
Reports of Meetings

Seventh International Symposium on Functional Equations September 1–13, 1969

The Seventh International Symposium on Functional Equations, September 1–13, 1969, was sponsored by the Faculty of Mathematics of the University of Waterloo with additional support from a University of Waterloo Research Grant, from the Department of University Affairs of Ontario and from I.B.M. The organizing committee consisted of J. Aczél, M. A. McKiernan, and A. M. Ostrowski. There were 51 participants from 13 countries. 42 papers and numerous short remarks and open problems were presented. A number of the invited were regrettably unable to attend but nevertheless submitted their papers, most of which could be presented and discussed at the meeting. This was made possible in particular by the extension of the conference to a two-week period, even though the work still had to be rather intensive (18 sessions).

The meeting was opened on Monday, September 1, 3:00 p.m. in the Mathematics and Computer Building of the University of Waterloo by Professor J. Aczél and then President H. E. Petch and Dean W. F. Forbes welcomed the participants in the name of the University of Waterloo. After this the scientific programme began.

A particular feature of this meeting was the emphasis on refined general methods applied to functional equations on very general spaces. Among the topics treated, the connections between functional equations and geometric algebra, axiomatic projective geometry, geometric objects and relativity, probability and information theory, algebras of functions, universal algebra, quasigroups and semigroups, Banach spaces, distributions, computer languages, convex functions and general inequalities should be mentioned.

There were animated discussions after the lectures, and in the periods set aside for general remarks and open problems. It was gratifying that essential progress could be made even on problems stated during the meeting. This was due largely to the somewhat less hectic pace permitted by the longer duration and the seclusion at Lumina Resort (Lake of Bays) during the second week.

The programme of the meeting included a tour of the University campus, another to the Computing Centre with presentation and trial of an enlarged programme on listing and classifying publications in functional equations, and a banquet and presentation by the City of Waterloo for the participants. Time was set aside for excursions to Toronto, Algonquin Park, Bigwin Island and other scenic points in the Muskoka

region. The participants also attended a performance of "Hamlet" in the Stratford Festival Theatre and viewed the Group of Seven paintings in the McMichael Collection of Art.

The housing of the participants, first in the Waterloo Cooperative Residences and then in the Lumina Resort, afforded ample opportunity for stimulating informal discussions, the results of which could already be partly presented at the meetings.

The full list of participants follows:

Australia:

J. B. Miller (Monash University, Melbourne, at present Trent University, Peterborough, Ontario, Canada)

Canada:

J. Aczél (University of Waterloo, Waterloo, Ontario)

S. Aczél (Waterloo, Ontario)

J. A. Baker (University of Waterloo, Waterloo, Ontario)

I. C. Chakravarty (Trent University, Peterborough, Ontario)

D. Ž. Djoković (University of Waterloo, Waterloo, Ontario)

L. Etigson (University of Waterloo, Waterloo, Ontario)

J. Hamilton (University of Waterloo, Waterloo, Ontario)

H. Haruki (University of Waterloo, Waterloo, Ontario)

Pl. Kannappan (University of Waterloo, Waterloo, Ontario)

B. Lawruk (McGill University, Montreal, Quebec)

D. Lovelock (University of Waterloo, Waterloo, Ontario)

M. A. McKiernan (University of Waterloo, Waterloo, Ontario)

J. P. Mokanski (University of Guelph, Guelph, Ontario)

C. T. Ng (University of Waterloo, Waterloo, Ontario)

T. Pietrzykowski (University of Waterloo, Waterloo, Ontario)

H. Schwerdtfeger (McGill University, Montreal, Quebec)

W. T. Tutte (University of Waterloo, Waterloo, Ontario) .

Czechoslovakia:

V. Havel (Technical University, Brno)

F. Neuman (University of Brno, Brno)

France:

J. Kampé de Fériet (Université de Lille, Lille)

Germany (E):

I. Fenyö (Universität Rostock, Rostock, DDR)

Germany (W):

W. Benz (Ruhr-Universität, Bochum, at present University of Waterloo, Waterloo, Ontario, Canada)

W. Leissner (Ruhr-Universität, Bochum)

U. Melchior (Ruhr-Universität, Bochum, at present University of Waterloo, Waterloo, Ontario, Canada)

Hungary:

P. Fischer (Automat. Kut. Int., Budapest, at present CNRS Laboratoire d'Automatique, Toulouse, France)

E. Vincze (Nehézip. Műsz. Egy., Miskolc-Egyetemváros)

Italy:

P. Benvenuti (Università di Roma, Roma)

B. Forte (Università di Pavia, Pavia)

Poland:

B. Choczewski (Mining Academy, Cracow)

St. Golab (Jagello University, Cracow)

M. Kucharzewski (Math. Inst. Pol. Acad. Sci., Warsaw)

M. Kuczma (University of Silesia, Katowice)

Z. Moszner (Pedagog. Inst., Cracow)

H. Swiatk (Jagello University, Cracow, at present McGill University, Montreal, Quebec, Canada)

A. Zajtz (Jagello University, Cracow)

Romania:

F. Radó (University Babeş-Bolyai, Cluj, at present University of Waterloo, Waterloo, Ontario, Canada)

Switzerland:

A. M. Ostrowski (Universität Basel, Basel)

U.S.A.:

J. Bailleul (Mass. Gen. Hospital, Dept. of Computer Science, Boston, Massachusetts)

A. Girod (Rochester, New York)

D. Girod (University of Rochester, Rochester, New York)

E. Hille (Yale University, New Haven, Conn. at present University of New Mexico, Albuquerque, New Mexico)

J. H. B. Kemperman (University of Rochester, Rochester, New York)

T. I. O'Hara (St. Francis College, Brooklyn, New York)

B. Schweizer (University of Arizona, Tucson, Arizona)

K. N. Sigmon (University of Florida, Gainesville, Florida)

A. Sklar (Illinois Institute of Technology, Chicago, Illinois)

G. I. Targonski (Fordham University, Bronx, New York)

J. M. Targonski (New York, New York)

D. Zupnik (Illinois Institute of Technology, Chicago, Illinois)

Yugoslavia:

S. Kurepa (University of Zagreb, Zagreb)

During the last session Professor J. Aczél thanked the sponsors, hosts and participants for their cooperation and effort and announced that the next meeting will be held at Oberwolfach, W. Germany, during the first half of August, 1970. Professor Kampé de Fériet thanked, in the name of the participants, the organizing committee and their co-workers to whose efforts both the scientific and social success of the meeting was due. The participants expressed their appreciation for the care and hospitality of the personnel at University of Waterloo and at Lumina Resort. The conference closed at 11:00 a.m. on Saturday, September 13, 1969.

The summaries of lectures and the presented problems and remarks follow in chronological order.

Summaries of Lectures

BRUNO FORTE: The solution of a functional equation problem in information theory.

The information theory without probability leads to the following problem. Notation: $\mathcal{B} = \{(u, x) \mid u \in (0, 1), x \in (-\infty, +\infty)\}$, $\mathcal{D} = \{(u, v, x, y) \mid (u, x) \in \mathcal{B}, (v, y) \in \mathcal{B}, u + v > 1\}$, $\mathfrak{M} = \{F \mid F \in C^0(\mathcal{B}), \forall u \in (0, 1) \ x \xrightarrow{F} R \text{ is monotonic non-increasing}\}$. The problem is to find the solutions in $\mathfrak{M}$ of the following system of functional equations

$$(A) \quad F(u, -x) = x + F(1 - u, x) \quad (u, x) \in \mathcal{B}$$

$$(B) \quad F\left[1 - u, x + F\left(\frac{1 - v}{u}, y\right)\right] + F\left(\frac{1 - v}{u}, y\right) = \\ = F\left[1 - v, y + F\left(\frac{1 - u}{v}, x\right)\right] + F\left(\frac{1 - u}{v}, x\right) \quad (u, v, x, y) \in \mathcal{D}.$$

The solutions are:

$$F(u, x) = -\frac{1}{\lambda} \log[u^{k\lambda} e^{\lambda x} + (1 - u)^{k\lambda}] \quad \lambda, k \in R$$

$$F(u, x) = \frac{1}{2} \left\{ \pm \left| x + k \log \frac{1 - u}{u} \right| - x + k \log u (1 - u) \right\} \quad k \in R$$

$$F(u, x) = -ux + k \{u \log u + (1 - u) \log(1 - u)\} \quad k \in R.$$

PIETRO BENVENUTI: On a system of functional equations in information theory.

The following system of functional equations is considered

$$\Psi(x, y, u, v) = \Psi(y, x, v, u)$$

$$\Psi[F(x, y), z, \Psi(x, y, u, v), w] = \Psi[F(y, z), x, \Psi(y, z, v, w), u]$$

$$\Psi(x + z, y + z, u + w, v + w) = w + \Psi(x, y, u, v)$$

and all its continuous solutions are determined.

The unknown function Ψ is defined for all quadruples of reals and the above equations are supposed to be valid for all real values of the variables in them. The function F is supposed known and the following two forms are considered for it:

$$F(x, y) = \inf(x, y), \quad F(x, y) = \sup(x, y).$$

The continuous solutions form a two-parameter set:

$$\Psi(x, y, u, v) = F_k(u - \lambda x, v - \lambda y) + \lambda F(x, y)$$

with $k \neq 0$, $-\infty \leq k \leq +\infty$, $-\infty < \lambda < +\infty$,

$$F_k(x, y) = k^{-1} \log(e^{kx} + e^{ky}),$$

$$F_{-\infty}(x, y) = \inf(x, y), \quad F_{+\infty}(x, y) = \sup(x, y).$$

The solutions of the above system of functional equations are of interest for the determination of the universal composition laws of entropies which are to be compatible with the only possible non-Shannonian universal composition law of event information.

CLAUDIO BAIocchi: Sur une équation liée à la théorie axiomatique de l'information.
(Presented by B. Forte.)

On se propose de caractériser, dans la classe des fonctions réelles d'une variable réelle $\Psi: \mathbf{R} \rightarrow \mathbf{R}$ partout définies et continues, les solutions de l'équation fonctionnelle

$$\Psi(u + \Psi(v)) - \Psi(u) = \Psi(v + \Psi(u)) - \Psi(v) \quad \forall \quad u, v \in \mathbf{R} \quad (*)$$

Ce problème se rattache d'un côté à l'étude d'équations «voisines» à l'équation de Cauchy; d'autre côté il se pose dans la construction axiomatique de la théorie de l'information, récemment donnée par MM. Joseph Kampé de Fériet et Bruno Forte. Le résultat qu'on obtient est le suivant:

Les solutions continues de () sont toutes et seules les fonctions suivantes:*

- I) les fonctions constantes;
- II) les fonctions du type $\Psi(x) = \Psi_{k,j}(x)$ où:

$$\Psi_{k,j}(x) = -\frac{1}{2} \{x - k + (1/j) \log[2 \cosh(jx - jk)]\}, \quad j, k \in \mathbf{R} \quad \text{avec} \quad j \neq 0;$$

III) les fonctions du type $\Psi(x) = \Psi_{k, +\infty}(x)$ et $\Psi(x) = \Psi_{k, -\infty}(x)$ où:

$$\Psi_{k, \pm\infty}(x) = \lim_{j \rightarrow \pm\infty} \Psi_{k, j}(x) = -\frac{1}{2} \{x - k \pm |x - k|\} \quad k \in \mathbf{R}.$$

F. RADÓ: Functional equations determining an involutory affine perspectivity in a translation plane and a characterization of Moufang planes.

Let $(Q, +, \cdot)$ be a Veblen-Wedderburn system, i.e. $(Q, +)$ forms an Abelian group, $Q^* = Q \setminus \{0\}$ is a loop with the neutral element 1, $(a+b) \cdot c = a \cdot c + b \cdot c$, $x \cdot 0 = 0$ and the equation $x \cdot a - x \cdot b = c$ has a unique solution for $a \neq b$, which we denote by $\varrho(c, a, b)$. We suppose also $1 + 1 \neq 0$. Let Π be the associate translation plane, i.e. the set of 'points' $Q \times Q$ together with the subsets

$$L_a = \{(a, y) \mid y \in Q\}, L_{m, b} = \{(x, y) \mid y = x \cdot m + b\},$$

called 'lines'. $(Q, +, \cdot)$ is a Moufang ring, if in addition we have: $c \cdot (a+b) = c \cdot a + c \cdot b$ and for $y \in Q^*$ there exists $y^{-1} \in Q^*$ such that $(xy)y^{-1} = x$ and $y^{-1}(yx) = x$; then Π is also called Moufang. An automorphism Φ of Π is an affine perspectivity with direction m_0 , if each line $L_{m_0, b}$ is mapped in itself. The fixed points of Φ form a line, the axis of Φ . The paper contains the following results:

A map $(x, y) \rightarrow (\varphi(x, y), \psi(x, y))$ of the translation plane Π is an affine perspectivity with direction m and $(0, 0)$ as fixed point if and only if there exist elements $\tilde{m}$, $\tilde{m} \in Q$ and a bijective function $f: Q \setminus \{\tilde{m}\} \rightarrow Q \setminus \{\tilde{m}\}$ such that

- 1) $\varphi(x, y) = \varphi_1(x) + \varphi_2(y)$, where $\varphi_1(x) = -\varrho(x\tilde{m}, \tilde{m}, m_0)$,
 $\varphi_2(x) = \varrho(x, \tilde{m}, m_0)$
- 2) $[\varphi_2(xm) + \varphi_1(x)] f(m) = [\varphi_2(xm) + \varphi_1(x)] m_0 + xm - xm_0$, $\forall x \in Q$,
 $\forall m \in \tilde{Q} = Q \setminus \{\tilde{m}\}$
- 3) $\psi(x, y) = [\varphi_1(x) - x] m_0 + \varphi_2(y) m_0 + y$.

A necessary and sufficient condition that the plane admits an involutory affine perspectivity with direction m is that there exist functions $f: \tilde{Q} \rightarrow \tilde{Q}$, $\varphi: Q \rightarrow Q$ satisfying the equation

$$\varrho(x, y, m) + \varrho(x, f(y), m) = \varphi(x), \quad \forall x \in Q, y \in \tilde{Q} \setminus \{m\}.$$

A necessary and sufficient condition that the plane Π be Moufang is that, for two arbitrary lines l, l^1 through $(0, 0)$ there exists an involutory affine perspectivity with a fixed direction transforming l onto l^1 and that another involutory affine perspectivity still exists.

VÁCLAV J. HAVEL: Contributions to a problem of J. Aczél.

Let P, P' be planes and let $Q \subset P$, $Q' \subset P'$ be sets consisting of all points lying on four generally situated lines in P or in P' respectively. Given a collinearity preserving

map κ from Q onto Q' , is it possible to extend κ to a homomorphism from P onto P' ? (The problem of J. Aczél.)

A supplementary question: Can we express such a κ purely algebraically and what can be said about the set of all such κ if P, P', Q, Q' are prescribed in any way?

The answer to the initial problem is positive when P, P' are desarguesian planes (J. Aczél – W. Benz, *Aequationes Math.* 1 (1968), p. 216, 3 (1969), 86–92) or if P, P' are Moufang planes (J. F. Rigby, *Mathematica* 10 (1968), 369–383). The answer to the supplementary question was given by F. Radó for Desarguesian planes P, P' (*Aequationes Math.* 3 (1969), 39–53) and in part by J. F. Rigby for Moufang planes P, P' (Rigby, article quoted above).

We shall consider the case of general planes P, P' or even the case in which the planes P, P' are removed and only the fundamental structure of the investigated geometric systems is retained: Following a remark of Aczél, we substitute the couples $(P, Q), (P', Q')$ from the initial problem by abstract 4-webs with generally situated vertices. This dual view point permits to establish a concept which seems to be more convenient than the direct one.

Such 4-webs can be described algebraically using double-systems $(S, 0, 1, +, \cdot)$ of the following form: S is a set; $0 \neq 1$ are distinguished elements of S , $+$ and $\cdot$ are binary operations in S such that $+$ has the domain $S \times S \setminus \{(x, 1) \mid x \in S \setminus \{0\}\}$ and $\cdot$ has the domain $S \times S$ and the following properties are fulfilled:

$(S \setminus \{0\}, \cdot)$ is a loop with neutral element 1;

$x \in M \Rightarrow x \cdot 0 = 0 \cdot x = 0, x + 0 = 0 + x = x$;

$b \in M \setminus \{1\}, c \in M \Rightarrow x + b = c$ is uniquely solvable for x ;

$a \in M \Rightarrow a + x = 1$ is uniquely solvable for x ;

$a, d \in M \setminus \{1\} \Rightarrow a \cdot u + v = d, u + v = 1$ is uniquely solvable with respect to (u, v) ;

$a, b, c \in M; a \neq b; b + c = 1 \Rightarrow x \cdot a = x \cdot b + c$ is uniquely solvable with respect to x .

To each 4-web there corresponds canonically its coordinatizing double-system and conversely, to each double-system there corresponds a uniquely determined 4-web (over it).

We define a *pseudohomomorphism* from $(S, 0, 1, +, \cdot)$ onto $(\mathbb{S}, \mathbb{0}, \mathbb{1}, \oplus, \circ)$ as a map σ from a subset of S onto $\mathbb{S}$ with the following properties (the symbols $\sim, \wedge, \vee, \neg, -$ mean that an element of S has an image, does not have an image, goes over into $\mathbb{0}$, does not go over into $\mathbb{0}$, goes over into $\mathbb{1}$ or does not go over into $\mathbb{1}$, respectively):

$\widehat{a}, \widehat{b} \Rightarrow a \cdot b \in (a^\sigma \circ b^\sigma)^{\sigma^{-1}}$;

$\widehat{a}, \widehat{b} \Rightarrow a + b \in (a^\sigma \oplus b^\sigma)^{\sigma^{-1}}$;

$\widehat{a}, \widehat{b}$ or $\widehat{a}, \widehat{b} \Rightarrow a \cdot b$;

if $a + b = 1$ then $\widehat{a} \Leftrightarrow \widehat{b}$;

$$\begin{aligned}
&\widehat{u}, x, u+v=1 \Rightarrow \widehat{x \cdot u + v}; \\
&x, \widehat{v}, u+v=1, x \cdot u + v = x \cdot U, x \cdot U \Rightarrow U \in (u^\sigma)^{\sigma^{-1}}; \\
&x, u, u+v=1, x \cdot u + v = x \cdot U, x \cdot U \Rightarrow U; \\
&u, u+v=1, \widehat{x \cdot u + v} \Rightarrow \widehat{x}.
\end{aligned}$$

THEOREM: Every homomorphism¹⁾ from a 4-web $\mathfrak{Q}$ onto a 4-web $\mathfrak{Q}'$ induces a pseudohomomorphism from the double-system of $\mathfrak{Q}$ onto a double-system of $\mathfrak{Q}'$. Conversely, each pseudohomomorphism from a double-system D onto a double-system D' determines canonically a homomorphism from the 4-web over D onto a 4-web over D' .

The question about the determination of all homomorphisms between any two given 4-webs is in general open. There exists a 4-web which is not embeddable in a plane without adding new points. Any pseudohomomorphism of a finite double-system D onto a double-system D' must be bijective. If the additions $+$, $\oplus$ of the investigated double-systems D , D' are group additions, then a much simpler definition of a pseudohomomorphism is possible. Further simplifications occur by adding various algebraic conditions upon D , D' . For instance, when D , D' are alternative division rings we get a characterization of all pseudohomomorphisms between them; this has geometrical importance for the description of all homomorphisms between two Moufang planes.

W. BENZ: A generalisation of a theorem of Aczél and McKiernan.

J. Aczél and M. A. McKiernan have proved (Math. Nachrichten 33 (1967), 315–337) a theorem concerning circles in the Gaussian plane and functional equations. This theorem is extended to other systems of curves in plane and space.

WERNER LEISSNER: Eine geometrische Anwendung der Funktionalgleichung von Hua.

Bis auf Isomorphie stimmt die Klasse der büschelhomogenen Lieebenen mit der Klasse der inneren Geometrien einer gewissen Quadrik Q des projektiven 4-dimensionalen Koordinatenraumes $\mathbf{P}^4(\mathfrak{f})$, $\mathfrak{f}$ ein kommutativer Körper, überein:

Der Menge der Zyklen einer büschelhomogenen Lieebene entspricht 1-1-deutig die Menge der Punkte $\langle x_0, x_1, x_2, x_3, x_4 \rangle$ des $\mathbf{P}^4(\mathfrak{f})$, die der Gleichung $-x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$ genügen. Zwei Zyklen X (Bild $\langle x_0, x_1, x_2, x_3, x_4 \rangle$) und Y (Bild $\langle y_0, y_1, y_2, y_3, y_4 \rangle$) berühren sich genau dann, wenn ihre Bilder konjugiert sind, d.h., wenn die Gleichung $-x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 = 0$ erfüllt ist.

Den Automorphismen einer Büschelhomogenen Lieebene entspricht demnach 1-1-deutig die Gruppe der inneren Automorphismen der Quadrik Q , d.h., die Menge

¹⁾ We suppose that this homomorphism transfers the i -th vertex of $\mathfrak{Q}$ into i -th vertex of $\mathfrak{Q}'$ ($i = 1, 2, 3, 4$).

der Permutationen von Q , die konjugierte Punktepaare wieder in konjugierte Punktepaare und nicht-konjugierte Punktepaare wieder in nicht-konjugierte Punktepaare überführten.

Wir zeigen: Jeder innere Automorphismus φ von Q lässt sich durch eine Abbildung φ beschreiben: $\langle x_0, x_1, x_2, x_3, x_4 \rangle \in Q \rightarrow \langle x_0^\sigma, x_1^\sigma, x_2^\sigma, x_3^\sigma, x_4^\sigma \rangle \cdot A$, A eine reguläre Matrix vom Typ $(5, 5)$, σ eine bijektive Abbildung des Körpers $\mathbb{F}$ auf sich, die Hua's Funktionalgleichungen

$$\begin{aligned} f(x+y) &= f(x) + f(y) \\ f\left(x \cdot \frac{1}{x}\right) &= f(x) \cdot f\left(\frac{1}{x}\right) \end{aligned}$$

genügt. σ ist also ein Körperautomorphismus und φ lässt sich somit zu einer Projektivität des $\mathbf{P}^4(\mathbb{F})$ erweitern.

Allgemeiner gilt: Jeder innere Automorphismus einer regulären Quadrik Q des $\mathbf{P}^4(\mathbb{F})$, die eine Gerade enthält, lässt sich zu einer Projektivität des $\mathbf{P}^4(\mathbb{F})$ erweitern, da solche Quadriken zur betrachteten projektiv äquivalent sind.

R. Wagner hat sogar gezeigt (Math. Z. 83 (1964), 336–344), dass jeder innere Automorphismus einer Quadrik in einem projektiven Pappus-Raum der Dimension $n \geq 3$ sich zu einer Projektivität des ganzen Raumes erweitern lässt, jedoch unter der zusätzlichen Voraussetzung, dass der innere Automorphismus koplanare Quadrikenpunkte jeweils wieder in koplanare Quadrikenpunkte überführt. Es stellt sich die Frage, wann man – wie im vorliegenden Fall – auf diese einschränkende Forderung verzichten kann.

M. KUCHARZEWSKI: Kovariante Ableitungen von Tensordichten.

Es sei

$$\begin{aligned} g^A K, \quad A(a_1, \dots, a_p), \quad K(k_1, \dots, k_q), \\ a_1, \dots, a_p, \quad k_1, \dots, k_q = 1, 2, \dots, n, \end{aligned} \quad (1)$$

eine Tensordichte der Valenz (p, q) und des Gewichtes $(-\alpha)$. Die g_K^A haben also die folgende Transformationsformel

$$g_{K'}^{A'} = \varphi(y) A_A^{A'} A_K^K g_K^A$$

wo $\varphi(y) = |y|^\alpha$ bzw. $\varphi(y) = (\text{sgn } y)|y|^\alpha$ ist, wenn (1) eine Weylsche bzw. gewöhnliche Dichte darstellt.

Die folgende Definition der kovarianten Ableitung der Tensordichte (1) wird gegeben.

DEFINITION. Die kovariante Ableitung von (1) ist jedes Funktionensystem

$$\begin{aligned} D_{L\lambda}^B, \quad B(b_1, \dots, b_p), \quad L(l_1, \dots, l_q), \\ b_1, \dots, b_p, \quad l_1, \dots, l_q, \quad \lambda = 1, 2, \dots, n, \end{aligned}$$

das die nachstehenden Bedingungen erfüllt.

1° Die $D_{L\lambda}^B$ sind nur von g_K^A und $\partial_\kappa g_{K,\kappa}^A = g_{K,\kappa}^A$, $\kappa = 1, 2, \dots, n$, abhängig, d.h.

$$D_{L\lambda}^B = D_{L\lambda}^B(g_K^A, g_{K\kappa}^A).$$

2° Die D_L^B sind hinsichtlich g_K^A additiv.

3° Für Produkte von beliebigen Skalaren σ und Tensordichten (1) gilt die Leibnizsche Regel, d.h.

$$D_L^B(\sigma g_K^A, (\sigma g_K^A), \kappa) = \sigma D_L^B(g_K^A, g_{K,\kappa}^A) + G_{L^\sigma, \lambda}^B$$

4° Die $D_{L\lambda}^B$ haben die folgende Transformationsformel

$$D_{L'x'}^{B'} = \varphi(y) A_B^{B'} A_L^L A_{\lambda'}^\lambda D_{L\lambda}^B.$$

Die Existenz und Eindeutigkeit der kovarianten Ableitung wird durch den folgenden Satz erledigt.

SATZ. Jede kovariante Ableitung der Tensordichte (1) hat die Form

$$D_{L\lambda}^B = g_{L,\lambda}^B + A_{AL\lambda}^{BK} g_K^A, \text{ wo } A_{AL\lambda}^{BK} \text{ nur von } \alpha \text{ abhängen.}$$

Die $A_{AL\lambda}^{BK}$ bilden ein geometrisches Objekt, und werden die Parameter der kovarianten Ableitung genannt.

Unter der Voraussetzung, dass die Leibnizsche Regel für die Produkte von Tensordichten beliebiger Valenz und beliebiger Gewichte erfüllt ist, werden gewisse Zusammenhänge zwischen Parametern der kovarianten Ableitung für verschiedene Tensordichte abgeleitet.

Alle Ergebnisse wurden durch das Lösen entsprechender Funktionalgleichungen erhalten.

Die Betrachtungen können auf beliebige lineare geometrische Objekte verallgemeinert werden.

M. A. MCKIERNAN and H. RICHARDS: Characterization of some tensor concomitants of the metric tensor and vector fields under restricted groups of transformations.

In a space V^n we consider objects which are concomitants of the metric tensor g_{ij} and of a vector field v^i , their partial derivatives, etc., under the restricted group of proper transformations $x^i \rightarrow x^{i'}$, that is, $A_i^{i'} \equiv \partial x^{i'} / \partial x^i$ and $\det(A_i^{i'}) > 0$. Relative to this class of transformations it is shown that:

1. The only scalar concomitant $T(g_{ij})$ of the metric tensor is a constant (which depends on the signature of the metric).
2. Any rank two covariant concomitant $T_{ij}(g_{rs})$ of the metric tensor in a V^n for

$n > 2$ is of the form $T_{ij}(g_{rs}) = k g_{ij}$ for constant k (again dependent on the signature of g_{ij}).

3. Any relative tensor concomitant of the form $T_{j_1 \dots j_s}^{\lambda_1 \dots \lambda_r}(g_{kl})$ in a V^n , for $r+s$ odd and $r+s < n$, is in fact a null tensor.

4. Any tensor concomitant of the form $T_{i_1 \dots i_n}(g_{kl})$ in a V^n where n is odd, is in fact a constant multiple of the Levi-Civita tensor $\varepsilon_{\lambda_1 \dots \lambda_n}$.

While the following assertions are well known, the proofs presented here permit generalization to arbitrary concomitants (not necessarily geometric) of class 1. Here ‘,’ denotes partial differentiation, ‘|’ covariant differentiation, R_{jkl}^i the curvature tensor, v^i a vector field.

If $T_{\dots}(g_{ij}; g_{ij,k}); T_{\dots}(g_{ij}; g_{ij,k}; g_{ij,kr}); T_{\dots}(g_{ij}; g_{ij,k}; g_{ij,kr}; v^i; v^i_{,j})$ are arbitrary concomitants of class 1, then they are in fact of the form $T_{\dots}(g_{ij}); T_{\dots}(g_{ij}; R_{jkl}^i); T_{\dots}(g_{ij}; R_{jkl}^i; v^i; v^i_{,j})$, respectively.

In the proofs of the above, considerable use was made of the following lemma. Let $S, \sigma, \bar{\sigma}, \tau$ be matrices ($n \times n$), k and l constants, and let S^T denote the transpose of S .

LEMMA. Let $\sigma \stackrel{\text{Def}}{=} \text{diag } [+1, \dots, +1, -1, \dots, -1]$ where there are p elements $+1$ and $n-p$ elements -1 . If, for given $\bar{\sigma}$, $S \bar{\sigma} S^T = \bar{\sigma}$ whenever $S \sigma S^T = \sigma$ and $\det S > 0$, then $\bar{\sigma} = k \sigma$ for $n > 2$ while for $n = 2$,

$$\bar{\sigma} = \begin{cases} \begin{bmatrix} k & l \\ -l & k \end{bmatrix} \sigma & \text{for } p = 0, 2 \\ \begin{bmatrix} k & l \\ l & k \end{bmatrix} \sigma & \text{for } p = 1 \end{cases}$$

ARTHUR MOÓR: Lie-Ableitung von Vektoren der M_n -Räume vom Standpunkt der geometrischen Objekte. (Bericht von M. A. McKiernan.)

Ein M_n -Raum ist eine Mannigfaltigkeit der Grundelemente (x^i, v^i) deren Grundtransformationen sich aus einer reinen Punkt- und aus einer darauffolgenden Richtungstransformation zusammensetzen. Ist diese Richtungstransformation die Identität, so reduziert sich der M_n -Raum auf den gewöhnlichen Linienelementraum der höheren Differentialgeometrie.

Die Lie-Ableitung der Vektoren der M_n -Räume kann durch zwei verschiedenartigen Methoden festgelegt werden. Die erste Methode ist die klassische Methode der infinitesimalen Transformationen. Die zweite geschieht mittels Funktionalgleichungssysteme mit Hilfe gewisser Hilfsobjekte. In unserem Aufsatz wird nun gezeigt, dass diese letztere Methode – abgesehen von einem skalaren Faktor – auch zu den klassischen Formeln der Lie-Ableitung führt, falls Differenzierbarkeit vorausgesetzt wird.

BERTHOLD SCHWEIZER: Compact semigroups on the space of distribution functions.

Let D^+ be the set of all functions F from the real line R into $[0, 1]$ which are left-continuous, non-decreasing and satisfy $F(0)=0$. The set D^+ is a commutative semigroup under any one of the three operations: convolution $(*)$, τ_T , and π_T , where T is a continuous, associative function (semigroup) on $[0, 1]$ and τ_T , π_T , are defined by $\tau_T(F, G; x) = \sup_{u+v=x} T(F(u), G(v))$, $\pi_T(F, G; x) = T(F(x), G(x))$, for any $F, G \in D^+$ and any $x \in R$. The function $\mathcal{L}: D^+ \times D^+ \rightarrow D^+$ defined by

$$\mathcal{L}(F, G) = \inf_h \left\{ F\left(x - \frac{h}{2}\right) - \frac{h}{2} \leq G\left(x + \frac{h}{2}\right) + \frac{h}{2} \text{ and } \right. \\ \left. G\left(x - \frac{h}{2}\right) - \frac{h}{2} \leq F\left(x + \frac{h}{2}\right) + \frac{h}{2}, \right. \\ \left. \text{for } -\frac{1}{h} - \frac{h}{2} < x < \frac{1}{h} + \frac{h}{2} \right\}$$

is a metric on D^+ and the metric space $(D^+, \mathcal{L})$ is compact. Each of the functions, $(*)$, τ_T , π_T is (uniformly) continuous on $(D^+, \mathcal{L})$. Hence $(D^+, \mathcal{L})$ is a compact semigroup under each of these operations. Various properties of these semigroups, particularly those determined by τ_T and π_T , are developed.

J. KAMPÉ DE FÉRIET: The composition law in information theory.

Given a set Ω , whose points ω represent elementary events, one wants to measure the amount of information $J(A)$ ($\in \bar{R}^+$) defined by an event $A \subset \Omega$; in the classical theory of N. Wiener and C. E. Shannon (1948) the definition of J is based on a probability law P , the set of events ω being a probability space (Ω, G, P) . But it is our opinion that a probability could be correctly applied to an event A only if A could be indefinitely repeated, in a *series of independent trials*; nevertheless it makes sense to compare the informations given by events which could not be repeated. Thus, in collaboration with Bruno Forte [Comptes Rendus Acad. Sc., Paris 265 (1967), 110–114, 142–146, 350–353], we have proposed a set of axioms (independent of the existence of any probability law on Ω) for $J(A)$. Axiom IV reads:

$$A_4: \text{if } A \cap B = \emptyset, J(A \cup B) = F[J(A), J(B)] = J(A) \top J(B)$$

but, as we pointed out, that is a very strong restriction on the general notion of information; there are useful types of information, which do not admit a composition law $\top$: for instance when the amount of information is defined by the accuracy of a measurement (Ω being a metric space, one can define: $J(A) = \psi[d(A)]$ where $d(A)$ is the diameter of A). Nevertheless the informations having a composition law $\top$ constitute a large and important class, to which for instance the Wiener-Shannon

information belongs as a particular case. The composition law must satisfy the following conditions:

$$(C_1) \quad x \top y = y \top x$$

$$(C_2) \quad x \top (y \top z) = (x \top y) \top z$$

$$(C_3) \quad y' < y'' \Rightarrow x \top y' \leq x \top y''$$

$$(C_4) \quad x \top (+\infty) = x$$

As a rule $x \top y$ is not defined on $\bar{R}^+ \times \bar{R}^+$ but on the subset

$$\Gamma = \{(x, y): A, B \in G, J(A) = x, J(B) = y, A \cap B = \emptyset\}$$

and $\Gamma^c = \bar{R}^+ \times \bar{R}^+ - \Gamma \neq \emptyset$. But it is always possible to extend the definition of $x \top y$ to $\bar{R}^+ \times \bar{R}^+$, by putting

$$x \top y = 0 \quad (x, y) \in \Gamma^c.$$

If one adds the condition:

$$(C_5) \quad x \top y \in C(\Gamma),$$

one sees that the composition law defines a *topological semi-group* on R^+ . Applying the results of P. S. Mostert – A. L. Shields (Ann. Math. 65 (1957), 117–143) and C. H. Ling (Publ. Math. Debrecen 12 (1965), 189–212), one obtains *the most general form of the composition law*.

The set of idempotents $A = \{x: x \top x = x\}$ is closed; thus A^c is the union of countably many open intervals $]a_n, b_n[$, ($a_n \in A, b_n \in A$). Then:

(a) if $\xi \in A$, one has

$$x \top y = \inf(x, y)$$

in the strips

$$B_1 = \{(x, y): 0 \leq x \leq \xi, \xi \leq y \leq +\infty\}$$

$$B_2 = \{(x, y): \xi \leq x \leq +\infty, 0 \leq y \leq \xi\}.$$

(b) In the square $[a_n, b_n] \times [a_n, b_n]$, one has

$$x \top y = \varphi_n[\theta_n^{-1}(x) + \theta_n^{-1}(y)]$$

where θ_n is submitted only to the conditions:

$$(\alpha) \quad \theta_n: [0, \bar{\mu}_n] \rightarrow \bar{R}^+, \bar{\mu}_n \leq +\infty$$

$$(\beta) \quad \theta_n(0) = b_n, \theta_n(\bar{\mu}_n) = a_n$$

$$(\lambda) \quad \theta_n \text{ is strictly decreasing}$$

$$(\delta) \quad \theta_n \text{ is continuous}$$

and:

$$\begin{cases} \varphi_n(x) = \theta_n(x) & x \in [0, \bar{\mu}_n] \\ = a_n & x \in [\bar{\mu}_n, +\infty]. \end{cases}$$

I. OLKIN and A. R. SAMPSON: Jacobians of matrix transformations and functional equations. (Presented by S. Kurepa.)

In the first part of this paper the functional equation $h(AB) = h(A)h(B)$ is con-

sidered where h is defined on a certain subgroup G of a full linear n -dimensional group, $h(A)$ is a scalar and h is a continuous function. In the second part these results are applied on the question of determining the determinant of some linear operators which are defined from the vector space M_m of all m -order matrices to the vector space M_n of all n -order matrices. These results are also extended to determine the determinant $\det f'(X)$ of a Jacobian of a differentiable map $f: M_m \rightarrow M_n$.

A. M. KAGAN and JU. V. LINNIK: Remarks on some non-linear functional equations encountered in mathematical statistics. (Presented by B. Schweizer.)

The authors exhibit several types of non-linear integral-functional equations which are considered in detail in their forthcoming book with C. R. Rao, 'Characterization Problems of Mathematical Statistics'. Their note will appear as a Short Communication to this journal.

A. M. OSTROWSKI: A functional equation solvable by asymptotic series.

A result by Kummer can be interpreted as the following theorem. If in the functional equation $G(z+1)=G(z)\varphi(z)-1$, $\varphi(z)$ has the asymptotic development

$\varphi(z) \sim \sum_{v=1}^{\infty} v_v/z^v$, then an asymptotic development can be found,

$G(z) \sim z \sum_{k=0}^{\infty} C_k/z^k$ ($C_0 \neq 0$), satisfying formally the functional equation.

The convergence problem of the development of $G(z)$ can be often solved, using the following theorem:

If φ is meromorphic and there exists a non polynomial solution $G(z)$ with an isolated singularity at ∞ , then for a certain α and a convenient integer $N > 0$ we have $\varphi(\alpha) = \infty$, $\varphi(\alpha + N) = 0$.

F. NEUMAN: Abel's equation in the theory of differential equations.

Consider

$$y'' = q(t) \quad y, \quad q \in C^0(a, b), \quad -\infty \leq a < b \leq \infty. \quad (1)$$

A continuous function α on (a, b) satisfying the relation $\tan \alpha(t) = y_1(t)/y_2(t)$ (y_1, y_2 independent solutions of (1)), is a phase of (1). Let $t_0 \in (a, b)$, $y_3(t)$ be a non-trivial solution of (1) such that $y_3(t_0) = 0$. Let $\varphi(t_0)$ (the dispersion of (1)) be the first zero of y_3 on the right of t_0 (if such a zero exists). Then $\alpha \in C^3$ (i.e., there exists a continuous derivative of the 3rd order), $\alpha'(t) \neq 0$, $\varphi \in C^3$, $\varphi'(t) > 0$, $\varphi(t) > t$, and Abel's equation

$$\alpha(\varphi(t)) = \alpha(t) + \pi \cdot \operatorname{sgn} \alpha' \quad (2)$$

is satisfied everywhere, where $t \rightarrow \alpha(\phi(t))$ is defined. A general solution of (1) can be expressed as

$$k_1 \frac{\sin[\alpha(t) + k_2]}{\sqrt{|\alpha'(t)|}}$$

and q is determined by

$$q(t) = -\{\alpha, t\} - \alpha'^2(t), \quad \{\alpha, t\} = \frac{1}{2} \frac{\alpha'''}{\alpha'} - \frac{3}{4} \left(\frac{\alpha''}{\alpha'} \right)^2.$$

For the basic definitions, properties and relations, see [1].

There is a close connection between special kinds of Abel's equation (2) and some properties of differential equations (1). On the basis of Abel's equation (2) all differential equations (1) with some prescribed distributions of zeros and extremes of their solutions, were given in [2], [3] and relations between distributions of zeros of solutions of (1) and the boundedness of these solutions, in [4].

The centroaffine invariants of closed (or simple closed) plane curves have been described; for convex centrosymmetric curves, Blaschke's and Santalò's and other, newer results in differential geometry were derived by a simple and unified method, in [5].

- [1] BORŮVKA, O., *Lineare Differentialtransformationen 2. Ordnung* (D.V.W., Berlin 1967).
- [2] NEUMAN, F., *Sur les équations différentielles linéaires du second ordre dont les solutions ont des racines formant une suite convexe*, Acta Math. Acad. Hung. 13, 281-287 (1962).
- [3] NEUMAN, F., *Construction of Second Order Linear Differential Equations with Solutions of Prescribed Properties*, Arch. Math. (Brno) 1, 229-246 (1965).
- [4] NEUMAN, F., *Relation between the Distribution of the Zeros of the Solutions of a 2nd Order Linear Differential Equation and the Boundedness of these Solutions*, Acta Math. Hung. 19, 1-6 (1968).
- [5] NEUMAN, F., *Centroaffine Invariants of Plane Curves in Connection with the Theory of the Second-order Linear Differential Equations*, to appear in Arch. Math. (Brno) 4 (1968).

GY. I. TARGONSKI: Linear endomorphisms of function algebras.

Every mapping of a function algebra into itself, induced by a scale transformation ω

$$\mathfrak{f}(P) \rightarrow (\mathfrak{f} \circ \omega)(P) \quad (1)$$

is a linear endomorphism. We raise the reverse question: is every linear endomorphism of a given function algebra induced by a scale transformation, that is, by a mapping of the underlying set into itself? The answer is positive in the case of the algebra of all polynomials over the real (complex) numbers, more generally in the algebra of all entire functions and also in $C(X)$, the algebra of all continuous real-valued functions on a compact Hausdorff space X . We show that the answer is

negative for the space of all bounded functions on e.g. the unit interval, since endomorphisms of the form

$$\mathfrak{f}(P) \rightarrow \chi_E(P) (\mathfrak{f} \circ \omega)(P) \quad (2)$$

(where χ is a characteristic function) cannot be reduced, in general, to the form $\mathfrak{f} \circ \tilde{\omega}$ for any $\tilde{\omega}$. We show, however, that (2) is the most general form of a linear endomorphism in the case of an inverse-closed, self-adjoint, separating Banach algebra (these terms being used as given by Katznelson in his *Harmonic Analysis*) of complex-valued functions on a compact Banach space. As a corollary the result follows that all linear endomorphisms of the Wiener algebra are induced by scale transformations. The results are applied to the algebra of all real (complex) n -tuples under 'component wise' addition, multiplication and scalar multiplication.

The $(n+1)^{\text{th}}$ -different linear endomorphisms are determined. The arising functional equation can be solved by classical methods in the real case; the solution for the complex case, which also follows from our theorem, does not seem to have a comparably simple 'classical' solution.

C. MIRA: Étude d'un cas critique pour une récurrence autonome du deuxième ordre. (Présentée par P. Fischer.)

La récurrence considérée est à variables réelles et a la forme:

$$r_{n+1} = r_n \cos \phi - s_n \sin \phi + R(r_n, s_n), \quad s_{n+1} = r_n \sin \phi + s_n \cos \phi + S(r_n, s_n) \quad (1)$$

où R et S sont deux fonctions analytiques dont les développements commencent par des termes de degré non inférieurs au second. Les multiplicateurs de (1) au point double 0 ($r=s=0$) sont égaux à $\exp(\pm i\phi)$ ($i=\sqrt{-1}$). La récurrence (1) correspond alors à un cas critique au sens de Liapunov et il est impossible de déduire la stabilité de 0 à partir de l'approximation linéaire de (1). Cette stabilité dépend des termes non linéaires R et S . L'extension de résultats dus à A. R. Cigala (*Annali Mat.* (3) 11 (1904), 67-78) permet alors de donner les conditions de stabilité asymptotique et d'instabilité de 0, à partir du signe d'une expression G_l faisant intervenir certains coefficients dans R et S . Pour certaines valeurs de ϕ commensurables avec 2π , $\phi = 2\pi p/q$ (cas d'exception), le signe de G_l ne permet pas de conclure. Une étude de ces cas d'exception conduit aux conclusions suivantes:

- si q est impair, les suites des points données par (1), pour différentes conditions initiales prises dans un voisinage suffisamment petit de 0, sont toujours situées sur des courbes avec $2q$ branches à allure «hyperbolique»;

- si q est pair, ces suites peuvent soit être situées sur des courbes avec $2q$ branches à allure «hyperbolique», soit sur des courbes dont les formes sont précisées.

B. CHOCZEWSKI: Asymptotic behaviour of continuous solutions of a linear functional equation.

Let us consider the functional equation

$$\phi[f(x)] = g(x)\phi(x) + h(x), \quad (1)$$

with the unknown function $\phi: I \rightarrow \mathbf{R}$, where I is an interval $\langle 0, a \rangle$. The given functions $f: I \rightarrow I$, $g: I \rightarrow \mathbf{R}^+$ and $h: I \rightarrow \mathbf{R}$ are assumed to be continuous in I . Moreover, we assume that $f(0)=0$, $0 < f(x) < x$ in $(0, a)$ and f is strictly increasing in I .

We were dealing with the problem of asymptotic behaviour at the point zero of these solutions ϕ of equation (1) (or its special cases), which are continuous in the interval I , under some assumptions on asymptotic behaviour of the given functions. To illustrate the results obtained we quote two theorems.

Denote by $f^n(x)$, $x \in I$, $n=0, 1, \dots$ the sequence of iterates of the function f and put $G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)]$.

THEOREM 1. Assume that there exist positive constants $m, \mu, a_{m+1}; p, v$ such that the formulae

$$\begin{aligned} f(x) &= x - a_{m+1} x^{m+1} + o(x^{m+1+\mu}) \\ g(x) &= 1 - pa_{m+1} x^m + o(x^{m+v}) \end{aligned} \quad (2)$$

hold for $x \rightarrow 0+0$. Then the function $\phi(x) = \lim_{n \rightarrow \infty} [f^n(x)]^p / G_n(x)$ is a solution of the homogeneous equation

$$\phi[f(x)] = g(x)\phi(x) \quad (3)$$

and $\phi \in C^0(I)$. Moreover, it possesses the following asymptotic property: (*) There exists a positive limit $\lim_{x \rightarrow 0+0} x^{-p}\phi(x)$. The functions $\phi_c(x) = c\phi(x)$, where $c \neq 0$ is an arbitrary real number are the only $C^0(I)$ -solutions of equation (3) possessing the property (*), however, the general $C^0(I)$ -solution of the equation (3) depends on an arbitrary function.

THEOREM 2. Assume that the function f has the form (2), and the functions g and h can be written in the form

$$\begin{aligned} g(x) &= 1 - b_k x^k + o(x^{k+\kappa}), \quad \text{where } b_k \neq 0, k > m, \kappa > 0 \\ h(x) &= o(x^l), \quad \text{where } l > m, \end{aligned}$$

for $x \rightarrow 0+0$. Then there exists a one-parameter family of $C^0(I)$ -solutions of equation (1). They are given by the formula

$$\Psi_c(x) = c \lim_{n \rightarrow \infty} 1/G_n(x) - \sum_{n=0}^{\infty} h[f^n(x)]/G_{n+1}(x)$$

and for every positive δ such that $\delta < \min(m, k-m, \mu, \kappa)$ the asymptotic relation

$$\Psi_c(x) = c \left(1 + \frac{b_k}{(k-m)a_{m+1}} x^{k-m} + O(x^{k-m+\delta}) \right) + O(x^{l-m})$$

holds for $x \rightarrow 0+0$.

Proofs are based on the theory of continuous solutions of the equation (1) that has been given in a joint paper of M. Kuczma and the present author (Fund. Math. 58 (1966), 163–175).

ZENON MOSZNER: Sur une hypothèse au sujet des fonctions subadditives.

En répondant à une question de M. P. Fischer (**P23S1**, Aeq. Math. 2, pp. 380–386) on montre qu'il existe des fonctions φ non-négatives d'une variable positive, pour lesquelles $\varphi(\lambda x) = \lambda \varphi(x)$ pour λ rationnel positif, qui remplissent l'inégalité $\varphi(x+y) \leq \leq \varphi(x) + \varphi(y)$ et qui ne sont pas de la forme $\varphi(x) = \|f(x)\|$, où f est une fonction additive d'une variable réelle, prenant des valeurs dans un espace normé. On peut démontrer qu'une fonction φ , remplissant des conditions formulés plus haut, doit être de la forme $\varphi(x) = \|f(x)\|$ si et seulement si l'on a $\varphi(|x+y|) \leq \varphi(|x|) + \varphi(|y|)$ pour chaque nombre réel x et y .

MAREK KUCZMA: Convex functions and Cauchy's functional equation.

A function $f: \Delta \rightarrow \mathbf{R}$ is called convex provided the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \quad (1)$$

holds for all $x, y \in \Delta$. Here Δ stands for an open real interval, though most results are easily extendable to the case of functions of several variables. A particular class of convex functions is provided by the solutions of Cauchy's functional equation

$$f(x+y) = f(x) + f(y). \quad (2)$$

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be classes of sets defined as follows:

$T \in \mathcal{A}$ iff every convex function $f: \Delta \rightarrow \mathbf{R}$, $T \subset \Delta$, bounded above on T is continuous in Δ . $T \in \mathcal{B}$ iff every additive function (i.e., satisfying (2)) $f: \mathbf{R} \rightarrow \mathbf{R}$ bounded above on T is continuous. $T \in \mathcal{C}$ iff every additive function bounded (from both sides) on T is continuous.

As recently proved by Marcin E. Kuczma, we have $\mathcal{A} = \mathcal{B}$. On the other hand, the inclusion $\mathcal{B} \subset \mathcal{C}$ is proper, (**P36S1**).

Some sufficient (but not necessary) conditions are given for a set T to belong to $\mathcal{A} = \mathcal{B}$, as well as a necessary (but not sufficient) condition for T to belong to $\mathcal{C}$.

These conditions are partially due to R. Ger. However, the problem **P35** of finding intrinsic conditions which are both necessary and sufficient for T to belong to $\mathcal{B}$ or $\mathcal{C}$ remains open. It is also an open problem whether the condition $m_e(J(T)) > 0$ is necessary for T to belong to $\mathcal{B}$. [The condition $m_i(J(T)) > 0$ is sufficient for $T \in \mathcal{B}$]. Here m_e and m_i denote the outer and inner Lebesgue measure, respectively, and $J(T)$ is the midpoint convex hull of the set T .

A function $f: \Delta \rightarrow \mathbf{R}$ is called *almost convex* iff (1) holds almost everywhere in $\Delta \times \Delta$ (in the sense of the planar Lebesgue measure). As an analogue of the results of W. B. Jurkat and N. G. de Bruijn concerning almost additive functions, we have the following theorem:

If $f: \Delta \rightarrow \mathbf{R}$ is almost convex, then there exists a unique convex function $g: \Delta \rightarrow \mathbf{R}$ such that $f = g$ almost everywhere (in the sense of the linear Lebesgue measure) in Δ . This function g is given by the formula

$$g(x) = \inf_{h \in \Delta_x} \text{ess } \frac{1}{2} [f(x+h) + f(x-h)],$$

where Δ_x is the largest open interval centered at x and contained in Δ .

It is of interest to investigate how far the condition

$$f(x) = \inf_{h \in \Delta_x} \text{ess } \frac{1}{2} [f(x+h) + f(x-h)] \quad (3)$$

is characteristic for the convexity of f . If $f: \Delta \rightarrow \mathbf{R}$ is convex, then (3) holds. Conversely, if (3) holds and either (i) f is measurable, or (ii) f is almost convex, then f is convex. It is an open problem to which extent the additional condition (i) or (ii) is essential.

Concerning further details and references, one may consult the following forthcoming papers:

- [1] ROMAN GER, *Some Remarks on Convex Functions*, to appear in *Fundamenta Mathematicae*.
- [2] ROMAN GER and MAREK KUCZMA, *On the Boundedness and Continuity of Convex Functions and Additive Functions*, to appear in *Aequationes Mathematicae*.
- [3] MARCIN E. KUCZMA, *On Discontinuous Additive Functions*, to appear in *Fundamenta Mathematicae*.
- [4] MAREK KUCZMA, *Almost Convex Functions*, to appear in *Colloquium Mathematicum*.

P. FISCHER: *On the inequality* $\sum g(p_i)f(p_i) \geq \sum g(q_i)f(q_i)$.

We deal with the inequality

$$\sum g(p_i)f(p_i) \geq \sum g(q_i)f(q_i) \quad (*)$$

where $0 < p_i < 1$, $0 < q_i < 1$ and $\sum p_i = \sum q_i = 1$, which is a generalization of the inequality

$$\sum p_i f(p_i) \geq \sum p_i f(q_i),$$

where $0 < p_i < 1$, $0 < q_i < 1$ and $\sum p_i = \sum q_i = 1$. We prove among others, the following.

THEOREM. *If g is a strictly increasing and continuous function, then the general solution of the inequality (*) can be written in the following form: ($n \geq 3$)*

$$f(p) = c \int \frac{dp}{g(p)} + d$$

where $c \geq 0$ and d is an arbitrary real number.

We give several generalizations of this theorem.

Z. DARÓCZY and L. LOSONCZI: Comparison of mean values. (Presented by J. Aczél.)

A joint paper of Z. Daróczy and L. Losonczi and three papers of L. Losonczi are reported upon. These give necessary and sufficient conditions for the validity of

$$\phi^{-1} \left(\frac{\sum f_i[k(x_i, y_i)] \phi[k(x_i, y_i)]}{\sum f_i[k(x_i, y_i)]} \right) \leq k \left[\psi^{-1} \left(\frac{\sum g_i(x_i) \psi(x_i)}{\sum g_i(x_i)} \right), \right. \\ \left. \mathcal{X}^{-1} \left(\frac{\sum h_i(y_i) \mathcal{X}(y_i)}{\sum h_i(y_i)} \right) \right]$$

and of its special cases, usually under differentiability conditions. For $f_i = f$, $g_i = g$, $k(x, y) = x$, however, only continuity (and strict monotony for ϕ , ψ , positivity of f , g) is supposed.

DAVID LOVELOCK: Certain functional equations occurring in general relativity.

In a detailed analysis of the role played by variational principles in general relativity (with a view to establishing that in a 4-dimensional space the Einstein field equations are the only second order field equations derivable from a Lagrange density of the type

$$L = L(g_{ij}; g_{ij,k}; g_{ij,kh}),$$

g_{ij} being the components of a symmetric tensor) a problem in functional equations is encountered, viz. to construct all covariant tensors of rank 4, (with components ψ_{ijkh}), which are concomitants of g_{ab} ; i.e. $\psi_{ijkh} = \psi_{ijkh}(g_{ab})$. In this case it can be shown that, under proper transformations,

$$\psi_{ijkh} = a g_{ij} g_{kh} + b g_{ik} g_{jh} + c g_{ih} g_{jk}$$

if $n > 2$ and $n \neq 4$, while if $n = 4$

$$\psi_{ijkh} = a g_{ij} g_{kh} + b g_{ik} g_{jh} + c g_{ih} g_{jk} + \alpha \epsilon_{ijkh}$$

where a, b, c, α are constants and ε_{ijkh} are the components of the 4-dimensional alternating tensor (D. Lovelock: Arch. Rational Mech. Anal. 33 (1969), 54–70).

In a slightly different context a similar problem arises, viz. to construct all ϕ^{ijkhrs} which are the components of a contravariant tensor of rank 6 with the following properties:

- (i) $\phi^{ijkhrs} = \phi^{jikhrs} = \phi^{ijhkr s} = \phi^{ijkhsr}$,
- (ii) $\phi^{ijkhrs} = \phi^{khi jrs} = \phi^{rskhi j}$,
- (iii) $\phi^{ijkhrs} + \phi^{kijhrs} + \phi^{jkihrs} = 0$,
- (iv) $\phi^{ijkhrs} = \phi^{ijkhrs} (g_{ab})$.

This functional equation may also be solved but in the process extensive use is made of (i)–(iii). From this it can be shown that in a 4-dimensional space the only divergence-free symmetric tensors of contravariant rank 2 which are concomitants of g_{ij} and its first two derivatives are the Einstein tensor and g^{ij} . It is possible to extend these results to n -dimensions but the corresponding functional equations become more complicated. (D. Lovelock, Divergence-Free Tensorial Concomitants, to appear in Aeq. Math. 4 (1970)).

E. HILLE: Cryptoanalysis.

Mathematics is loaded with codes which have to be deciphered. In particular, every functional equation contains a message more or less well hidden. All information about the solution is built-in into the equation and becomes available by asking the right questions. This is a priori information available before explicit solutions have been found. Some properties are categorical, pertaining to all solutions, others conditional, valid only under restrictive assumptions. Thus $w(0) w(z+a) = w(a) w(z)$ is a categorical property of solutions of $w'(z) = w(z)$, while linearity of the solutions of Cauchy's equation is conditional since it is valid iff the solution is measurable. The purpose of this communication is to call attention to this range of ideas and to the need of general methods for a priori analysis. Several special cases are studied.

For solutions of the resolvent equation in a Banach algebra local boundedness implies local analyticity. – For solutions of the Kolmogorov-Nagumo equation in the theory of mean values

$$f[A(s, t)] = \frac{1}{2} [f(s) + f(t)],$$

where $A(s, t)$ is a mean value, continuity implies uniqueness for two end-point conditions (J. Aczél) and that the solution, if not a constant, is strictly monotone. – Solutions

of the Thomas-Fermi equation $y''(x) = x^{-1/2} [y(x)]^{3/2}$ with $y(b) > 0$, $y'(b) \geq 0$, $b > 0$, become infinite as x increases to some finite value $x = a$ in such a manner that $y(x) < 400 a (x - a)^{-4}$. This is the right order of magnitude but $x = a$ is *not* a pole of the solution.

J. B. MILLER: Linear operators on Banach algebras.

A number of functional equations for bounded linear operators on a Banach algebra $\mathbf{A}$ with identity e have been studied in recent years. The consequences of *Baxter's equation*

$$Tx \cdot Ty = T(Tx \cdot y + x \cdot Ty - \theta xy) \quad \text{all } x, y \in \mathbf{A},$$

which arises in the theory of queues and in the description of cumulative sums of equally distributed random variables, can be described in some detail provided the parameter θ is taken to be either 0 or 1. The spectrum of T , $Sp(T)$, and a formula for the resolvent $R(\lambda, T) \equiv (\lambda I - T)^{-1}$ can be stated, and a form of representation for T given. When θ is an element of $\mathbf{A}$ other than 0 or e , there is a more complicated formula for $R(\lambda, T)$, but the spectrum is unknown.

The *Reynolds equation*

$$Tx \cdot Ty = T(Tx \cdot y + x \cdot Ty - Tx \cdot Ty) \quad \text{all } x, y \in \mathbf{A},$$

extracted by J. Kampé de Fériet from the work of O. Reynolds on turbulence, has a rich but incomplete theory. A formula for $R(\lambda, T)$ is known for the case when $\|Te\| < 1$, T then being quasinilpotent so that $Sp(T) = \{0\}$, but this is by no means the typical case, and examples with more interesting spectra have been constructed. If $Sp(T)$ lies outside the annular region bounded by two circles having 0 and 1 as inverse points, there is a representation formula for T ,

$$T = Q + (I - D)^{-1} - (I - G)^{-1}$$

where $QD = DQ = D$, $QG = GQ = 0$, $Q^2 = Q$; and if Q and $I - Q$ are averaging operators, D is a derivation (that is, $D(xy) = Dx \cdot y + x \cdot Dy$) and G satisfies Baxter's equation with $\theta = 0$.

An *averaging operator* A is one for which

$$A(Ax \cdot y) = A(x \cdot Ay) = Ax \cdot Ay \quad \text{all } x, y \in \mathbf{A}$$

Fairly complete spectral analysis of averaging operators is possible.

Certain other connections between these and other types of operators are known.

ABE SKLAR: Reduction of a class of functional equations to conjugacy equations.

Let S be a non-empty set, d and r positive integers, and F a function mapping a non-empty subset of S^d into S^r . Then we say that F is a *function over S of degree d and rank r* , and we write $d = \delta F$, $r = \varrho F$ (cf. B. Schweizer and A. Sklar, *Aequationes Math.* 2 (1969), 62–85). Now for a fixed positive integer m , let G, H be functions over S with $\delta G - \varrho G = \delta H - \varrho H = m$, and let F be a function over S with $\varrho G \leq \delta F \leq \delta G$, $\varrho F = \varrho H$. Then many functional equations are of the form

$$F(G(x_1, \dots, x_{\delta G}), \dots, x_{m+\delta F}) = H(x_1, \dots, x_m, F(x_{m+1}, \dots, x_{m+\delta F})). \quad (1)$$

E.g., the equations of homogeneous functions, the translation equation, and the associativity equation and some of its generalizations, are all special cases of (1).

Now let F, G, H be (given or unknown) functions over S with degrees and ranks interrelated in the manner specified above. Define functions F_1, G_1, H_1 such that $\delta F_1 = \delta G + m$, $\varrho F_1 = \delta H + m$, $\delta G_1 = \varrho G_1 = \delta G + m$, $\delta H_1 = \varrho H_1 = \delta H + m$, via:

$$F_1(x_1, \dots, x_{\delta G+m}) = (x_1, \dots, x_m, F(x_{m+1}, \dots, x_{m+\delta F}), x_{\delta G+1}, \dots, x_{\delta G+m}),$$

$$G_1(x_1, \dots, x_{\delta G+m}) = (x_1, \dots, x_m, G(x_1, \dots, x_{\delta G}), x_{\delta G+1}, \dots, x_{\delta G+m}),$$

$$H(x_1, \dots, x_{\delta H+m}) = (x_1, \dots, x_m, H(x_1, \dots, x_{\delta H}), x_{\delta H+1}, \dots, x_{\delta H+m}).$$

Then we have the following Theorem: The functions F, G, H satisfy the equation (1) if and only if the functions F_1, G_1, H_1 satisfy the following (conjugacy) equation:

$$F_1(G_1(x_1, \dots, x_{\delta G+m})) = H_1(F_1(x_1, \dots, x_{\delta G+m})). \quad (2)$$

DAVID ZUPNIK: Cayley-functions.

In connection with the functional equation of associativity, the following question is of interest: Given a one-place function f , does there exist a semigroup (associative 2-place function) F on $\text{Dom } f$, and an element $a \in \text{Dom } f$, such that $f x = F a x$ for all $x \in \text{Dom } f$? If such an F and a exist, then we call f a *Cayley-function*.

Before stating the theorem, which fully characterizes Cayley-functions it will be convenient to introduce the following definitions:

A function f is *non-range-stable* if $\text{Ran } f^{m+1} \subsetneq \text{Ran } f^m$ for every $m \geq 0$. A function f is *eventually range-stable* if there exists an integer $s \geq 0$, such that $\text{Ran } f^m = \text{Ran } f^{m+1}$ if and only if $m \geq s$, where s is called the *stabilizer of f* .

Let f be eventually range-stable with stabilizer s . Then the following is easily verified: The function $g = f|_{\text{Ran } f^s}$, the *basic function of f* , is a function of equal domain and range (viz. $\text{Ran } f^s$). Furthermore, f^s is an *inflation* of g^s ; i.e., $f^s = g^s p$,

where p the inflation factor of f is a unique idempotent function such that $\text{Ran } p = \text{Rang} = \text{Dom } g = \text{Dom } f = \text{Dom } p$. Every ev. range-stable function has at least one characteristic element a with the following properties: $a \in \text{Ran } f$, and $f^m a \in \text{Ran } f^s$ iff $m \geq s - 1$.

THEOREM. A function f is a Cayley-function if and only if $\text{Ran } f \subset \text{Dom } f$ and f satisfies one of the following conditions:

1. f is non-range-stable, and there exists an element $a \in \text{Dom } f$ such that for every $x \in \text{Dom } f$, $f^m x = f^n a$ implies $n \geq m$, and $f^m a = f^n a$ implies $m = n$,

2. f is eventually range, stable with stabilizer s and inflation factor p

and either: a) its basic function is invertible and there exists a characteristic element a such that $f^m pa = pa$ implies $f^m = p$.

or: b) its basic function is noninvertible, and there exists a characteristic element a such that for every $n > 1$, there exists in $\text{Ran } f^s$ elements $x_{n-1} \neq f^{n-1} pa$ such that $fx_{n-1} = f^n pa$.

JOHN BAILLIEUL: Green's relations in some finite function semigroups.

DEFINITION 1. F_n will denote the set of functions on a finite set of n elements. That is, F_n will consist of all functions whose domain and range are subsets of a given n element set.

DEFINITION 2. We define Green's relations as follows:

(1) We shall say that $f, g \in F_n$ are $\mathcal{R}$ related if they have the same range. ($f\mathcal{R}g$ if $Lf = Lg$)

(2) We shall say $f, g \in F_n$ are $\mathcal{L}$ related if they have the same domain and partition this common domain in the same way. ($f\mathcal{L}g$ if (i) $Rf = Rg$ and (ii) $f(x) = f(y) \Leftrightarrow g(x) = g(y)$).

(3) We shall say $f, g \in F_n$ are $\mathcal{H}$ related if they are both $\mathcal{R}$ related and $\mathcal{L}$ related.

(4) We shall say $f, g \in F_n$ are $\mathcal{D}$ related if their ranges have the same cardinality.

DEFINITIONS 3. Stirling numbers of the second kind are denoted by symbols of the form $S(m, n)$, where $S(m, n)$ is defined to be the number of ways of partitioning a finite set of n elements into m cells.

THEOREM 1. i) $S(0, n) = \delta_{0, n}$

ii) $S(1, n) = S(n, n) = 1$

iii) $S(m, n+1) = mS(m, n) + S(m-1, n)$

iv) $S(k+1, n+1) = \sum_{j=k}^n \binom{n}{j} S(k, j)$

THEOREM 2. *The number of $\mathcal{R}$ classes of functions in F_n with k element ranges is $\binom{n}{k}$.*

THEOREM 3. *The number of $\mathcal{L}$ classes of functions in F_n with k element ranges is $S(k+1, n+1)$.*

THEOREM 4. *The number of $\mathcal{D}$ classes in F_n is $n+1$.*

THEOREM 5. *The number of $\mathcal{H}$ classes of functions in F_n with k element ranges is $\binom{n}{k} S(k+1, n+1)$.*

For convenience we shall denote the $\mathcal{D}$ class whose functions have k element ranges by D_k .

THEOREM 6. i) *The number of functions in each $\mathcal{R}$ class contained in D_k is $k!S(k+1, n+1)$.*

ii) *The number of functions in each $\mathcal{L}$ class in D_k is $\binom{n}{k} k!$.*

iii) *The number of functions in each $\mathcal{H}$ class in D_k is $k!$.*

iv) *The number of functions in D_k is $k! \binom{n}{k} S(k+1, n+1)$.*

THEOREM 7. *The number of idempotent functions in D_k is*

$$\sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} k^j$$

J ACZÉL: On a theorem of Belousov.

A distributive type hyperidentity is of the form $x1(y2z) = (x3y)4(x5z)$ and is supposed valid whenever arbitrary elements of a fundamental set S are put in place of the variables (free elements) x, y, z and arbitrary elements of a given set Σ of binary operations defined on S are put in place of the free operations 1, 2, 3, 4, 5 (not necessarily all different).

One of the most interesting theorems in V. D. Belousov's paper (Russian Math. Surveys 20 (1965), No. 1, 75-143) can be formulated as follows: The usual distributivity $x1(y2z) = (x1y)2(x1z)$ and the self-distributivity $x1(y1z) = (x1y)1(x1z)$ are the only distributive type hyperidentities possible, if Σ has at least two elements and if, for every element $\circ$ of the operation-set Σ , $(S, \circ)$ is a quasigroup.

Unfortunately, the proof in Belousov's paper is incomplete. – After having filled those gaps, the present author has found a simpler proof which has been presented here.

D. GIROD and J. H. B. KEMPERMAN: On the functional equation $\sum_{j=0}^n a_j f(x + T_j y) = 0$.

Let A and B be torsion-free Abelian groups, and let $\mathcal{F}(A, B)$ denote the set of all functions $f: A \rightarrow B$. Let $\bar{A}$ and $\bar{B}$ be minimal divisible extensions of A and B respectively, and let H be a basis for $\bar{A}$ as a vector space over the rationals $\mathcal{Q}$; for each $x \in \bar{A}$, $x = \sum_{\alpha \in H} x_\alpha \alpha$. Define Γ to be the set of all functions $\gamma: H \rightarrow \mathbb{Z}^+$ with $|\gamma| = \sum_{\alpha \in H} \gamma(\alpha) < \infty$. Finally let $x \in \bar{A}$, and define $x^\gamma = \prod_{\alpha \in H} x_\alpha^{\gamma(\alpha)}$. Then we have the following

THEOREM 1. *Let $f \in \mathcal{F}(A, B)$. The following are equivalent:*

- i) $\Delta_h^q f = 0$ for all $h \in A$
- ii) $\Delta_{h_1} \Delta_{h_2} \dots \Delta_{h_q} f = 0$ for all $h_1, h_2, \dots, h_q \in A$.
- iii) f has a unique extension to a function $\bar{f}: \bar{A} \rightarrow \bar{B}$, satisfying i) for all $h \in \bar{A}$, and $\bar{f}$ has a unique representation of the form $\bar{f}(x) = \sum_{|\gamma| < q} x^\gamma b_\gamma$ for some choice of the $b_\gamma \in \bar{B}$.

For convenience define $\{f \in \mathcal{F}(A, B) \mid \Delta_h^q f = 0 \text{ for all } h \in A\} = \mathcal{F}_q$. Theorem 1 shows that the functions in $\mathcal{F}_q$ have an interesting algebraic structure. Moreover, results of Kemperman imply that if $A = \mathbb{R}^n$ and $B = \mathbb{R}$, then $f \in \mathcal{F}_q$ and bounded on a set of positive Lebesgue measure $\Rightarrow f$ is a polynomial of degree $< q$.

Now let X and Y be torsion-free Abelian groups and let K be a field of characteristic 0. Let $\mathcal{F}(X, K)$ be as before and denote by $\mathcal{L}(X)$ the algebra of operators on $\mathcal{F}$ generated over K by the shift operators E_h , $h \in X$ ($E_h f(x) = f(x + h)$). As usual, the multiplication in $\mathcal{L}(X)$ is composition of operators. One can prove the following theorem by elementary vector-space reasoning.

THEOREM 2. *Let $\mathcal{A} \subset \mathcal{L}(X)$ and let $[\mathcal{A}]$ denote the ideal generated by $\mathcal{A}$.*

Let

$$\Phi(\mathcal{A}) = \{f \in \mathcal{F}(X, K) \mid Uf = 0 \text{ for all } U \in \mathcal{A}\}.$$

Then

$$[\mathcal{A}] = \{U \in \mathcal{L}(X) \mid Uf = 0 \text{ for all } f \in \Phi(\mathcal{A})\}.$$

Now let $T_0, T_1, \dots, T_n$ be given homomorphisms: $Y \rightarrow X$ and let $a_0, a_1, \dots, a_n$ be given elements of K . We are interested in the solutions in $\mathcal{F}$ to the functional equation $\sum_{j=0}^n a_j f(x + T_j y) = 0$ for all $y \in Y$; in operator notation this becomes $U_y f = 0$ for all $y \in Y$, where $u_y = \sum_{j=0}^n a_j E_{T_j y}$. As before, let $\Phi = \{f \mid U_y f = 0 \text{ for all } y \in Y\}$. We solve this functional equation by deriving conditions on $h \in X$ which are necessary and sufficient for the operator Δ_h^q to be in $J = [\{U_y \mid y \in Y\}]$, and then applying theorem 1.

Let σ be a surjective homomorphism: $X \rightarrow G$, G some Abelian group, and define $\hat{\sigma}: \mathcal{L}(X) \rightarrow \mathcal{L}(G)$ by

$$\hat{\sigma}(V) = \hat{\sigma}\left(\sum_{i=1}^m b_i E_{h_i}\right) = \sum_{i=1}^m b_i E_{\sigma(h_i)}.$$

Note that

$$\hat{\sigma}(V) = 0 \Leftrightarrow \sum_{\sigma(h_i) = \sigma(h_j)} b_i = 0 \quad \text{for all } j.$$

We call σ a *collapsing homomorphism* for the U_y if $\hat{\sigma}(U_y) = 0$ for all $y \in Y$; Σ is the set of all collapsing homomorphisms for U_y . Our main result is the following.

THEOREM 3. *There exists a positive integer q such that $\Delta_n^q \in J \Leftrightarrow h \in N$, where $N = \bigcap_{\sigma \in \Sigma} \ker \sigma$. Moreover, if $h \in N$, then $\Delta_n^q \in J$ for all $q > 2^{n-1}$. In particular, if $N = X$, then $\Phi \subset \mathcal{F}_q$ as soon as $q > 2^{n-1}$.*

J. H. B. KEMPERMAN: On a generalized difference property.

Let X be an abelian group and g a linear and translation invariant class of functions $\psi: X \rightarrow R$. We say that g has property D_q if for each $f: X \rightarrow R$ the relation

$$\Delta_h^q f \in g, \quad \text{for each } h \in X, \quad (1)$$

implies that $f - P \in g$ for some $P: X \rightarrow R$ with

$$\Delta_h^{q+1} f = 0, \quad \text{for each } h \in X. \quad (2)$$

Under a slight regularity condition on f this often implies $P \in g$ hence $f \in g$. We say that g has property W_q if (1) together with a specified weak regularity condition implies $f \in g$. There follows an extensive discussion with examples of the properties D_q , W_q and some related properties.

The above is very important in connection with the functional equation

$$\psi_y = \sum_{j=0}^n a_j f(x + T_j y) \in g, \quad \text{for each } y \in Y. \quad (3)$$

Here, $f: X \rightarrow R$ is the unknown function, Y is a group, $T_j: Y \rightarrow X$ a homomorphism while $a_j \in R$. ($j = 0, \dots, n$).

Assume that (*) for each homomorphism $\sigma: X \rightarrow G$ with $\sigma X \neq \{0\}$ there exists an index k with $\sum_{\sigma T_j = \sigma T_k} a_j \neq 0$. Then (3) implies that (1) holds with $q = \binom{n}{2} - 1$ as follows from unpublished work of D. Girod and the author; (usually (*) is also necessary). Afterwards, one may be able to conclude as above that $f \in g$, provided f satisfies an appropriate regularity condition.

S. KUREPA: Remarks on functional equations in vector spaces.

Functional equations which characterize some elementary functions are taken as

a starting point for defining such functions on a vector space. Different properties as measurability, boundedness on small sets imply continuity of functions which satisfy those equations. Representations of groups or semigroups are considered as solutions of multiplicative Cauchy equation. If such a representation of G is uniformly bounded on a Banach space or in a Banach algebra, then all elements are isometries or have norm equal to one in a suitable norm equivalent to the original norm. If A is a Banach algebra $\mathfrak{a}$ and $A^2=1$ or $A^2=A$ or $A^3=A$ then $\exp itA$ ($t \in \mathbb{R}$) is a group and $\|\exp itA\|=1$, where $\|\cdot\|$ is a norm on $\mathfrak{a}$ equivalent to the original one. If $e^B=1$ then $\|e^{tB}\|=1$ ($t \in \mathbb{R}$) in a suitable equivalent norm on $\mathfrak{a}$. Precise statements and results will appear elsewhere.

JOHN A. BAKER: A difference analogue of the wave equation.

The only continuous solutions of the functional equation $f(x+h, y) + f(x-h, y) - f(x, y+h) - f(x, y-h) = 0$ are of the form $f(x, y) = \alpha(x+y) + \beta(x-y)$ where α and β are continuous. We prove this result of Sakovič as follows. Choose a sequence $\{\phi_n\}$ of $C^\infty(\mathbb{R}^2)$ functions with $\iint \phi_n(x, y) dx dy = 1$ and with support of ϕ_n in a circle centered at the origin with radius $1/n$. If f satisfies the above functional equation and is continuous then $f_n = f * \phi_n$, the convolution of f with ϕ_n , also satisfies the equation. Furthermore, each f_n satisfies the wave equation and $\{f_n\}$ converges to f uniformly on compact subsets of $\mathbb{R}^2$. Also $f_n \in C^\infty(\mathbb{R}^2)$. The result follows readily from these observations. This method can be used to find continuous solutions or to obtain qualitative information concerning continuous solutions of related functional equations.

I. FENYŐ: On a generalisation of a problem of M. Hosszú.

As a generalisation of the functional equation $f(x+y-xy) + f(xy) = f(x) + f(y)$ considered first by M. Hosszú, we treat the functional equation

$$f(r_0 + (r_1x + r_2)(r_3y + r_4)) + g(s_0 + (s_1x + s_2)(s_3y + s_4)) = h(x) + k(y) \quad (1)$$

where f, g, h, k are unknown functions, r_i, s_i given constants, $r_1r_3 \neq 0, s_1s_3 \neq 0$. We replace the unknown functions by distributions and rewrite (1) into a distributional functional equation by means of suitable transformations. The following theorem is proved. The most general solutions of (1) in the domain of distributions are the following:

a) If $(r_1s_2 - r_2s_1)^2 + (r_3s_4 - r_4s_3)^2 \neq 0$ they are the distributions generated by the following functions

$$f = l_1(t - r_0) + l_2 \log |t - r_0| + l_3; \quad g = -\frac{r_1r_3}{s_1s_3} l_1(t - s_0) + k_2 \log |t - s_0| + k_3;$$

$$h = vt + l_2 \log |r_1t + r_2| + k_2 \log |s_1t + s_2| + \mu;$$

$$k = qt + l_2 \log |r_3t + r_4| + k_2 \log |s_3t + s_4| + \tau.$$

The functions above produce at the same time the general solution of (1) in the domain of locally integrable functions.

(b) If $(r_1 s_2 - r_2 s_1)^2 + (r_3 s_4 - r_4 s_3)^2 = 0$, it is the system of the following distributions: g arbitrary distribution,

$$f = -T_1 \left[\frac{s_1 s_3}{r_1 r_3}, s_0 - \frac{s_1 s_3}{r_1 r_3} (r_0 + r_2 r_4) \right] g + \kappa_1 \log |t - r_0| + \kappa_2,$$

$$h = \kappa_1 \log |r_1 t + r_2| + \kappa_3; k = \kappa_1 \log |r_3 t + r_4| + \kappa_2 - \kappa_3.$$

$T_1(\alpha, \beta) g (\alpha \neq 0)$ means the following transformation: if ϕ is an arbitrary Schwartz-testfunction, then

$$T_1(\alpha, \beta) g \cdot \phi = \frac{1}{|\alpha|} g \cdot \phi \left(\frac{t - \beta}{\alpha} \right).$$

HALINA ŚWIATAK: On the regularity of the continuous and locally integrable solutions of a class of functional equations.

The problem of regularity of the continuous and locally integrable solutions of the functional equations

$$\sum_{i=1}^k a_i(x, t) f(\varphi_i(x, t)) = F(x, f(\lambda_1(x)), \dots, f(\lambda_p(x))) \quad (*)$$

$$+ G(f(x + \omega_1(t)), \dots, f(x + \omega_s(t))) + b(x, t),$$

where $x \in R^n$, $t \in \Omega \subset R^n$, $\varphi_i(x, t) = (\varphi_{i1}(x, t), \dots, \varphi_{in}(x, t))$, $\lambda_i(x) = (\lambda_{i1}(x), \dots, \lambda_{in}(x))$, $\omega_i(t) = (\omega_{i1}(t), \dots, \omega_{in}(t))$, $n \geq 1$, $r \geq 1$, is considered.

If the functions $a_i(x, t)$, $b(x, t)$, $\varphi_i(x, t)$, $\lambda_i(x)$, $\omega_i(t)$, $F(x, u_1, \dots, u_p)$, and $G(u_1, \dots, u_s)$ satisfy suitable regularity conditions and if there exist an $\alpha \in \Omega$ and a multi-index p such that

$$\varphi_i(x, \alpha) \equiv x \quad \text{for } i = 1, \dots, k,$$

$$D^q \omega_i(\alpha) = 0 \quad \text{for } i = 1, \dots, s, |q| \leq |p|$$

and the distributional differential equation

$$D_t^p \left(\sum_{i=1}^k a_i(x, t) \Gamma(\varphi_i(x, t)) \right) = 0 \quad (T \in \mathcal{D}')$$

is of constant strength for $t = \alpha$, hypoelliptic for some x_0 , then every continuous solution f of the equation (*) has to be a function of class C^∞ . Some additional assumptions allow one to prove that all the locally integrable solutions f are equal almost everywhere to some solutions of class C^∞ (or that they are functions of class C^∞).

S. GOLAB: On the functional equation $\phi(x+y)=H[\phi(x), x, y]$.

A geometric problem conduced me to a functional equation of the type

$$\phi(x+y) = H[\phi(x), x, y]. \quad (1)$$

This type falls indeed under the more general type

$$\phi(x+y) = F[\phi(x-y), \phi(x), \phi(y), x, y] \quad (2)$$

which is discussed in details in the literature, but it is so special that it can be treated individually, the more so that the question is the existence of solutions in the neighbourhood of the point (0, 0) only.

The equation (1) can possess no solutions, can possess only one solution (or a finite number) or at least a one-parameter family of solutions according to the form of the given function H . The last case can occur if the function H fulfils the condition

$$H[z, x, 0] \equiv z. \quad (3)$$

The condition (3), however, is not sufficient. The function H has to fulfil moreover a certain functional equation of order 4 of the iterative type.

In our geometric problem the function H has the form

$$H(z, x, y) = \frac{qz |\sin x|}{\sqrt{W}} \quad (4)$$

where

$$W(z, x, y) = q^2 \sin^2(x+y) + z^2 \sin^2 y - 2qz \sin y \cdot \sin(x+y) \sqrt{1 - q^2 \sigma^2 z^2 \sin^2 x} \quad (5)$$

(q, σ are two parameters with geometric meaning). For this function the condition (3) is fulfilled.

The general solution of (1) depends in this case on an arbitrary constant C .

TOMASZ PIETRZYKOWSKI: Some applications of λ -calculi and bracket free (TPL2) notation to functional equations.

In the following there are presented examples of certain unformal facts, which subsequently will be described in TPL2.

(a) to describe what are the legal substitutions of variables in respectively $(\bar{a}i)$, $(\bar{a}i)$ ($i=1, 2, 3$) such that respective formulae will not loose the relation between them namely that the function f described in $(\bar{a}i)$ is a solution of $(\bar{a}i)$.

$$(a1) \quad f(u) = a \cdot u \quad \text{where } a \text{ is real}$$

$$(a2) \quad f(u) = a \cdot u - z \quad \text{where } a, z \text{ are real}$$

$$(a3) \quad f(u, v) = a \cdot u - v \quad \text{where } a \text{ is real}$$

and

$$(\bar{a}1) \quad f(x+y) = f(x) + f(y)$$

$$(\bar{a}2) \quad f(x+y) = f(x) + f(y) + z$$

$$(\bar{a}3) \quad f(x+y, z) = f(x, z) + f(y, z) + z$$

(b) to describe the property of the function f which is: $\phi(x) = f(x, y)$ implies that ϕ is monotonic for all y and $\psi(y) = f(x, y)$ implies that is continuous for all x .

Let us assume that there is already given a TPL2 description at the predicate calculi with equality. Adding to it the following sentences:

LET $X; Y; Z; U; V; (X); X \cdot Y; X - Y; X + Y; F$ W BE REAL

LET $F; (F); \lambda X$ FUNCTION; λX REAL BE FUNCTION WHERE X BOUND.

LET λF PHRASE BE PREDICATE WHERE F IS BOUND.

LET REAL; $W; W, X; W X$ BE LIST.

LET $X; Y; Z; U; V; W; F$ BE VARIABLE.

In the following there are the TPL2 sentences which express formally the respective unformal contents of (ai) and ($\bar{a}i$) ($i=1, 2, 3$):

(A) LET F IS SOLUTION OF P IFF PF BE AXIOM (it defines the solution, where P is predicate and $P F$ is phrase)

(A1) $F = \lambda V U \cdot V$ IMPLIES F IS SOLUTION OF $\lambda F F(X+Y) = FX + FY$.

(A2) $F = \lambda V U \cdot V - Z$ IMPLIES F IS SOLUTION OF $\lambda F F(X+Y) =$
 $FX + FY + Z$.

(A3) $F = \lambda V Z U \cdot V - Z$ IMPLIES F IS SOLUTION OF $\lambda F F(X+Y, Z) =$
 $FX, Z + FY, Z + Z$.

Having the above theorems it may be formally verified which substitution of variables are legal.

The property (b) may be expressed as follows:

$\lambda X F X, Y$ IS CONTINUOUS AND $\lambda Y F X, Y$ IS MONOTONIC, providing that predicates IS CONTINUOUS and IS MONOTONIC are defined in TPL2 notation.

H. HARUKI: On a functional equation characterizing conic the sections.

The functional equation

$$|f(x+y) - f(x-y)| = |f(x+\bar{y}) - f(x-\bar{y})|, \quad (1)$$

where f is an entire function of a complex variable z and x, y are complex variables is considered. Equation (1) is connected with Ivory's following Theorem in geometry:

For a family of confocal conics, let P, Q, R, S be the four vertices of a curvilinear rectangle formed by any four conics arbitrarily chosen. Then $\overline{PR} = \overline{QS}$ holds.

The result is a characterization of the conic sections.

PL. KANNAPPAN: Some relations between additive functions.

Let α be the set of all additive functions $f: R \rightarrow R$, that is, $(1) f(x+y) = f(x) + f(y)$, for all $x, y \in R$ (reals). Then α is a vector space over the reals R . Let B be the vector space spanned by all continuous solutions of (1) and by all derivatives on R , that is functions f which satisfy (1) and $(2) f(xy) = f(x)y + xf(y)$, $x, y \in R$.

Problem. When do conditions of the form

$$\sum_{i=1}^n P_i(x) f_i(U_i(x)) = 0$$

where U_i 's are rational functions in x , P_i 's are continuous on R except on singular points of U_i , f_i 's solutions of (1), imply that the f_i 's are linearly dependent relative to B ?

THEOREM. If $f, g \in \alpha$ and if there exists a number α , two continuous functions $P_1 \neq 0, P_2 \neq 0$, A, B constants such that for all $x \neq \alpha$

$$f\left(\frac{1}{x-\alpha}\right) + f\left(\frac{A}{(x-\alpha)^2}\right) = P_1(x) g\left(\frac{1}{x-\alpha}\right) + P_2(x) g\left(\frac{B}{(x-\alpha)^2}\right),$$

then f and g are linearly dependent relative to the subspace B .

E. VINCZE: Über ein allgemeines Lösungsverfahren von Funktionalgleichungen für Skalar-Funktionen auf einem n -dimensionalen Vektor-Gebiet.

Verf. skizziert ein allgemeines Lösungsverfahren für die verallgemeinerte Cauchy-sche Funktionalgleichung

$$f(\xi * \eta) = f(\xi) + f(\eta) \quad (1)$$

$$[\xi = \{x_1, \dots, x_n\}, \eta = \{y_1, \dots, y_n\}, \xi * \eta \in \mathcal{M}_n; f: \mathcal{M}_n \rightarrow I_0(b_0, c_0)],$$

wobei die Vektorkomponenten x_k, y_k ($k = 1, 2, \dots, n$) und die gesuchte Funktion f reell sind, weiterhin die Vektoroperation $\xi * \eta$ explicit angegeben ist. Im Falle $n=2$ geht (1) mit den Bezeichnungen $f(\xi) = F(x_1, x_2)$ und $\xi * \eta = \{H_1(x_1, y_1, x_2, y_2), H_2(x_1, y_1, x_2, y_2)\}$ in die Funktionalgleichung

$$\left. \begin{aligned} F[H_1(x_1, y_1, x_2, y_2), H_2(x_1, y_1, x_2, y_2)] &= F(x_1, x_2) + F(y_1, y_2) \\ [x_k, y_k \in I_k(b_k, c_k); H_k: I_1 \times I_1 \times I_2 \times I_2 \rightarrow I_k; k &= 1, 2; F: I_1 \times I_2 \rightarrow I_0(b_0, c_0)] \end{aligned} \right\} \quad (1')$$

über, wobei H_1, H_2 gegebene, (und nach der Voraussetzung von Verf.) in allen Veränderlichen stetige und monotone (aber nicht unbedingt streng monotone) Funktionen sind, bzw. F die gesuchte Funktion bezeichnet. – Falls die Funktion F in ihren

beiden Veränderlichen stetig und streng monoton ist, und wenn die Gleichungen

$$\begin{aligned} H_1(a_1, a_1, x_2, y_2) &\equiv a_1, \\ H_2(x_1, y_1, a_2, a_2) &\equiv a_2; \\ \left. \begin{aligned} H_1(x_1, a_1, a_2, x_2) &= u \\ H_2(x_1, a_1, a_2, x_2) &= v \end{aligned} \right\} &\Rightarrow \begin{cases} x_1 = G_3(u, v), \\ x_2 = G_4(u, v); \end{cases} \\ G_1(x_1, y_1) &\stackrel{\text{def}}{=} H_1(x_1, y_1, a_2, a_2) \neq \text{konst.}, \\ G_2(x_2, y_2) &\stackrel{\text{def}}{=} H_2(a_1, a_1, x_2, y_2) \neq \text{konst.} \end{aligned}$$

mit irgendeinem festen Vektor $\alpha = \{a_1, a_2\}$ erfüllt sind, dann sind die Funktionen G_1, G_2 assoziativ $[G_k(G_k(x_k, y_k), z_k) = G_k(x_k, G_k(y_k, z_k)); k = 1, 2]$ und die Lösung der Funktionalgleichung (1') hat die Form

$$\begin{aligned} F(u, v) &= k_1 \cdot g_1[G_3(u, v)] + k_2 \cdot g_2[G_4(u, v)], \\ G_1(x_1, y_1) &= g_1^{-1}[g_1(x_1) + g_1(y_1)], \\ G_2(x_2, y_2) &= g_2^{-1}[g_2(x_2) + g_2(y_2)], \end{aligned}$$

wobei g_1, g_2 stetige und streng monotone Funktionen sind und k_1, k_2 ($|k_1| + |k_2| > 0$) beliebige reelle Konstanten bezeichnen. – Ähnliche Sätze gelten auch im allgemeineren Falle $n > 2$.

ANDRZEJ ZAJTZ: On some matrix functional equations occurring in the theory of group representations.

Let G be a group and $s, t \in G$. We consider the following matrix functional equations

- (i) $F(st) = F(s)F(t)$
- (ii) $g(st) = F(s)g(t) + g(s)$
- (iii) $f(F(s)u) = H(s)f(u),$

where F, H are square matrices and f, g, u are vectors. In equations (ii) and (iii) the functions $F(s)$ and $H(s)$ are given and fulfil the equation (i). The solutions of (i) and (ii) determine some linear and affine representations of the group G respectively; the equation (iii) is connected with the similarity problem for group representations.

The structure of solutions of the above equations were given in the cases where G is a linear group $GL(n)$ or a full differential group L^n . In the former case we are mostly interested in the non-measurable solutions, especially for equations (i) and (ii). The results have immediate applications to the theory of geometric objects.

Problems and Remarks

1. Remark and Problem. An axiomatic characterization of the quantity

$$-\sum_{k=1}^n p_k \log q_k / \sum_{k=1}^n p_k$$

of *inaccuracy* by Prem Nath leads to the system

$$F(pq, rs) = F(p, r) + F(q, s) \quad \text{and} \quad (1)$$

$$(p+q)F(p+q, r) = pF(p, r) + qF(q, r) \quad (2)$$

of functional equations ($\{p, q, r, s\} \in (0, 1]^4$). Since it is natural to suppose $F(p, r) \geq 0$, (Prem Nath has supposed F to be continuous) it was not difficult for the undersigned to give the general solution. For instance, (1) gives

$$F(pq, 1) = F(p, 1) + F(q, 1), \quad F(p, 1) = a \log p \quad (3)$$

$$F(1, rs) = F(1, r) + F(1, s), \quad F(1, r) = b \log r \quad (4)$$

$$F(p, s) = F(p, 1) + F(1, s) \quad \text{and so} \quad (5)$$

$$F(p, s) = a \log p + b \log s$$

and putting this into (2) we get $a=0$

$$F(p, r) = b \log r, \quad (6)$$

in particular

$$F(p, 1) = 0 \quad (7)$$

Now, we can get (7) (from which, of course, (6) follows immediately by (4) and (5)) also in the following way: use (2) and (1) for $r=s=1$, then $(p+q)F(p+q, 1) = pF(p, 1) + qF(q, 1)$ and $F(pq, 1) = F(p, 1) + F(q, 1)$ or, for $f(p) = pF(p, 1)$,

$$f(p+q) = f(p) + f(q), \quad (8)$$

$$f(pq) = pf(q) + qf(p) \quad (9)$$

that is, f is a *derivative*. Of course, the only continuous (or non-negative) derivative is the function identically 0, which proves (7). This proof shows that it is enough to suppose $p \rightarrow F(p, 1)$ and $r \rightarrow F(1, r)$ to be continuous (or non-negative, bounded from one side on a set of positive measure, etc.).

For possible further weakening of the regularity suppositions on $p \rightarrow F(p, 1)$ (i.e., f), the following *problem* (P49) is of interest: Are there regularity conditions on f under which (8) and (9) (i.e. f being a derivation) imply that f is continuous but the Cauchy equation (8) *alone* does not imply the continuity of f under the same regularity conditions?

J. ACZÉL

2. Remark on the Landau-Kallman-Rota inequality. Let f be continuous together with its first two derivatives in the interval $[0, a]$ and denote $\max |f(t)|$, $\max |f'(t)|$, $\max |f''(t)|$ by M_0, M_1, M_2 , respectively. It was shown by E. Landau that

$$M_1^2 \leq 4 M_0 M_2 \quad (1)$$

and the same inequality holds for the L_p -norms.

In a recent unpublished note R. Kallman and G. C. Rota have generalized the inequality to a particular class of semi-groups. Let X be a B -space over the complex field, $\{T(s)\}$ a one-parameter semi-group of linear bounded transformations from X to itself such that $\|T(s)\| \leq 1$ for all $s \geq 0$. Let A be the infinitesimal generator of the semi-group, $D[A] \subset X$ its domain of definition. Then for any $f \in D[A^2]$

$$\|Af\|^2 \leq 4 \|f\| \|A^2f\|. \quad (2)$$

Proof. For $f \in D[A^2]$ we have

$$T(s)f = f + sAf + \int_0^s (s-u) T(u) A^2f du \quad (3)$$

whence

$$s \|Af\| \leq \|f\| + \|T(s)f\| + \int_0^s (s-u) \|T(u) A^2f\| du \leq 2 \|f\| + \frac{1}{2} s^2 \|A^2f\|$$

since $\|T(s)\| \leq 1$. Hence

$$\|Af\| \leq \frac{2}{s} \|f\| + \frac{1}{2} s \|A^2f\|. \quad (4)$$

If $A^2f \neq 0$, the minimum of the right hand side is reached for

$$s = 2 \left\{ \frac{\|f\|}{\|A^2f\|} \right\}^{1/2}$$

and the minimum value is

$$2 [\|f\| \|A^2f\|]^{1/2}$$

which gives (2).

Landau's inequality is obtained by taking $X = C[0, \infty]$ or $L_p[0, \infty]$ and defining

$$T(s)[f](t) = f(t+s), \quad s \geq 0, \quad A[f](t) = f'(t). \quad (5)$$

So far this is Kallman and Rota. Hille remarked that other unimodular semi-groups are generated by higher derivatives. Thus $A[f](t) = f''(t)$ generates the Gauss-Weierstrass semi-group which is uni-modular. This gives

$$\|f''\|^2 \leq 4 \|f\| \|f^{(4)}\|, \quad (6)$$

if $f^{(4)}$ belongs to the space $C[-\infty, \infty]$ or $L_p(-\infty, \infty)$. More generally we can use as infinitesimal generator one of the operators

$$(-1)^{k-1} f^{(2k)} \quad \text{or} \quad if^{(2k+1)}. \quad (7)$$

This gives

$$\|f^{(n)}\|^2 \leq 4 \|f\| \|f^{(2n)}\| \quad (8)$$

for any positive integer n . The space is $C[-\infty, \infty]$ or $L_p[-\infty, \infty]$. If n is even, a semi-group is generated, if n is odd, a group. Prof. Reuben Hersh (UNM) contributed to the discussion of (7).

E. HILLE

3. Remark on the Landau inequality.

§ 1. Let X be a Banach space, A an infinitesimal operator which generates a strongly continuous semigroup $\{T(s) \mid s > 0\}$. R. Kallman and G. C. Rota have proved that the Landau type inequality

$$\|AX\|^2 \leq 4 \|X\| \|A^2X\|, \quad X \in D(A^2), \quad A^2X \neq 0 \quad (1)$$

holds true provided that

$$\|T(s)\| = 1, \quad \forall s > 0. \quad (2)$$

In general we have

$$\|T(s)\| \leq M \exp \omega s, \quad \forall s > 0, \quad (3)$$

where M and ω are constants. Now we have the inequality

$$\|(A - \omega I)x\|^2 \leq 2M(1 + M) \|(A - \omega I)^2x\|, \quad (4)$$

provided that $x \in D(A^2)$, $(A - \omega I)x \neq 0$ and I is the identity operator. In the case $\omega = 0$, $M = I$ we get (1).

For a real number $c > 0$ set

$$M_c = \sup \|T(s)\| \quad (s \leq c). \quad (5)$$

Then we have

$$\|Ax\| \leq \frac{(1 + M_c) \|x\|}{s} + \frac{M_c \|A^2x\|}{2} s, \quad s \leq c. \quad (6)$$

The function

$$g(s) = \frac{a}{s} + bs, \quad s > 0 \quad (7)$$

is decreasing for $s < \sqrt{a/b}$, increasing for $s > \sqrt{a/b}$, has minimum $2\sqrt{ab}$ for $s = \sqrt{a/b}$. Set $a = (1 + M_c) \|x\|$, $b = \frac{1}{2} M_c \|A^2x\| > 0$.

If $\sqrt{a/b} \leq c$, then (6) implies $\|Ax\|^2 \leq 2(1 + M_c) M_c \|x\| \cdot \|A^2x\|$.

If $\sqrt{a/b} > c$, then (6) implies:

$$\|Ax\| < \frac{1 + M_c}{c} \|x\| + \frac{M_c \|A^2x\|}{2} c.$$

By use of (3) instead of (6) one can get:

$$\|Ax\| < \frac{1 + Me^{\omega s}}{s} \|x\| + M \|A^2x\| \frac{e^{\omega s} - 1 - \omega s}{\omega^2 \cdot s}.$$

from which some other types of inequalities may be derived.

§ 2. If A is an operator in a Banach space X and if A generates a cosine function $s \mapsto T(s)$, such that $\|T(s)\| \leq M$, $\forall s > 0$ then

$$\|Ax\|^2 \leq \frac{2}{3}(1 + M) M \|x\| \cdot \|A^2x\|.$$

If $M=1$, then

$$\|Ax\|^2 \leq \frac{4}{3} \|x\| \|A^2x\|, \quad x \in D(A^2), \quad A^2x \neq 0.$$

S. KUREPA

4. Remark concerning the talk of Professor Forte. All continuous, associative, surjective functions $F: S \times S \rightarrow S$, where S is connected subset of the real line, are known under any of several (very weak) additional hypotheses. (See C. R. Storey, Trans. Amer. Math. Soc. 130 (1968), 185–207 and references there.)

Based on this, one can easily show that any continuous function $F: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ satisfying the conditions (i) F is associative and (ii) $F(x+z, y+z) = F(x, y) + z$, given by Professor Forte, must be of one of the following forms.

- a) $F(x, y) = \min \{x, y\}$
- b) $F(x, y) = \max \{x, y\}$
- c) $F(x, y) = x$
- d) $F(x, y) = y$
- e) $F(x, y) = f^{-1}(f(x) + f(y))$ where $f: \mathbf{R} \rightarrow \langle 0, \infty \rangle$ is some homeomorphism.

The solutions of the functional equation resulting from combining (ii) and e) have been given by Professor Aczél so that an explicit expression for the function f can be given.

KERMIT SIGMON

5. Remark and Problem. The above remark does not cover the case we considered in the information theory. As a matter of fact, the problem (P50) has been stated in the following form:

Let $\mathcal{F}$ be the family of the subsets $\Gamma \subset \bar{\mathbf{R}}^+ \times \bar{\mathbf{R}}^+$ which have the further properties:

- a) $\forall x \in]0, +\infty[\exists y < +\infty: (x, y) \in \Gamma$
- b) $(x, y) \in \Gamma \Rightarrow (y, x) \in \Gamma$
- c) $(x, y) \in \Gamma, y' > y \Rightarrow (x, y') \in \Gamma$.

Let $\mathcal{M}$ be the class of functions $F: \Gamma \rightarrow \bar{\mathbf{R}}^+$, which have the following properties:

- a) $y \rightarrow F$ is nondecreasing
- b) $\forall x \in [0, +\infty] \exists y_x \in [0, +\infty]: x \top y_x = 0$.

Stating the foundations of information theory without probability we solved the problem which consists in finding the couple Γ, F in $\mathcal{F}$ and $\mathcal{M}$ respectively, such that

the function F satisfies on Γ the functional system:

$$\text{I} \quad F(x, y) = F(y, x)$$

$$\text{II} \quad F[F(x, y), z] = F[x, F(y, z)]$$

$$\text{III} \quad F(x+z, y+z) = F(x, y) + z \quad \forall z \geq 0.$$

However, it would be also of some interest in the information theory if one could give a representation theorem for the solutions in $(\mathcal{F}, \mathcal{M})$ of I–II only. This last problem can be solved in the class of the functions which are not only monotonic but also continuous on Γ , using some results by P. S. Mostert and A. L. Shields (Ann. of Math. 65 (1957), 128–130, cf. the remark 4 made by K. Sigmon) concerning the general representation of topological semigroups on a closed interval.

B. FORTE

6. Problems. The following seem to be natural conditions for the measures of information

$$H_n(p_1, \dots, p_n) \quad \left(\sum_{k=1}^n p_k = 1, p_k \in [0, 1]; \quad k = 1, 2, \dots, n; \quad n = 2, 3, \dots \right):$$

(1) unit:

$$H_2\left(\frac{1}{2}, \frac{1}{2}\right) = 1,$$

(2) symmetry:

$$H_n(p_1, \dots, p_n)$$

is invariant under permutation of $\{p_1, \dots, p_n\}$ ($n = 2, 3, \dots$),

(3) decisivity:

$$H_n(0, \dots, 0, 1) = 0 \quad (n = 2, 3, \dots),$$

(4) expansibility:

$$H_{n+1}(p_1, \dots, p_n, 0) = H_n(p_1, \dots, p_n) \quad (n = 2, 3, \dots),$$

(5)–(6) additive equality and inequality

$$H_{mn}(P_1, \dots, P_{1n}, \dots, P_{m1}, \dots, P_{mn}) \leq H_m(p_1, \dots, p_m) + H_n(q_1, \dots, q_n)$$

where

$$p_j = \sum_{k=1}^n P_{jk}, \quad q_k = \sum_{j=1}^m P_{jk} \quad (j = 1, 2, \dots, m, \quad k = 1, 2, \dots, n), \quad m, n = 2, 3, \dots$$

with equality if

$$P_{jk} = p_j q_k \quad (j = 1, 2, \dots, m; \quad k = 1, 2, \dots, n),$$

(7) nonnegativity

$$H_n(p_1, \dots, p_n) \geq 0 \quad (n = 2, 3, \dots),$$

(8) maximality

$$H_n(p_1, \dots, p_n) \leq H_n\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right) \quad (n = 2, 3, \dots)$$

and a rather natural regularity condition is

$$(9) \quad \lim_{p \rightarrow 0} H_2(p, 1-p) = H_2(0, 1) (= 0 \text{ by (3)}).$$

The Shannon-entropy

$$(10) \quad H_n(p_1, \dots, p_n) = - \sum_{k=1}^n p_k \log p_k \quad (n = 2, 3, \dots)$$

has all these properties. – Questions: (**P51**, **P52**, **P53**, **P54** respectively)

- (i) Are there quantities other than (10) which satisfy the conditions (1)–(8)?
- (ii) Are there quantities other than (10) which satisfy the conditions (1)–(9)?
- (iii) Determine *all* quantities satisfying (1)–(8).
- (iv) Determine *all* quantities satisfying (1)–(9).

P. Benvenuti has pointed out that Rényi's entropy of order 0

$$H_0(p_1, \dots, p_n) = \lim_{\alpha \rightarrow 0} \log \left(\sum_{k=1}^n p_k^\alpha \right)$$

gives an answer to (i), (**P51S1**).

J. ACZÉL

7. Remark. A proof of the following conjecture of G. Laube (Cologne) was given: If $f: R^n \rightarrow R$ is (1) continuous and (2) $f(x) = f(x')$ implies $f(x+y) = f(x'+y)$ for all $y \in R^n$, then there exists a strictly monotonic and continuous function $g: R \rightarrow R$ such that $f(x) = g((a, x)) = g(\sum a_k x_k)$ for all $x \in R^n$. It was also remarked that every injection $f: R^n \rightarrow R^n$ satisfies (2), while every $f: R^n \rightarrow R^m$ satisfying (1) and (2) is on lines of any given set of parallel straight lines in R^n either constant or injective.

J. ACZÉL

8. Remark and Problem. A proof of the following theorem by A. Rényi and the undersigned was presented. Every continuous real function f satisfying

$$f(x) = f(y) \Leftrightarrow f(x+1) = f(y+1) \text{ and } f(x) = f(y) \Leftrightarrow f(x+\alpha) = f(y+\alpha) \quad (1)$$

(α a positive irrational constant number) is either constant or strictly monotonic (cf. my remark 7). The original *problem* (**P55**) of B. Peleg (Jerusalem) which has given rise to the above weaker result is still *unsolved*: Has every continuous real function f satisfying

$$f(x) = f(y) \Leftrightarrow [f(x+1) = f(y+1) \text{ and } f(x+\alpha) = f(y+\alpha)], \quad (2)$$

where α is again a positive irrational constant (this condition is weaker than (1)), also to be either constant or strictly monotonic? As M. Simonovits has remarked, our proof works also to prove the analogue of Peleg's conjecture if α is a *negative* irrational constant number.

J. ACZÉL

9. Remark. Let f denote the quadratic Čebyšev polynomial, i.e., $f(x) = 2x^2 - 1$. Then f maps the closed interval $[-1, 1]$ onto itself, and for almost all $x \in [-1, 1]$, the sequence of iterates $\{x, f(x), f(f(x)), \dots\}$ is aperiodic and distributed on $[-1, 1]$ in the density δ , where $\delta(t) = 1/(\pi\sqrt{1-t^2})$ for $t \in (-1, 1)$. This means that if A is any measurable subset of $[-1, 1]$ and $A_n(x)$ denotes the number of terms out of the first n terms of the sequence $\{x, f(x), f(f(x)), \dots\}$ that are in A , then $\lim_{n \rightarrow \infty} A_n(x)/n$ exists and equals $\int_A \delta(t) dt$. The same result holds for any Čebyšev polynomial of degree at least 2.

This result can be proved in various ways. It follows, e.g., from a theorem of R. L. Adler and T. J. Rivlin [Proc. Amer. Math. Soc. 15 (1964), 794–795] together with the Birkhoff ergodic theorem. It is also possible to prove it with somewhat less ponderous machinery by using the Borel theorem that almost all real numbers are normal to every base, plus a characterization of normal numbers due to D. D. Wall [both theorems may be found, e.g., in I. Niven's Carus Monograph on Irrational Numbers]. Conversely, any advance that is made toward a characterization of the 'almost all' set or the exceptional set will yield information about normal numbers. A trivial example is the following: Every Čebyšev polynomial of degree at least 2 has periodic orbits of every period. (In particular, f has

$$\frac{1}{m} \sum_{d|m} \mu(d) 2^{m/d}$$

orbits of period m for $m \geq 1$, where μ denotes the Möbius function). The points in the periodic orbits correspond to the rational numbers, which are of course not normal.

A. SKLAR

10. Remark. There are several theorems about the non-measurability of the composition of form $f(g(x))$, where the f and g are real functions, which are defined for all real x . Let us state one more: If f is a continuous function, which is different from constant, and if g is such a function, that for every set A of positive measure $g(A)$ is everywhere dense, then $x \rightarrow f(g(x))$ is a nonmeasurable function.

As a consequence of this theorem we get: If g is such a function, that for every set A of positive measure $g(A)$ is everywhere dense, then g is a non-measurable function.

P. FISCHER

11. Problem. (P56) (In connection with M. Kuczma's talk.) If

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad (1)$$

for all $\{x, y\} \in S \subset R^2$ where $m(R^2 \setminus S) = 0$, does there exist an everywhere Jensen-convex g , i.e.

$$g\left(\frac{x+y}{2}\right) \leq \frac{g(x) + g(y)}{2} \quad (2)$$

for all $\{x, y\} \in R^2$, such that

$$g(t) = f(t) \quad \text{for all } t \in \{x \mid \exists y; (x, y) \in S\} \cup \{y \mid \exists x; (x, y) \in S\} \cup \left\{ \frac{x+y}{2} \mid (x, y) \in S \right\} \quad (3)$$

The similar question with $f(x+y) = f(x) + f(y)$ instead of (1), and the respective changes in (2) and (3) was posed by me at previous occasions.

J. ACZÉL

12. Remark. Proofs found by M. A. Taylor, of the general quasigroup solutions of $F[G(x, y), z] = H[x, K(y, z)]$ and of $F[G(x, y), H(u, v)] = K[L(x, u), M(y, v)]$ by means of reduction to the algebraic generalizations of Reidemeister's and Thomsen's theorems on the regularity of nets, and possible generalizations were mentioned.

J. ACZÉL

13. Remark to Professor E. Hille's lecture. I wish to point out a particularly elegant application of the functional equation (1) $R(s) - R(t) = (t-s)R(s)R(t)$ discussed by Professor Hille in his lecture. The theorem is due to B. Misra and has been communicated in my Seminar on Functional Operators and Equations, (Springer Lecture Notes No. 33, p. 80).

Let $R(s)$ be the resolvent of an operator in $L^2(a, b)$. It is interesting to know (also in view of physical applications, in scattering theory) whether $R(s)$ is a Carleman operator, i.e. of the form $\int_a^b dy \varrho(x, y)$ where, for almost all fixed values of x , ϱ as a function of y satisfies $\varrho \in L^2(a, b)$. Misra's theorem states: if $R(s_0)$ is a Carleman operator for one value of $s = s_0$ lying in the resolvent set, then $R(s)$ is a Carleman operator for all values s in the resolvent set. The proof is an ingenious application of the functional equation (1).

GY. TARGONSKI

14. Problem concerning generalized nets. A (generalized) net is any triple of coverings $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, resp. of sets X , Y , Z , resp. A *closure condition* for a net $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is any implication of the following form: some finite set of statements $p \in \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$ with variables $p, l, \dots$ implies a statement of the same type.

Let A, B, C be sets. A relation $R \subseteq A \times B \times C$ is called *saturated* if

$$\begin{aligned} \forall a \in A \exists b_1 \in B, \quad c_1 \in C \quad (a, b_1, c_1) \in R, \\ \forall b \in B \exists a_2 \in A, \quad c_2 \in C \quad (a_2, b, c_2) \in R, \\ \forall c \in C \exists a_3 \in A, \quad b_3 \in B \quad (a_3, b_3, c) \in R. \end{aligned}$$

An *admissible condition* for a saturated relation $R \subseteq A \times B \times C$ is any implication of the form: for a subset of R with triples composed from variables some triple of $A \times B \times C$ composed of some preceding variables must lie in R .

A net $\mathfrak{N}_R = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ corresponding to a given saturated relation $R \subseteq A \times B \times C$ is composed from sets

$$\begin{aligned}\mathcal{X} &= \left\{ \{(x, y) \in A \times B \mid x = a\} \mid a \in A \right\}, \\ \mathcal{Y} &= \left\{ \{(x, y) \in A \times B \mid y = b\} \mid b \in B \right\}, \\ \mathcal{Z} &= \left\{ \{(x, y) \in A \times B \mid (x, y, z) \in R\} \mid z \in C \right\}.\end{aligned}$$

Each admissible condition $\mathfrak{C}$ for R can be then equivalently expressed as an implication $\mathfrak{C}_{\mathfrak{N}_R}$ in terms of $\mathfrak{N}_R$.

1st question: (P57) To characterize the set of such admissible conditions $\mathfrak{C}$ with respect to a given saturated relation $R \subseteq A \times B \times C$ for which $\mathfrak{C}_{\mathfrak{N}_R}$ are closure conditions for $\mathfrak{N}_R$!

2nd question: (P58) Let an admissible condition $\mathfrak{C}$ with respect to a given saturated relation $R \subseteq A \times B \times C$ be given. To find additional assumptions under which $\mathfrak{C}_{\mathfrak{N}_R}$ is a closure condition for $\mathfrak{N}_R$!

V. HAVEL

15. Remark. Let $x \rightarrow f(x)$ be an n -dimensional point-transformation such that $f(0)=0$ and there exists a G neighbourhood of the point $x=0$ for which $f(G)$ is bounded. If there exists for this f a continuous function W such that

$$\begin{aligned}W(0) &= 0, \\ W(x) &> 0, \quad \text{if } x \neq 0, \\ W(f(x)) - W(x) &< 0,\end{aligned}$$

then there exist G_1, G_2 neighbourhoods of the point $x=0$ such that for all $x \in G_1$ the sequence

$$x, f(x), f(f(x)), \dots$$

belongs to G_2 .

P. FISCHER

16. Remark. Mean values and a geometric extremal problem. In the theory of generalized transfinite diameters (Fekete, Leja) there is the question of maximizing the average distance between n points belonging to sets S in a metric space X . In the following X is a Hilbert space, S the unit sphere.

The point of departure is the identity which for $n=2$ reduces to the parallelogram law and which is valid in any inner product space

$$\sum_{1 \leq j < k \leq n} \|x_k - x_j\|^2 + \left\| \sum_1^n x_j \right\|^2 = n \sum_1^n \|x_j\|^2.$$

If the x_j satisfy $\|x_j\| = 1, \forall j$, this gives the inequality

$$\sum_{1 \leq j < k \leq n} \|x_k - x_j\|^2 \leq n^2$$

with equality iff $\sum x_j = 0$. If $\|x_k - x_j\| = d_{jk}$, the quadratic mean average of the lengths of the edges of an n -simplex inscribed in the unit sphere becomes

$$\mathcal{M}_2(d_{jk}) \leq \sqrt{\frac{2n}{n-1}}$$

with equality iff the centroid of the vertices lies at the center of the sphere. In particular, the regular n -simplex satisfies this condition but it is not unique.

For $0 < p < 2$ we use that

$$\mathcal{M}_p(d_{jk}) \leq \mathcal{M}_2(d_{jk})$$

with equality iff all d_{jk} are equal. This gives

$$\max \mathcal{M}_p(d_{jk}) = \sqrt{\frac{2n}{n-1}}, \quad \lim_{n \rightarrow \infty} \mathcal{M}_p(d_{jk}) = \sqrt{2}$$

where the maximum is reached iff the simplex is regular.

For $p > 2$ the regular simplex does not give the maximum. If the number of edges is even, the maximum is given by a multiple diameter and

$$\lim_{n \rightarrow \infty} \mathcal{M}_p(d_{jk}) = 2^{(p-1)/p}.$$

The maximum can also be determined for the following classes of mean values of the Kolmogorov–Nagumo type. Let f be the function which generates the mean value so that

$$f(M(d_{jk})) = \frac{2}{n(n-1)} \sum f(d_{jk}).$$

If either

- 1) f is decreasing and concave upward or
- 2) f is increasing and is concave downward

then the regular simplex gives the maximum value for the average

$$\max M(d_{jk}) = \sqrt{\frac{2n}{n-1}}, \quad \lim_{n \rightarrow \infty} \max M(d_{jk}) = \sqrt{2}$$

and the maximum value is reached in the first place by the regular n -simplex and by no other configuration.

This gives the transfinite diameter of the unit sphere in Hilbert space as

$$\sqrt{2} \quad (\text{or } 2^{(p-1)/p}, p > 2)$$

if the average is based on $\mathcal{M}_p$ for $p \leq 2$ with $\mathcal{M}_0$ representing the geometric mean or on a function f satisfying either 1) or 2).

The transfinite diameters of the unit sphere of $C[0, 1]$, $L_1(0, 1)$, $L_\infty(0, 1)$ are all equal to 2 for any mean value.

See a paper of E. Hille (J. Austral. Math. Soc. 6 (1966), 122–128), where references to the literature are to be found.

E. HILLE

17. Remark (P59R1) on the following problem (P59) of M. Hosszú: What can be said about the general construction of the *continuous* solutions of the functional equation $f[g(x)] = f(x)$, where g is a given, say, complex function?

In such a general setting one cannot say much. A function f satisfies the equation in question if and only if it is constant on every orbit of the function g . Thus the answer depends on the structure of orbits of g . If g has an orbit which is everywhere dense in its domain, as is the case, e.g., for the Čebyšev polynomial $g(x) = 2x^2 - 1$, $x \in [-1, 1]$, or for the hat-function

$$g(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{for } x \in [\frac{1}{2}, 1] \end{cases},$$

then the only continuous solutions are the constant functions. Similarly, if g has an attractive fixed point ξ , then every solution which is continuous at ξ must be constant on the domain of attraction of ξ . On the other hand, it may happen that a continuous solution may be prescribed arbitrarily on an open set. It is so, for example, in the case $g(x) = x + 1$ (the family of continuous solutions consists here of all continuous periodic functions of period 1), or, more generally, if there exists a continuous and invertible solution α of Abel's functional equation $\alpha[g(x)] = \alpha(x) + 1$.

It is an open problem whether besides these two possibilities (constant functions as the only continuous solutions or the continuous solutions depending on an arbitrary function) there may still happen something else.

M. KUCZMA

18. Problem. (P60) To determine whether or not there exists a nonmeasurable function $f: R \rightarrow R$ which satisfies

$$f(x+h) + f(x+2h) \geq 2f(x), \quad (h > 0). \quad (1)$$

Remarks. (P60R1) A measurable solution of (1) is necessarily nondecreasing. This follows from results in my recent paper (Trans. Amer. Math. Soc. 135 (1968), 69–93). Section 7 of this paper is concerned with the more general inequality

$$\sum_{j=0}^n a_j f(x + T_j h) \geq 0, \quad (h > 0). \quad (2)$$

Recently, D. Girod exhibited a nonmeasurable solution for a large class of inequalities of type (2).

In the original problem it would be no loss of generality to assume that f is non-negative and bounded on an interval. Namely, if f satisfies (1) then also the function f_1 defined by $f_1(x) = \max(0, c + f(x))$ for $x \geq 0$, ($c = \text{constant}$); $f_1(x) = 0$ for $x < 0$, satisfies (1).

J. H. B. KEMPERMAN

19. Remark (P60R2) to the above problem. Putting in equation (1)

$$f(x + h) - f(x) = g_x(h),$$

it follows from (1):

$$g_x(2h) + g_x(h) \geq 0.$$

If this inequality should only have non-negative solutions, it would follow, that all solutions of (1) are monotonic.

A. OSTROWSKI

20. Remark. (P60R3) Professor Ostrowski asked (in connection with Professor Kemperman's problem) whether the functional inequality

$$g(y) + g(2y) \geq 0 \quad \text{for } y > 0$$

can have solutions g that are not non-negative in $(0, \infty)$. Now, the above inequality is certainly satisfied by all solutions of the functional equation

$$g(y) + g(2y) = 1 \quad \text{for } y > 0.$$

The general solution of the equation is

$$g(y) = F(t) + p(t) \sin \pi t,$$

where

$$t = \log_2 y, F(t) = (-1)^{[t]} \left(t - [t] + \frac{(-1)^{[t]} - 1}{2} \right),$$

$[t]$ is the greatest integer not exceeding t , and $p(t)$ is an arbitrary periodic function of period 1. By a suitable choice of the function $p(t)$ we may make g to change the sign in $(0, \infty)$ infinitely many times.

Unfortunately, this sheds no light onto the original problem of Professor Kemperman.

M. KUCZMA

21. Remark on the formal definition of the ‘hyperidentity’ in J. Aczél’s proof of the Belousov theorem. Let Q, Ω be respectively sets of free elements and free operators, while Q, Ω are sets of elements and operators. Let W be a set of words over Q, Ω . Each $w \in W$ can be identified with a mapping of $Q^Q \times \Omega^\Omega$ into Q in the following way: if $q \in Q^Q$ and $\omega \in \Omega^\Omega$ then

$$w(q, \omega) = \begin{cases} q(x) & \text{if exists } x, \quad x \in Q \quad \text{and} \quad x = w \\ w_1(q, \omega) \omega(0) w_2(q, \omega) & \text{where} \quad w = w_1 \circ w_2, \quad \circ \in \Omega. \end{cases}$$

The hyperidentity relation between $w_1, w_2 \in W$ holds, which is denoted

$$w_1 \stackrel{H}{=} w_2 \quad \text{iff} \quad \forall q \forall \omega (q \in Q^Q \text{ and } \omega \in \Omega^\Omega) \quad \text{implies} \quad w_1(q, \omega) = w_2(q, \omega).$$

T. PIETRZYKOWSKI

22. Remark on the equation $f(x+y-xy) + f(xy) = f(x) + f(y)$. The equation

$$f(x+y-xy) + f(xy) = f(x) + f(y), \quad (1)$$

considered first by M. Hosszú then by Z. Daróczy, I. Fenyő and H. Świątak, admits, under certain regularity assumptions for $f: R \rightarrow R, f(x) = Ax + B$ as the most general solution (A and B are arbitrary constants). Recently D. Blanuša proved the equivalence of (1) and

$$2f(x+y) = f(2x) + f(2y) \quad (2)$$

without any additional assumptions, when $f: R \rightarrow R$ and $f: C \rightarrow C$. The same result was obtained by Z. Daróczy for $f: R \rightarrow R$ with other methods.

J. Aczél noticed that the proof of Blanuša for complex variables can be extended for $f: Q \rightarrow Q$, where Q is a quadratically closed field with characteristic different from 2 and 3.

The following theorem can be stated: Let $f: Q \rightarrow G$, where Q is a field with characteristic different from 2 and 3 and G an abelian group. If

$$(*) \quad \begin{cases} \text{for every } g \in G \text{ there exists an additive mapping} \\ \gamma: Q \rightarrow G \text{ such that } \gamma(1) = g, \end{cases}$$

then equations (1) and (2) are equivalent.

If $G = Q$, then $(*)$ holds evidently.

PROBLEM (P61) Are (1) and (2) equivalent if $(*)$ does not hold?

PROBLEM (P62) Let Q be an abelian group with respect to the operation '+' and '◦', '*' operations from $Q \times Q$ into Q such that

$$(x \circ y) + (x * y) = x + y.$$

Let G be an abelian group and $f: Q \rightarrow G$. What additional assumptions on Q and G are necessary and sufficient to prove the equivalence of the equation

$$f(x \circ y) + f(x * y) = f(x) + f(y)$$

and Jensen's functional equation (2)?

H. ŚWIATAK

23. Problem. (P63) Let ϕ be a solution of the Schröder equation

$$\phi \circ \omega = \lambda \phi \tag{1}$$

here ϕ and ω are real-valued functions of a real variable, λ a real constant. One immediately verifies $(\phi \circ \omega)^n = \lambda^n \phi^n$ and $\phi \circ \omega_n = \lambda^n \phi$ (where ω_n is the n -th iterate of ω). From this follows

$$(\phi \circ \omega)^n = \phi^{n-1}(\phi \circ \omega_n) \quad n = 1, 2, 3, \dots \tag{2}$$

For $n=1$ (and incidentally for $n=0$) (2) is an identity.

Question: is (2) equivalent to (1), that is, given (2), does there follow the existence of a real constant λ such that (1) holds? If so, the system (2) would amount to a 'scalar-free' representation of the Schröder equation (1).

Does the equivalence perhaps hold if (2) is assumed to hold for certain values of n (and if this conjecture is true, then hold for all n)? Can one prove a corresponding statement for entire complex functions ϕ and ω , with λ a complex constant?

Provided the equivalence of (1) and (2) turns out to be valid, this would not only provide an alternative form of the Schröder equation, but also a generalization of the equation to rings which are not necessarily linear spaces; in a sense a 'scalar-free' and thus more general formulation of the Schröder equation, the eigenvalue equation of scale transformation operators.

GY. TARGONSKI

24. Remark. We consider the following functional equation:

$$f(x+t, y) + f(x-t, y) = f(x, y+t) + f(x, y-t), \tag{1}$$

where $f(x, y)$ is a real-valued function of two real variables x, y and t is a real variable. J. A. Baker proved the following

THEOREM. *If f is continuous on the whole xy -plane and satisfies (1), then and*

only then

$$f(x, y) = \alpha(x + y) + \beta(x - y),$$

where $\alpha(x)$, $\beta(x)$ are arbitrary real-valued continuous functions of a real variable x on $-\infty < x < +\infty$.

In this remark we give another proof of the above theorem.

H. HARUKI

25. Remark. In the theory of information as developed by J. Kampé de Fériet and B. Forte the information assigned to an event is a non-negative real number. One can generalize this concept, in the same way that probabilistic metric spaces are generalizations of ordinary metric spaces, by assigning a distribution function (i.e., an element of the set D^+ defined in the abstract of my communication) to each event. Using known results from the theory of probabilistic metric spaces, one readily extends the axioms of Kampé de Fériet and Forte to this more general setting. This leads to a new theory and to some interesting functional equations on the space D^+ as well.

B. SCHWEIZER

26. Problem. (P64) Was sind die Lösungen der Funktionalgleichung

$$f(x + y) = f(x) + yg[M(x, x + y)],$$

wobei $x, x+y$ in einem offenen Intervall (a, b) liegen, die unbekannten Funktionen f und g reell, stetig und streng monoton sind, weiterhin M eine feste sog. interne Funktion ist, d.h. für jedes $u < v$ gilt $u \leq M(u, v) \leq v$. Verf. kennt die Lösungen im Falle, wenn M ein quasiarithmetisches Mittel von der Form $M(u, v) = \phi^{-1}[p\phi(u) + q\phi(v)]$ ($p+q=1$) ist. Die Frage ist zuerst, für welche (von dem vorangehenden verschiedenen) Funktionen M die vorliegende Gleichung (mindestens) eine Lösung besitzt, und zweitens, wie man die Funktionen f und g in den Formen

$$f(x) = F[M(c_1, x), M(x, c_2)], \quad g(x) = G[M(c_3, x), M(x, c_4)]$$

darstellen kann, wobei c_1, c_2, c_3, c_4 irgendwelche feste Konstanten sind.

E. VINCZE

27. Remark. (P64R1) The functional equation investigated by E. Vincze

$$\frac{f(x) - f(y)}{x - y} = g(H(x, y)) \quad (*)$$

gives opportunity to define a generalized derivative of a function f . If in $(*)$ f is given, $H(x, y)$ is an internal function, g, H are unknowns, then, if $(*)$ has a solution with

respect to g and H , g can be considered as generalized derivative e.g., if $f(x) = |x|$, and $x=0$, then $g(H(0, y)) = \text{sign } y$. So it might be possible to define the generalized derivative in one point.

I. FENYŐ

28. Remark. If we consider the semi-group $(F_n, \circ)$ of all partial functions on an n element set, a naturally arising functional equation is

$$f^n(x) = x$$

which is due to Babbage and is a special case of

$$f^{m+1}(x) = f(x). \quad (1)$$

THEOREM 1. *If f satisfies (1) then:*

- i) f^m is idempotent
- ii) $f^m \mathcal{H} f$.

Let D_k denote the $\mathcal{D}$ -class of functions in F_n whose ranges have cardinality k .

THEOREM 2. *The number of idempotent functions in D_k is*

$$\sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} k^j.$$

Combining this with the fact that each $\mathcal{H}$ class in D_k has $k!$ elements we get:

THEOREM 3. *The number of functions in D_k which can be solutions of equations of type (1) is*

$$\sum_{j=0}^{n-k} k! \binom{n}{k} \binom{n-k}{j} k^j$$

J. BAILLIEUL

29. Problem. (P65) For $q \in C^\circ(a, b)$, $b \leq \infty$, every solution $\alpha \in C^3(a, b)$, $d\alpha/dt > 0$, of the equation

$$-\frac{1}{2} \frac{\alpha'''}{\alpha'} + \frac{3}{4} \left(\frac{\alpha''}{\alpha'} \right)^2 - \alpha'^2 = q(t), \quad ' = \frac{d}{dt},$$

defines the same function (dispersion) $t \rightarrow \phi(t) \stackrel{\text{def}}{=} \alpha^{-1}(\alpha(t) + \pi)$. The problem consists in finding an algorithm (not requiring computation of α) which assigns to a given $q \in C^\circ(a, b)$ the 'best' (in some sense) q^* from the set of all functions of $C^\circ(a, b)$ having the same (or asymptotically same for $t \rightarrow b_-$) dispersion as q .

Possibly, as the first step, some necessary conditions would be useful, under which Abel's equation

$$\alpha(\phi(t)) = \alpha(t) + \pi$$

$$(\text{or } \alpha(\tilde{\phi}(t)) = \alpha(t) + \pi, \tilde{\phi} \sim \phi \text{ as } t \rightarrow b_-)$$

has a unique solution $\alpha^* \in C^3(a, b)$, $\alpha^{*'}(t) > 0$, for a given $\phi \in C^3$, $\phi'(t) > 0$, $\phi(t) > t$.

For connections with the theory of differential equations, see O. Borůvka: Lineare Differentialtransformationen 2. Ordnung, VEB, Berlin 1967, and articles quoted there.

F. NEUMAN

30. Remark. The method outlined by G. Targonski for finding continuous real solutions of the equation $f(g(x)) = f(x)$ [see the remark 17 of M. Kuczma] when g is invertible can be extended to certain cases where g is non-invertible. Sufficient conditions for the validity of the method are: g continuous, $g|_{\text{Rang}}$ invertible, and $\text{Ran}(g|_{\text{Rang}}) = \text{Rang}$. These conditions are satisfied, for example, when $g(x) = \sin(\pi/2x)$ for all real x .

A. SKLAR

31. Remarks on the translation equation. We start with a function $F(u, v): \mathbf{R}^2 \rightarrow \mathbf{R}$ which is a solution to the translation equation:

$$F(u, v + w) = F[F(u, v), w].$$

We choose u_0 fixed. We then give conditions on F which will make $v \rightarrow F(u_0, v)$ continuous for all u_0 .

Claim 1: Let F be such that for *each* fixed u_1 , $v \rightarrow F(u_1, v)$ is continuous at the same fixed value $v = \xi$. Then $v \rightarrow F(u_0, v)$ is continuous for all v .

Now we assume $u \rightarrow F(u, v)$ to be continuous everywhere for each fixed value of v . Then we get:

Claim 2: Let F be such that $u \rightarrow F(u, v_0)$ is continuous for each fixed v_0 . Choose u_0 fixed and let $v \rightarrow F(u_0, v)$ have at least one point of continuity. Then $v \rightarrow F(u_0, v)$ is continuous for all v .

Claim 3: Let F be such that $u \rightarrow F(u, v_0)$ is continuous for each fixed v_0 . Choose u_0 fixed. Let $v \rightarrow F(u_0, v)$ have at least one point $v = b$ such that $F(u_0, b-0)$ and $F(u_0, b+0)$ both exist. Then $v \rightarrow F(u_0, v)$ is continuous for all v .

Claim 4: Let F be such that $u \rightarrow F(u, v_0)$ is continuous for each fixed v_0 . Choose u_0 fixed. Let $v \rightarrow F(u_0, v)$ be 1-1 and let the range of $v \rightarrow F(u_0, v)$ contain an open interval (a, b) . Also let d_1, d_2 be such that $F(u_0, d_1-0)$ and $F(u_0, d_2+0)$ both exist and let $F(u_0, d_1-0) \in (a, b)$ and $F(u_0, d_2+0) \in (a, b)$. Then $v \rightarrow F(u_0, v)$ is continuous for all v .

Now let F be a solution to the translation equation such that for each fixed v_0 , $u \rightarrow F(u, v_0)$ has at worst discontinuities of the first kind, assumed isolated. Now choose u_0 fixed. Then:

Claim 5: Let $v \rightarrow F(u_0, v)$ be locally non-constant. Then the discontinuities of $v \rightarrow F(u_0, v)$ are either isolated or everywhere dense.

Claim 6: Let $v \rightarrow F(u_0, v)$ have at least one point $v=a$ such that $F(u_0, a-0)$ and $F(u_0, a+0)$ both exist. Let $v \rightarrow F(u_0, v)$ be monotonic in a neighborhood of $v=a$. Then for every real $v=b$, $F(u_0, b-0)$ and $F(u_0, b+0)$ both exist.

JAMES HAMILTON

32. Problems.

(P66) It is well known that a given functional equation (FE) sometimes possesses only regular solutions and sometimes solutions of different kinds of regularity. Let a (FE) be given and the set of S of all functions be defined in the domain of existence of (FE).

Definition: We shall say that (FE) has the property $[S_1, S_2]$, if

- (i) S_1, S_2 are subsets of S such that $S_1 \subset S_2 \subset S$, $S_2 - S_1 \neq \phi$
- (ii) (FE) has solutions belonging to S_1 and solutions belonging to $S_2 - S_1$.

Given the pair (S_1, S_2) we can ask for the existence of a (FE) having the property $[S_1, S_2]$. I don't believe that general sufficient conditions can be found. But it would be desirable to construct a possibly large list of such pairs $[S_1, S_2]$ for different types of (FE).

(P67). The second question deals with the meta-theory of (FE).

Everybody will agree with the following assertion. 'When for the Cauchy equation we denote by S_1 the set of all analytic (or continuous) functions and by S_2 the whole set S , then the solving of (FE) in the class S_1 is "easier" than in the class S_2 because the second problem involves the knowledge of the notion of the Hamel-basis'. Of course the word 'easier' can't be defined in a strict manner.

In general if a (FE) fulfils the property $[S_1, S_2]$, then it is easier to solve it in the class S_1 , than in the class S_2 . But it is not always so and counterexamples can be given.

Therefore the question arises, for which (FE) and for which pairs (S_1, S_2) such that (FE) possesses the property $[S_1, S_2]$ it is easier to solve the (FE) in the class S_1 than in the class S_2 .

S. GOLAB

33. Problem. (P68) Let R be the field of all real numbers, i.e. R is an ordered Archimedean complete field (any bounded set $A \subset R$ has the $\sup A \in R$). Prove without use of topological notions that for any number $a > 0$ there exists a unique ordered pair (C_a, S_a) of two functions from R into R such that:

- 1) $C_a(x+y) = C_a(x)C_a(y) - S_a(x)S_a(y), \quad x, y \in R$
 $S_a(x+y) = S_a(x)C_a(y) + S_a(y)C_a(x), \quad x, y \in R$

- 2) $C_a(0) = 1, S_a(0) = 0$
 $C_a(x+4a) = C_a(x), \quad \forall x \in R$
 $S_a(x+4a) = S_a(x), \quad \forall x \in R$
- 3) S_a and C_a are strictly monotonic functions on the interval $[0, a]$.

S. KUREPA

34. Problem. (P69) Construct all quantities which under proper transformations are

- (a) tensors of covariant rank m , and
 (b) concomitants of a symmetric tensor of rank 2.

The problem has been solved if

- (i) m is odd and $m \leq n$ (n the dimension of the space)

[S. Golab and M. Kucharzewski, *Acta Math. Acad. Sci. Hung.* 11 (1960), 173–174;
 M. A. McKiernan and H. Richards, *Ann. Polon. Math.* (1970)].

- (ii) $m=4$ [D. Lovelock, *Arch. Rational Mech. Anal.* 33 (1969) 54–70].

D. LOVELOCK

35. Remark. In the article ‘On the algebra of queues’ (J. Appl. Probability 3 (1966), 285–326), J. F. C. Kingman asks (p. 322) for a functional equation satisfied by the linear operator $\Gamma: \mathcal{M} \rightarrow \mathcal{M}$, where $\mathcal{M}$ is the algebra under convolution of all finite signed measures on R^2 , and where

$$(\Gamma\mu)(E) = \mu(\Phi^{-1}(E)) \quad \text{for all } \mu \in \mathcal{M}, \text{ all Borel sets } E,$$

Φ being the point mapping of the plane into itself

$\Phi(x_1, x_2)$ = point with coordinates x_1^+, x_2^+ taken in increasing order. (Here $x^+ = \max(x, 0)$.)

The point mapping Φ may be factored as $\Phi = \Phi_3 \Phi_2 \Phi_1$ where $\Phi_1(x_1, x_2) = (x_1^+, x_2)$, $\Phi_2(x_1, x_2) = (x_1, x_2^+)$, and Φ_3 is a folding map,

$$\Phi_3(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{if } x_1 \leq x_2, \\ (x_2, x_1) & \text{if } x_1 > x_2. \end{cases}$$

Let $\Gamma_1, \Gamma_2, \Gamma_3$ be the corresponding linear operators on $\mathcal{M}$. Then Γ_1 and Γ_2 satisfy the summation identity, i.e.

$$\Gamma_1 \lambda \cdot \Gamma_1 \mu = \Gamma_1 [\Gamma_1 \lambda \cdot \mu + \lambda \cdot \Gamma_1 \mu - \lambda \cdot \mu] \quad \text{for all } \lambda, \mu \in \mathcal{M}, \quad (1)$$

with a similar identity for Γ_2 ; also Γ_3 satisfies

$$\begin{aligned} \Gamma_3 [\Gamma_3 \lambda \cdot \Gamma_3 \mu \cdot \Gamma_3 \nu - (\Gamma_3 \lambda \cdot \Gamma_3 \mu \cdot \nu + \Gamma_3 \lambda \cdot \mu \cdot \Gamma_3 \nu + \lambda \cdot \Gamma_3 \mu \cdot \Gamma_3 \nu) \\ + (\Gamma_3 \lambda \cdot \mu \cdot \nu + \lambda \cdot \Gamma_3 \mu \cdot \nu + \lambda \cdot \mu \cdot \Gamma_3 \nu) - \lambda \cdot \mu \cdot \nu] = 0. \end{aligned} \quad (2)$$

To obtain a description of the queue G1/G/2 as proposed by Kingman, one needs to use (1), (2), the idempotence of the Γ 's, and the commutativity properties $\Gamma_1 \circ \Gamma_2$,

$\Gamma_1 \Gamma_2 \cup \Gamma_3$ to obtain a functional equation of this general type for

$$\Gamma = \Gamma_3 \Gamma_2 \Gamma_1 = \Gamma_1 \Gamma_3 \Gamma_1,$$

and then to find a formula for $(\alpha I - \Gamma L_\kappa)^{-1}$ when α is a complex number, and L_κ denotes convolution with a measure κ .

A two-variable analogue of the identity (2) was the subject of a communication to this journal by V. D. Belousov, S. L. Segal and G. Szekeres; see *Aequationes Math.* 2 (1969), 116–117.

J. B. MILLER

(Report compiled by F. RADÓ)