

Supporting Spheres for Families of Independent Convex Sets

By

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Definition 1. There will be said that the family $\mathcal{K}$ of sets in the Euclidean space $\mathbf{R}^n$ has a *supporting sphere*, if there exists a sphere S in $\mathbf{R}^n$ having common points with each member of the family $\mathcal{K}$ and the interior of S contains no point of any member of $\mathcal{K}$.

Definition 2. The family $\mathcal{K}$ of sets in $\mathbf{R}^n$ will be said to be independent, if for any $n + 1$ pairwise distinct members $K_1, \dots, K_{n+1}$ of $\mathcal{K}$, any set of points $p_1, \dots, p_{n+1}$, where $p_i \in K_i$, $i = 1, \dots, n + 1$ determines a simplex of dimension n , or equivalently the vectors $p_2 - p_1, \dots, p_{n+1} - p_1$ are linearly independent.

If M' and M'' are compact sets in $\mathbf{R}^n$, we denote

$$d(M', M'') = \min \{d(p', p'') : p' \in M', p'' \in M''\},$$

where $d(p', p'')$ is the Euclidean distance of p' from p'' .

In the case of $M'' = \{p''\}$, a point p' in M' for which $d(M', p'') = d(p', p'')$, is called a nearest point of M' to p'' .

In some problems of the geometry of convex sets can be applied with success the Brouwer's fixed point theorem (see for example the proof of Helly's theorem in the paper of G. D. Chakerian [1]). This will be the basic tool also in the proof of the existence part in our following theorem.

Theorem 1. Let be $K_1, \dots, K_{n+1}$ independent convex and compact sets in $\mathbf{R}^n$. Then this family of convex sets admits exactly one supporting sphere.

This theorem states in fact that there exists a single point p in $\mathbf{R}^n$ such that $d(p, K_1) = \dots = d(p, K_{n+1})$.

A somewhat similar result was proved by H. Rademacher and I. J. Schoenberg (Theorem 3 in [4]): if $K_1, \dots, K_m$ ($m \geq 2$) are convex and compact sets in the Euclidean space $\mathbf{R}^n$ such that $\bigcap_{j=1}^m K_j = \emptyset$, then there exists an integer s , $2 \leq s \leq n + 1$, the sets $K_{i_1}, \dots, K_{i_s}$ and a point p , with the properties $\bigcap_{j=1}^s K_{i_j} = \emptyset$, and

$$d(p, K_{i_1}) = d(p, K_{i_2}) = \dots = d(p, K_{i_s}).$$

Proof of Theorem 1. The existence. Suppose that $p_i \in K_i$, $p_i = (p_i^1, \dots, p_i^n)$, $i = 1, \dots, n + 1$. The center $c(p_1, \dots, p_{n+1})$ of the sphere determined by these

points has the coordinates

$$c^i(p_1, \dots, p_{n+1}) = \frac{1}{2} \frac{D_i(p_1, \dots, p_{n+1})}{D(p_1, \dots, p_{n+1})},$$

where D is the determinant of order $n + 1$ having the general line

$$(1, p_j^1, \dots, p_j^n),$$

$j = 1, \dots, n + 1$, and D_i is the determinant of the same order with the general line of the form

$$\left(1, p_j^1, \dots, p_j^{i-1}, \sum_{k=1}^n (p_j^k)^2, p_j^{i+1}, \dots, p_j^n\right),$$

$j = 1, \dots, n + 1$. From the independence of the points $p_1, \dots, p_{n+1}$, it follows that $D(p_1, \dots, p_{n+1}) \neq 0$, and, from the compactness of the sets $K_1, \dots, K_{n+1}$ it follows that the continuous functions $c^i(p_1, \dots, p_{n+1})$ are bounded on the direct product $K_1 \times \dots \times K_{n+1}$. Let be B a compact ball containing all the points $c(p_1, \dots, p_{n+1})$ with $p_i \in K_i$, $i = 1, \dots, n + 1$. Let be now the mapping $f: B \rightarrow B$ defined in the following way: for $q \in B$ consider the points $p_1(q), \dots, p_{n+1}(q)$, where $p_i(q)$ is the nearest point of K_i to q . It is well known that $p_i(q)$ is unique and depends continuously on q (see for instance [3], Proposition 2.3). Let now be

$$f(q) = c(p_1(q), \dots, p_{n+1}(q)),$$

the center of the sphere determined by the points $p_1(q), \dots, p_{n+1}(q)$. Then the mapping f is well defined and continuous. From Brouwer's fixed point theorem it follows that there exists a point q_0 , such that

$$f(q_0) = q_0,$$

i.e. q_0 is the center of the sphere determined by the nearest points $p_i(q_0)$ of K_i to q_0 , $i = 1, \dots, n + 1$. Taking now $p = q_0$, the existence of a point with the required property is proved.

The unicity. Suppose that there are two points p' and p'' , the first at the distance d' from the sets K_i , the second at the distance d'' from K_i , $i = 1, \dots, n + 1$. Denote by S' and S'' the spheres of the radius d' , respectively d'' with the centre p' , respectively p'' . Let be $d' \leq d''$. There are three possibilities:

- (i) $d(p', p'') \leq d'' - d'$,
- (ii) $d(p', p'') > d' + d''$,
- (iii) $d'' - d' < d(p', p'') \leq d' + d''$.

From (i) it follows that S' is contained in the ball $B(p'', d'')$, and it follows that at least one of the distances $d(p'', K_i)$ is smaller as d'' , which contradicts the definition of the supporting sphere.

If (ii) holds, then there exists a separating hyperplane H between the convex sets $B(p', d')$ and $B(p'', d'')$. Denote by

$$q'_i = B(p', d') \cap K_i \quad \text{and} \quad q''_i = B(p'', d'') \cap K_i.$$

Because the hyperplane H determines two open halfspaces H' and H'' and we have

$B(p', d') \subset H'$, $B(p'', d'') \subset H''$, it follows that the segment $q'_i q''_i$ intersects the hyperplane H in a point q_i . From the convexity of K_i it follows that q_i is in K_i , $i = 1, \dots, n+1$. But this means that H contains points from each set K_i , $i = 1, \dots, n+1$, which is a contradiction with the independence of this family of sets.

(iii) In this case S' and S'' determine a hyperplane H which is or the common tangent hyperplane, if $d(p', p'') = d' + d''$, or is the hyperplane which contains the intersection of S' and S'' . Because q'_i , $i = 1, \dots, n+1$ cannot be contained in the interior of $B(p'', d'')$ and q''_i , $i = 1, \dots, n+1$ cannot be contained in the interior of $B(p', d')$, they are in different closed halfspaces determined by the hyperplane H and a contradiction is obtained (see Fig. 1 in the case $n = 2$) with the independence of the sets K_i , $i = 1, \dots, n+1$, as in case (ii). This completes the proof of the unicity.

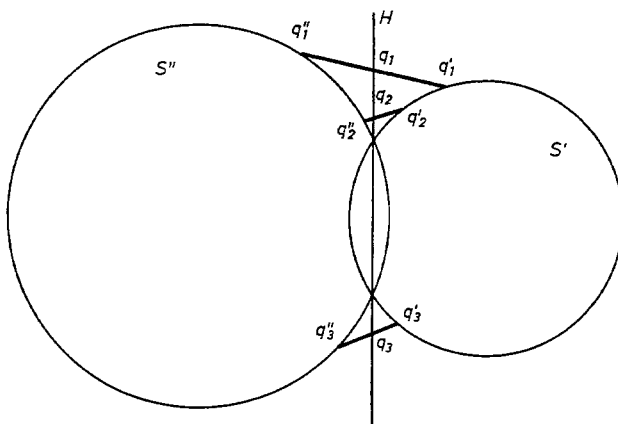


Fig. 1

This theorem has the following immediate consequence:

Corollary. Let $\mathcal{K}$ be a family of compact convex sets in $\mathbb{R}^n$, having the following two properties:

- (i) there are $n+1$ members of $\mathcal{K}$, which are independent convex sets;
- (ii) for any $n+2$ members of $\mathcal{K}$ there exists a supporting sphere;

then there exists exactly one supporting sphere for the family $\mathcal{K}$ of convex sets.

From the theorem of Helly about the common points of the convex sets (see for instance [2] or [5]), it follows that if we have a family $\mathcal{K}$ of compact convex sets in $\mathbb{R}^n$ with the property that any $n+1$ members have a supporting sphere of radius r , then there exists a point at the distance $\leq r$ to each member of $\mathcal{K}$. We ask about the existence of a supporting sphere of radius r for the family $\mathcal{K}$, under the hypothesis that the members in $\mathcal{K}$ are independent, any $n+1$ members of the family have a supporting sphere of radius r , and the cardinal of $\mathcal{K}$ is $\geq n+3$. In the case, when the family has $n+2$ members, the answer is generally negative. We illustrate this in the case $n=2$ by the following figure:

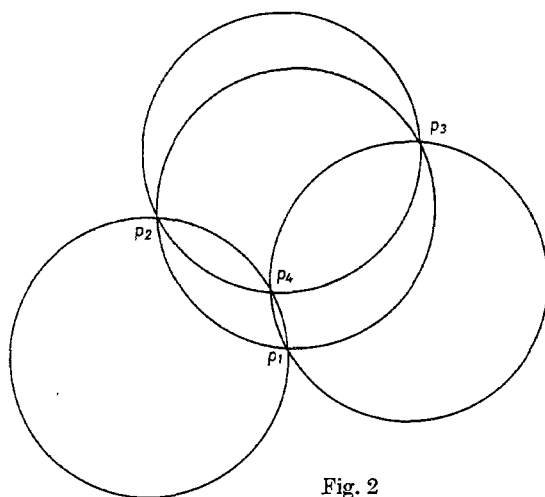


Fig. 2

where the members of $\mathcal{K}$ are the points p_1, p_2, p_3 and p_4 .

We can give an affirmative answer to this question in the particular case $n = 2$. More precisely, we will prove the following Helly-type theorem:

Theorem 2. *Let $\mathcal{K}$ be a family of compact, independent, convex sets in the Euclidean plane $\mathbb{R}^2$. Suppose that the number of members in $\mathcal{K}$ is ≥ 5 . If for any three members in $\mathcal{K}$ there exists a supporting circle of a given radius r , then there exists a supporting circle of radius r for all the family $\mathcal{K}$.*

We need in the proof of the theorem the following auxiliary result:

Lemma. *Let be K' and K'' two compact, disjoint convex sets in the plane $\mathbb{R}^2$, such that $d(K', K'') < 2r$. Then there exist at most two supporting circles of radius r for these two sets.*

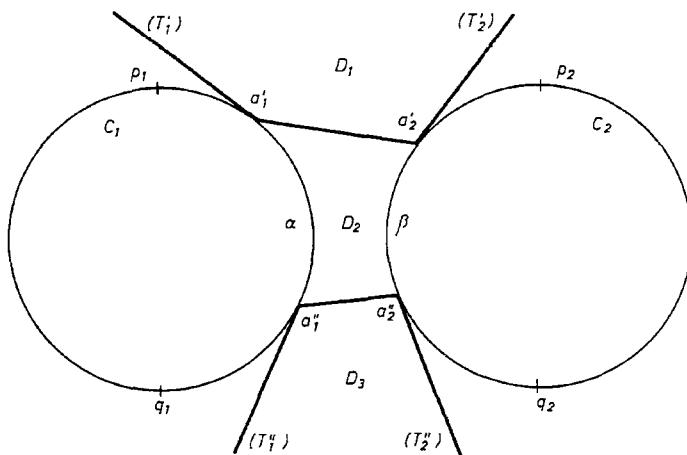


Fig. 3

Proof. Suppose the contrary: there exist three distinct supporting circles, C_1, C_2, C_3 , of radius r for K' and K'' . We suppose that C_1 and C_2 are given and ask for the possible positions of C_3 . In Fig. 3 the case when C_1 and C_2 are disjoint is considered; the case when C_1 and C_2 have a nonvoid intersection may be treated in an analogous way.

Suppose that $a'_i = K' \cap C_i$, $a''_i = K'' \cap C_i$, $i = 1, 2, 3$. From the definition of the supporting circles it follows that a'_i and a''_i (see Fig. 3) must be on the semicircle $p_1 \alpha q_1$ and a'_2, a''_2 must be on the semicircle $p_2 \beta q_2$ (where p_1 and p_2 , respectively q_1 and q_2 are the contact points of the exterior tangents to C_1 and C_2). Furthermore, the segments $a'_1 a'_2$ in K' and $a''_1 a''_2$ in K'' are disjoint as K' and K'' are disjoint. Denote by (T'_i) , (respectively (T''_i)) the tangent to C_i from a'_i (respectively a''_i) as in Fig. 3. From the definition of the supporting circle and the convexity of K' and K'' it follows that a'_3 and a''_3 must be in the domain between the lines $(T'_1)a'_1 \alpha a'_1(T'_1)$ and $(T'_2)a'_2 \beta a'_2(T'_2)$. The segments $a'_1 a'_2$ and $a''_1 a''_2$ determine the domains D_1, D_2 , and D_3 . If $a''_3 \in D_1$ then the segments $a''_3 a''_1$ and $a''_3 a''_2$ in K'' intersect either the segment $a'_1 a'_2$, or one of the circles C_1 or C_2 in a point distinct from a''_i ($i = 1, 2$), which is a contradiction. Similarly, a'_3 cannot be in D_3 .

Suppose that a'_3 belongs to D_1 , but is not on $a'_1 a'_2$. If then a''_3 is in D_2 or D_3 , by a simple geometrical reasoning it may be seen that C_3 must contain in its interior points of $a'_1 a'_2$, which contradicts the definition of the supporting circle.

It follows that a'_3 and a''_3 belong to the closed domain D_2 . If one of these points, say a'_3 , would be in the interior of D_2 , then it would follow that C_3 contains in its interior points of the segment $a'_1 a'_2$. Therefore $a'_3 \in a'_1 a'_2$ and $a''_3 \in a''_1 a''_2$. If

$$d(a'_1 a'_2, a''_1 a''_2) < 2r,$$

then C_3 will contain in its interior points of $a'_1 a'_2 \cup a''_1 a''_2$. If $d(a'_1 a'_2, a''_1 a''_2) = 2r$, then we have $a'_i = p_i$ and $a''_i = q_i$, $i = 1, 2$, and from the definition of the supporting circles it follows that $d(K', K'') = d(a'_1 a'_2, a''_1 a''_2) = 2r$, in contradiction with the hypothesis. This completes the proof of the lemma.

Proof of Theorem 2. Consider five members of $\mathcal{K}$: $K_1, \dots, K_5$. The sets K_1, K_2, K_3 have a supporting circle of radius r . Then we obviously have either

$$d(K_1, K_2) < 2r, \quad \text{or} \quad d(K_1, K_3) < 2r.$$

Without loss of generality, we may suppose $d(K_1, K_2) < 2r$. Then, by the lemma, K_1 and K_2 have at most two supporting circles C_1 and C_2 of radius r . For a given i , $3 \leq i \leq 5$, the supporting circle of K_1, K_2, K_i must coincide either with C_1 or with C_2 . We may suppose then that C_1 is the supporting circle for K_1, K_2, K_3, K_4 , and C_2 the supporting circle for K_1, K_2, K_5 . We have either $d(K_5, K_1) < 2r$ or $d(K_5, K_2) < 2r$. Suppose for example $d(K_5, K_1) < 2r$. Let be C'_1 and C'_2 the only possible supporting circles of radius r for the sets K_5, K_1 (see the lemma). As above one of these supporting circles, say C'_1 , must be a supporting circle for the sets K_1, K_5, K_i, K_j , where i and j are different indices, $2 \leq i, j \leq 4$. But then for K_1, K_i and K_j are supporting circles both C_1 and C'_1 . By Theorem 1, it follows that these circles coincide and because C_1 is a supporting circle for K_1, K_2, K_3, K_4 ,

C'_1 is a supporting circle also for K_5 , it follows that these five sets have a common supporting circle. By the Corollary after Theorem 1 the same follows for all the family $\mathcal{K}$, which completes the proof of the theorem.

We observe that the hypothesis of the independence of family $\mathcal{K}$ cannot be weakened to one of the form of that in the Corollary, point (i). A counter example is given in the Fig. 4.

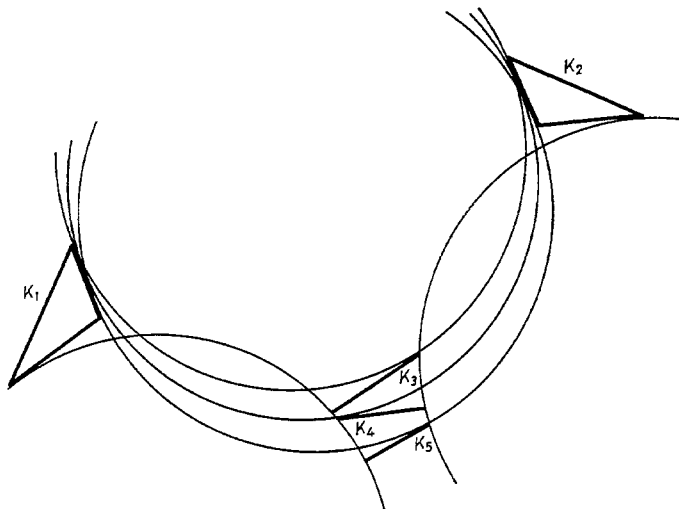


Fig. 4

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Eingegangen am 9. 5. 1972

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