VERY NONPROXIMINAL SETS IN c_0

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Abstract. It is proved that the Banach space c_0 contains a locally uniformly convex, closed, bounded, symmetric convex body S so that no point outside S has a nearest point in S, giving so a positive answer to a question of M. Edelestein and A. C. Thompson [4].

Let X be a normed linear space, S a subset of X and x an element in X. An element $s_0 \in S$ such that

$$\|x - s_0\| = \inf\{\|x - s\| : s \in S\} \tag{1}$$

is called a *best approximation element* of x by elements in S (or a *nearest point* to x in S).

Sometimes we shall use the notation $d(x, S) = \inf\{\|x - s\| : s \in S\}$ to denote the *distance* from x to S. We say that the set S is *very nonproximinal* if no element in $X \setminus S$ has an element of best approximation in S (see [8, p. 348]). We shall denote by $\mathrm{Prox}(S)$ the set of all elements in $X \setminus S$ which have an element of best approximation in S. In this Note we shall be concerned with convex very nonproximinal subsets in c_0.

The theory of best approximation in normed spaces by elements in linear subspaces is well established, little less being known about the best approximation by elements in arbitrary subsets. We shall present some results obtained in this direction by M. Edelstein. M. Edelstein has proved in [2] that if S is a closed subset of a uniformly convex Banach space X, then $\mathrm{Prox}(S)$ is dense in X. In [3] it is shown that if X is a separable conjugate Banach space (that is $X = E^*$ for some normed linear space E), then for every real number $d \geq 0$ there exist $x \in X$ and $s_0 \in S$ such that $d = \|x - s_0\| = d(x, S)$. A natural question is to ask what happens if we renounce to the condition that X is a conjugate space. Taking $X = c_0$, the Banach space all all sequences of real numbers converging to 0 with the norm defined by

$$\|x\| = \max\{|x(i)| : i \in \mathbb{N}\}, \tag{2}$$

for every $x = (x(i))_{i \in \mathbb{N}} \in c_0$, M. Edelstein and A. C. Thompson have shown in [4] that c_0 contains a bounded closed symmetric convex body S such that $\mathrm{Prox}(S) = \emptyset$. (By a convex body we understand a convex set with nonempty interior). This shows that c_0 is of N_2-type, in the classification given by V. Klee [5], that is a linear normed space containing a closed

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convex bounded and very nonproximinal set. M. Edelstein shows that one can construct a strictly convex very nonproximinal closed symmetric convex body and asks whether there exists a locally uniformly convex one with these properties. We give a positive answer to this question. (Following A. R. Lovaglia [6] we say that a normed space is locally uniformly convex if $x_n, x \in X$, $\|x_n\| = \|x\| = 1$ and $\|x_n + x\| \to 2$ implies $x_n \to x$). We use the notations from [4]. We denote by δ_i, $i \in \mathbb{N}$, the continuous linear functionals defined on c_0 by

$$\delta_i(x) = x(i), \quad x \in c_0, \tag{3}$$

and by g_i, $i \in \mathbb{N}$, the continuous linear functionals defined by

$$g_i(x) = x(i) + \sum_{k=1}^{\infty} 2^{-k-1} x(2^{i-1}(2k+1)), \quad x \in c_0. \tag{4}$$

Consider the operator $A : c_0 \to c_0$ defined by

$$Ax(i) = g_i(x), \quad i \in \mathbb{N}, \;\; x \in c_0. \tag{5}$$

In [4] one shows that A is a topological isomorphism of c_0 onto c_0 and its adjoint A^*, which is an topological isomorphism of $c_0^* = \ell_1$ onto c_0^* satisfies the conditions

$$A^*\delta_i = g_i, \quad i \in \mathbb{N}. \tag{6}$$

As usual, we denote by ℓ_1 the Banach space of all absolutely summable sequences of real numbers with the norm

$$\|x\| = \sum_{i=1}^{\infty} |x(i)|, \quad x \in \ell_1. \tag{7}$$

M. Day [1] has constructed a strictly convex norm on c_0, equivalent to (2), in the following way: for every $x \in c_0$ the sequence $\{x(i)\}$ is rearranged such that

$$|x(i_1)| \geq |x(i_2)| \ldots, \tag{8}$$

and the norm is defined by

$$p(x) = \left(\sum_{j=1}^{\infty} \left[2^{-j} x(i_j)\right]^2 \right)^{1/2}. \tag{9}$$

J. Rainwater [7] has shown that the norm p is further locally uniformly convex.

Now we can enounce our result.

Theorem. *The Banach space c_0 contains a locally uniformly convex, closed, bounded, symmetric, and very nonproximinal convex body.*

Proof. Let p be Day's norm given by (9) and $A : c_0 \to c_0$ the isomorphism defined by (5). Define

$$q(x) = p(Ax), \quad x \in c_0. \tag{10}$$

Because A is a topological isomorphism, q is a norm on c_0 equivalent to p, and so to (2), too. Let us show that q is locally uniformly convex. Let $x_n, x \in c_0$ such that $q(x_n) = q(x) = 1$ and $q(x_n + x) \to 2$. It follows that $p(Ax_n) = p(Ax) = 1$ and $p(Ax_n + Ax) = p(A(x_n + x)) \to 2$. Since p is locally uniformly convex, it follows $Ax_n \to Ax$ and so, as A is a topological isomorphism, $x_n \to x$.

Let

$$S = \{x \in c_0 : q(x) \le 1\}. \tag{11}$$

Because q is a norm on c_0 equivalent to (2), S is a closed, bounded, symmetric, convex body. Moreover, as we have shown it is locally uniformly convex, too.

We shall show now that S is very nonproximinal. Suppose that it would exist an element $x \in c_0$ with $q(x) > 1$ and $s_0 \in S$ such that

$$\|x - s_0\| = \inf\{x - s\| : s \in S\}.$$

By the definition of Day's norm and of the operator A, it follows that there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$|g_{\sigma(i+1)}(s_0)| \ge |g_{\sigma(i)}(s_0)|, \quad i = 1, 2 \ldots,$$

and

$$q(s_0) = \left(\sum_{i=1}^{\infty} \left[2^{-i} g_{\sigma(i)}(s_0)\right]^2 \right)^{1/2}. \tag{12}$$

We make the following remark concerning the bijections $\sigma : \mathbb{N} \to \mathbb{N}$.

(*) *It is impossible that the relation*

$$\exists k_0,\ \forall k > k_0,\ \ \sigma(2^i(2k+1)) \le k \tag{13}$$

holds for two different indices i.

Suppose the contrary. Then there would exist two indices $i \ne j$ in $\mathbb{N}$ and $k_0 \in \mathbb{N}$ such that

$$\forall k > k_0,\ \sigma(2^i(2k+1)) \le k \quad \text{and} \quad \sigma(2^j(2k+1)) \le k.$$

Let $m > k_0$ and $k = k_0 + m$. Then for all l, $k_0 < l \le k$,

$$\sigma(2^i(2l+1)) \le l \le k \quad \text{şi} \quad \sigma(2^j(2l+1)) \le l \le k.$$

But the set $M = \{2^i(2l+1) : l = k_0 + 1, \ldots, k_0 + m\} \cup \{2^j(2l+1) : l = k_0 + 1, \ldots, k_0 + m\}$ contains $2m$ elements and the set $N_k = \{1, 2, \ldots, k\}$ contains $k = k_0 + m < 2m$ elements. The inclusion $\sigma(M) \subset N_k$ implies that σ is not a bijection. The assertion (*) is proved.

Based on this assertion the proof of the theorem is similar to that given in §3 (ii) from [4]. If only one $g_i(s_0)$ is non-zero, then we can define the bijection σ by $\sigma(k) = k+1$ for $k = 1, 2, \ldots, i-1$, $\sigma(i) = 1$, and $\sigma(k) = k$ for $k > i$. If there are two indices i for which $g_i(s_0) \neq 0$, then we choose that i for which (*) does not hold for $i-1$. Let $n = 2^{i-1}(2k+1)$, $\varepsilon > 0$ and

$$s_0' = (s_0(1), s_0(2), \ldots, s_0(n) - \varepsilon, \ldots).$$

Then $g_j(s_0) = g_j(s_0')$ for $j \neq i$, $j \neq n$, and

$$\begin{aligned} g_n(s_0')^2 - g_n(s_0)^2 &= \varepsilon^2 - 2\varepsilon g_n(s_0), \\ g_i(s_0')^2 - g_i(s_0)^2 &= \varepsilon^2 2^{-2k-2} - 2^{-k}\varepsilon g_i(s_0), \end{aligned}$$

so that

$$\begin{aligned} q(s_0')^2 - q(s_0)^2 = {} & 2^{-\sigma(n)}\varepsilon^2 + 2^{-2\sigma(i)-2k-2}\varepsilon^2 - \\ & - \varepsilon 2^{-2\sigma(n)+1} g_n(s_0) - \varepsilon 2^{-2\sigma(i)-k} g_i(s_0). \end{aligned} \tag{14}$$

Without restricting the generality we can suppose $g_i(s_0) > 0$.

Suppose now that ε is chosen such that $2\varepsilon < \max\{\|x - s_0\|, \|s_0\|\}$ and that k is sufficiently large in order that:

(i) $\sigma(2^{i-1}(2k+1)) \geq k$;

(ii) $2^{-k}\varepsilon + 2^{-2\sigma(i)-k-2}\varepsilon + 2^{-k+1}|g_n(s_0)| < 2^{-\sigma(i)} g_i(s_0)$;

(iii) $|x(n) - s_0(n)| < \dfrac{1}{2}\|x - s_0\|$ and $|s_0(n)| < \dfrac{1}{2}\|s_0\|$.

Then by (14), (i) and (ii),

$$\begin{aligned} & q(s_0')^2 - q(s_0)^2 \leq \\ & \leq 2^{-2k}\varepsilon^2 + 2^{-2\sigma(i)-2k-2}\varepsilon^2 + \varepsilon|g_n(s_0)|2^{-2k+1} - 2^{-2\sigma(i)-k}\varepsilon g_i(s_0) \\ & = 2^{-k}\varepsilon\left[2^{-k}\varepsilon + 2^{-2\sigma(i)-2k-2}\varepsilon + 2^{-k+1}|g_n(s_0)| - 2^{-2\sigma(i)} g_i(s_0)\right] < 0. \end{aligned}$$

It follows, $q(s_0') < q(s_0) = 1$ and, from (iii), $\|x - s_0'\| = \|x - s_0\| = d(x, S)$. But this is impossible, because x cannot have a best approximation element situated in the interior of the set S. The theorem is completely proved.

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