Proof: Taking into account that $f \in W$, differentiating the relation (1), we have

$$\begin{aligned} (S_n f)'(x) &= n e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} [f(\frac{k+1}{n}) - f(\frac{k}{n})] = \\ &= e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} [\frac{k}{n}, \frac{k+1}{n}; f] = (S_n D_1^f)(x). \end{aligned}$$

As a rule we suppose that for s -derivative is true that

$$(3) \quad (S_n f)^{(s)} = s! S_n D_s^f.$$

Making use of (3) we can write

$$\begin{aligned}
 (S_n f)^{(s+1)}(x) &= s! n e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left\{ \left[\frac{k+1}{n}, \frac{k+2}{n}, \dots, \frac{k+s+1}{n}; f \right] - \right. \\
 &\quad \left. - \left[\frac{k}{n}, \frac{k+1}{n}, \dots, \frac{k+s}{n}; f \right] \right\} = \\
 &= (s+1)! e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left[\frac{k}{n}, \frac{k+1}{n}, \dots, \frac{k+s+1}{n}; f \right] = \\
 &= (s+1)! (S_n D_{s+1}^f)(x),
 \end{aligned}$$

and the theorem is proved.

THEOREM 2: *If $f \in W$ then the Szász-Mirakyan operator preserves the convexity of any order of f , the concavity of any order f , the non-convexity of any order of f , the non-concavity of any order of f , and the polynomiality of f .*

Proof. If a function g has a $k+1$ -derivative on $[a, b]$, then the convexity of k -order may be expressed as $g^{(k+1)} > 0$ on the interval $[a, b]$. From (2) it is readily verified that for $k = -1, 0, 1, \dots$, the sign of D_{k+1}^f implies the sign of $(S_n f)^{(k+1)}$ and in accordance with the definition 2 we conclude that the S_n operator preserves the shape of any order.

THEOREM 3: *On the space W the Szász-Mirakyan operator has the following form*

$$(4) \quad (S_n f)(x) = \sum_{k=0}^{\infty} \left[0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{k}{n}; f \right] x^k.$$

Proof. Using the fact that the series

$$\sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

is absolutely convergent on $[0, a]$ for any $a > 0$, by developing the function e^{-nx} in power series we have

$$e^{-nx} = \sum_{k=0}^{\infty} (-1)^k \frac{n^k}{k!} x^k$$

and the Szász-Mirakyan operator defined by (1) can be written in the following manner

$$\begin{aligned} (S_n f)(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{n^k}{k!} x^k \sum_{k=0}^{\infty} \frac{n^k}{k!} f\left(\frac{k}{n}\right) x^k = \\ &= \sum_{k=0}^{\infty} x^k \sum_{j=0}^k (-1)^{k-j} \frac{n^k}{(k-j)! j!} f\left(\frac{j}{n}\right) = \\ &= \sum_{k=0}^{\infty} D_k^f(x) x^k, \end{aligned}$$

and we give (4).

COROLLARY: *When f is a polynomial of s -degree, then $S_n f$ is itself a polynomial of the degree $\leq s$.*

This statement results taking into account that for a polynomial f of the s -degree (which is evidently an element of the space W), we have

$$[0, 1/n, 2/n, \dots, k/n; f] = 0 \quad \text{for } k > s.$$

3. SOME PROPERTIES OF THE BASKAKOV OPERATORS

In Order to construct sequences of linear positive operators with the aid of sequences of functions, V.A. BASKAKOV [1] stated the following conditions. Let consider the sequence $\{\varphi_n\}$, $n = 1, 2, \dots$, of real functions, each function having the following properties on the interval $[0, a]$, $a > 0$,

- I. $\varphi_n(0) = 1$
- II. φ_n is completely monotonic on $[0, a]$ that is φ_n has derivatives of all orders there and $(-1)^j \varphi_n^{(j)} \geq 0$, $j = 0, 1, \dots$,
- III. there exists a positive integer m_n not depending on j , such that for $x \in [0, a]$ we have

$$-\varphi_n^{(j)}(x) = \psi(n, x) \varphi_{m_n}^{(j-1)}(x) [1 + E_{j,n}(x)], \quad j = 1, 2, \dots,$$

where

$$\lim_{n \rightarrow \infty} E_{j,n} = 0, \quad x \in [0, a], \quad j = 1, 2, \dots,$$

$$\lim_{n \rightarrow \infty} \frac{\psi(n, x)}{n} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\psi(m_n, x)}{n} = 1, \quad x \in [0, a].$$

Let W_I be a linear set of real functions which are defined on $[0, \infty)$ and such that the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k \varphi_n^{(k)}(x) x^k}{k!} f\left(\frac{k}{n}\right)$$

is uniformly and absolutely convergent on $[0, a]$ for $a > 0$. The Baskakov operators are defined on the space W_I by

$$(5) \quad (L_n f)(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \varphi_n^{(k)}(x) x^k}{k!} f\left(\frac{k}{n}\right)$$

In [5] the author has shown that this operator preserves the convexity, the polynomiality respectively the concavity of any order of a function $f \in W_I$. Also it is proved that the operator L_n defined by (5) on a certain subspace of W_I has the following form

$$(L_n f)(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \varphi_n^{(k)}(0)}{n^k} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right] x^k,$$

and moreover for $f \in W_I$,

$$V(L_n f; I) \leq V(f; R_T)$$

where $I = [0, a]$, $a > 0$ and $R_T = [0, \infty)$, that is L_n is a variation diminishing operator.

Now, our purpose is to carry out a new property of the Baskakov operator.

THEOREM 4: *The L_n operator defined on the space W_I is variation diminishing of order r ($r = 1, 2, \dots$).*

Prof. For $f \in W_I$, differentiating the formula (5) we obtain:

$$(6) \quad (L_n f)^{(r)}(x) = \frac{r!}{n^r} \sum_{k=0}^{\infty} h_{nkr}(x) \left[\frac{k}{n}, \frac{k+1}{n}, \dots, \frac{k+r}{n}; f \right] x^k$$

where

$$0 \leq h_{nkr}(x) = \frac{(-1)^{k+r} \varphi_n^{(k+r)}(x)}{k!}, \quad x \in [0, a].$$

Because $f \in W_I$, the function $L_n f$, n fixed, has a continuous r -derivative, and we obtain with $I = [0, a]$, $a > 0$,

$$(7) \quad V_r(L_n f; I) = V\{(L_n f)^{(r)}; I\}.$$

E.W. CHENEY and A. SHARMA [2] showed that if

$$h(x) = \sum_{k=0}^{\infty} a_k x^k$$

is uniformly convergent on $[0, a]$, then $V(h; I) \leq V(\{a_k\})$.

Now, from (7) with the notation

$$Q : 0 < 1/n < 2/n < \dots < k/n < \dots$$

it is clear that (6) implies the inequality

$$\begin{aligned} V\{(L_n f)^{(r)}; I\} &\leq V(D_{\frac{1}{n}}^r f_S; Q) \leq \lim_{n \rightarrow \infty} V(D_{\frac{1}{n}}^r f_S; Q) \leq \\ &\leq \sup_Q \lim_{n \rightarrow \infty} V(D_{\frac{1}{n}}^r f_S; Q) = V_r(f; R) \end{aligned}$$

and the theorem is proved.

By means of many suitable choices of the sequence $\{\varphi_n\}$, we give some well-known operators which are variation diminishing of order r .

- i) Let $\varphi_n(x) = (1-x)^n$, $x \in [0, 1] = I$, $W_I = C[0, 1]$; the operator generated by this sequence of functions is the Bernstein operator.

- ii) For $\varphi_n(x) = e^{-nx}$, $x \in [0, a]$, $a > 0$, $W_1 = W$, the corresponding operator is the Szász-Mirakyan operator. B. WOOD [12] proved that S_n is variation diminishing operator of r order. Also he gives this property and for a general class of linear positive operators but which not include the Bernstein operator.

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