

ANTIPROXIMAL SETS IN SOME BANACH SPACES

ŞTEFAN COBZAŞ

1. INTRODUCTION

All the spaces will be considered over the real numbers. Let X be a normed linear space, $M \subset X$ and $x \in X$. We use the following notations:

$$\begin{aligned} d(x, M) &= \inf\{\|x - y\| : y \in M\}, \\ P_M(x) &= \{y \in M : \|x - y\| = d(x, M)\} \quad \text{-- the metric projection,} \\ E(M) &= \{x \in X : P_M(x) \neq \emptyset\}, \\ Tc(M) &= \{x \in X : \text{card}(P_M(x)) = 1\}. \end{aligned}$$

The set M is called *proximal* if $E(M) = X$ and *antiproximal* if $E(M) = M$. The term antiproximal was proposed by R. R. Holmes [10]. I. Singer [13] proposes the term very non-proximal, which we have used in the note [2]. Following V. Klee [11], let us denote by N_1 the class of all normed linear spaces which contain an antiproximal, nonempty, closed, convex set, and by N_2 the class of all normed linear spaces which contain an antiproximal, nonempty, bounded, closed, convex set. Evidently, $N_2 \subset N_1$, and, as it was observed in [11], a Banach space belongs to the class N_1 if and only if it is non-reflexive. The characterization of the Banach spaces belonging to the class N_2 is more complicated. By a result of M. Edelstein [7], it follows that the class N_2 does not contain separable conjugate Banach spaces. The first example of a Banach space of class N_2 was given by M. Edelstein and A. C. Thompson [8]: $c_0 \in N_2$. For convenience, we agree to call *convex cell* a bounded, closed, symmetric and convex body. By a convex body we mean a convex set with non-void interior. Then the exact result in [8] is the following: the space c_0 contains an antiproximal convex cell. In [4] it was shown that the Banach space c contains also an antiproximal convex cell, so that $c \in N_2$. This result and the above result of M. Edelstein and A. C. Thompson [8] can be generalized as follows: If S is a compact Hausdorff space such that $C(S)$ is isomorphic to c_0 and I is an infinite dimensional closed ideal in $C(S)$, then $I \in N_2$ ([4]). M. Edelstein and A. C. Thompson [8] showed further that c_0 contains an antiproximal, strictly convex, convex cell. A problem in [8] is: there exists in c_0 an antiproximal, locally uniformly convex, convex cell? A positive answer to this question was given in [2]. In this note we show that the example from [2] gives also a negative answer to a question of S. B. Stečkin [14]. In the last part of this Note we give a sufficient condition for a space $L_1(S, \Sigma, \mu)$ to be not of class N_2 .

Concerning the Banach space theory we follow [5] and [6].

Let X be a normed space and X^* its dual. We say that a functional $f \in X^*$ supports the set $M \subset X$ if there exists a point $x \in M$ such that $f(x) = \sup f(M)$ or $f(x) = \inf f(M)$. We will denote by $\mathcal{S}(M)$ the set of all support functionals of the set M . In the following we shall need a result from [8]:

Presented at the 5th Balkan Mathematical Congress (Beograd, 24–30 June, 1974).
Published in: **Mathematica Balkanica** 4 (1974), 79–82.

Lemma 1.1 ([8], Prop. 1.(iii)). *Let X be a Banach space, M a nonempty, bounded, closed, convex subset of X and B the closed unit ball of X . Then M is antiproximinal if and only if*

$$\mathcal{S}(M) \cap \mathcal{S}(B) = \{0\}.$$

2. ON A PROBLEM OF S. B. STECKIN

S. B. Stečkin [14] proved that if X is a uniformly convex Banach space and M is a closed subset of X , then $X \setminus Tc(M)$ is of first Baire category in X (i.e., a countable union of nowhere dense subsets of X). In particular, by the Baire category theorem (see, e.g., [9, Th. 11.6.54]), $Tc(M)$ is dense in X . A question in [14] is: remains this true for a locally uniformly convex Banach space? We will show that the answer is no.

Indeed, let B_1 be the antiproximinal, locally uniformly convex, convex cell in c_0 , whose existence was proved in [2]. Then the Minkowski functional $\|\cdot\|_1$ is a locally uniformly convex norm on c_0 equivalent to the usual norm (which we denote by $\|\cdot\|$), and

$$B_1 = \{x \in c_0 : \|x\|_1 \leq 1\}.$$

If B denotes the unit ball in $(c_0, \|\cdot\|)$, then, by Lemma 1.1,

$$(1) \quad \mathcal{S}(B) \cap \mathcal{S}(B_1) = \{0\}.$$

But, again by Lemma 1.1, the relation (1) means also that B is an antiproximinal convex cell in the locally uniformly convex Banach space $(c_0, \|\cdot\|_1)$. Then $E(B) = Tc(B) = B$, so that $c_0 \setminus Tc(B)$ is not of first Baire category in c_0 .

3. THE SPACE $L_1(S, \Sigma, \mu)$

Let (S, Σ, μ) be a positive measure space, i.e., S is an abstract set, Σ a σ -algebra of subsets of S , and μ is a countably additive, non-negative measure on Σ . An atom in (S, Σ, μ) is a set $S_1 \in \Sigma$ of positive measure and such that $\mu(M) = 0$ or $\mu(M) = \mu(S_1)$ for every $M \in \Sigma$, $M \subset S_1$. We denote by $L_1(S, \Sigma, \mu)$ the Banach space of all real-valued, absolutely integrable functions on S with the norm

$$\|x\| = \int_S |x(s)| \mu(ds), \quad x \in L_1(S, \Sigma, \mu),$$

and by $L_\infty(S, \Sigma, \mu)$ the Banach space of all essentially bounded, measurable functions on S with the norm

$$\|f\| = \text{ess sup}_S |f(s)| = \inf \left\{ \sup_{s \in S \setminus M} |f(s)| : M \in \Sigma, \mu(M) = 0 \right\}, \quad f \in L_\infty(S, \Sigma, \mu).$$

As usual, by a function we mean an equivalence class of measurable functions coinciding μ -almost everywhere (μ -a.e. in short). We say that the conjugate space $L_1^*(S, \Sigma, \mu)$ of the space $L_1(S, \Sigma, \mu)$ is canonically isometric to the space $L_\infty(S, \Sigma, \mu)$ if there exists an isomorphism $\varphi : L_1^*(S, \Sigma, \mu) \rightarrow L_\infty(S, \Sigma, \mu)$ such that for every $f \in L_1^*(S, \Sigma, \mu)$

$$\begin{aligned} f(x) &= \int_S \varphi(f) x d\mu, \quad x \in L_1(S, \Sigma, \mu), \quad \text{and} \\ \|\varphi(f)\| &= \|f\|_\infty. \end{aligned}$$

For the sake of the convenience in notations, we will identify $\varphi(f)$ with f . The space $L_1^*(S, \Sigma, \mu)$ is not always canonically isometric to $L_\infty(S, \Sigma, \mu)$ (see, e.g., [9, V.20.17]). If (S, Σ, μ) is a decomposable measure space then $L_1^*(S, \Sigma, \mu)$ is canonically isometric to $L_\infty(S, \Sigma, \mu)$ ([9, V.20.20]).

Now we can state our result.

Theorem 3.1. *Let (S, Σ, μ) be a positive measure space containing at least one atom and such that $L_1^*(S, \Sigma, \mu)$ is canonically isometric to $L_\infty(S, \Sigma, \mu)$. Then $L_1(S, \Sigma, \mu)$ is not of class N_2 .*

Proof. Let us assume, on the contrary, that $L_1(S, \Sigma, \mu)$ contains an antiproximal, bounded, closed, convex set M . Then, by Lemma 1.1,

$$(2) \quad \mathcal{S}(M) \cap \mathcal{S}(B) = \{0\},$$

where B denotes the closed unit ball of $L_1(S, \Sigma, \mu)$. By a result of E. Bishop and R. R. Phelps [1], the set $\mathcal{S}(M)$ is dense in $L_\infty(S, \Sigma, \mu)$. Put $K = L_\infty(S, \Sigma, \mu) \setminus \mathcal{S}(B)$. Then by (2), $\mathcal{S}(M) \subset K \cup \{0\}$. We will show that $K \cup \{0\}$ is not dense in $L_\infty(S, \Sigma, \mu)$. Then, a fortiori, $\mathcal{S}(M)$ will be not dense in $L_\infty(S, \Sigma, \mu)$, which is a contradiction.

Obviously, in order to prove that $K \cup \{0\}$ is not dense in $L_\infty(S, \Sigma, \mu)$ it suffices to prove that K is not dense in $L_\infty(S, \Sigma, \mu)$. Let us suppose, again by contradiction, that K is dense in $L_\infty(S, \Sigma, \mu)$. By hypothesis, there exists an atom $S_1 \in \Sigma$. Let χ_{S_1} be the characteristic function of the set S_1 . Since we have supposed that K is dense in $L_\infty(S, \Sigma, \mu)$, it follows that there exists an $f \in K$ such that

$$(3) \quad \|\chi_{S_1} - f\|_\infty < 1/4,$$

so that

$$(4) \quad \|f\| \geq \|\chi_{S_1}\|_\infty - \|\chi_{S_1} - f\|_\infty > 1 - 1/4 = 3/4,$$

By a result of R. R. Phelps, $g \in \mathcal{S}(B)$ if and only if

$$\mu(\{s \in S : |g(s)| = \|g\|_\infty\}) > 0,$$

so that

$$(5) \quad \mu(\{s \in S : |f(s)| = \|f\|_\infty\}) = 0.$$

Since S_1 is an atom and f is measurable, there exists a real number α such that $f(s) = \alpha$, μ -a.e. on S_1 . By (5), $\|\alpha\| < \|f\|_\infty$, so that

$$\|f\|_\infty = \text{ess sup}_{S \setminus S_1} |f(s)|.$$

Since $\chi_{S_1}(s) = 0$ for $s \in S \setminus S_1$, (3) and (4) imply

$$1/4 > \|\chi_{S_1} - f\| \geq \text{ess sup}_{S \setminus S_1} |\chi_{S_1}(s) - f(s)| = \text{ess sup}_{S \setminus S_1} |f(s)| = \|f\|_\infty > 3/4.$$

This contradiction completes the proof of the theorem. $\square$

Corollary 3.2. *The space $\ell_1(\Gamma)$ is not of class N_2 .*

When Γ is a countable set, then $\ell_1(\Gamma)$ is a separable conjugate space ($\ell_1(\Gamma) = c_0(\Gamma)^*$), and the above Corollary follows also from the result of M. Edelstein [7] quoted in the introduction.

NOTE. This file was created in January 2012, explaining the somewhat paranormal quotations in a paper from 1974 of some papers published in 1975 and 1976.

REFERENCES

- [1] E. Bishop and R.R. Phelps, *The support functionals of a convex set*, in Convexity, Proc. Sympos. Pure Math. vol.7, pp.27-35, Amer.Math.Soc. Providence RI 1963.
- [2] S. Cobzaş, *Very non-proximinal sets in c_0* Rev. Anal. Numer. Theor. Aprox. **2**, no. 2 (1973), 137–141 (Romanian).
- [3] S. Cobzaş, Convex antiproximinal sets in the spaces c_0 and c , *Matem. Zametki* **17** (1975), 449-457.
- [4] S. Cobzaş, Antiproximinal sets in Banach spaces of continuous functions, *Anal. Numér. Théorie Approx.* **5** (1976), 127-143.
- [5] M. M. Day, *Normed Linear Spaces*, Third edition, Springer-Verlag, New York-Heidelberg, 1973.

- [6] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Interscience Publishers, Inc., New York, 1958.
- [7] M. Edelstein, *A note on nearest points*, Quart. J. Math. **21** (1970), 403-407.
- [8] M. Edelstein and A. C. Thompson, *Some results on nearest points and support properties of convex sets in c_0* , Pacific J. Math. **40** (1972), 553-560.
- [9] E. Hewitt and K. Stromberg, *Real and Abstract Analysis, A Modern Treatment of the Theory of Functions of a Real Variable*, Second printing corrected, Springer-Verlag, New York-Berlin 1969.
- [10] R. B. Holmes, *A Course on Optimization and Best Approximation*, Lect. Notes Math. Vol. 257. Springer-Verlag, Berlin-New York, 1972.
- [11] V. Klee, *Remarks on nearest points in normed linear spaces*, in Proc. Colloq. Convexity, Copenhagen 1965, pp. 161-176, Copenhagen 1967.
- [12] R. R. Phelps, *Subreflexive normed linear spaces*, Arch. Math. (Basel) **8** (1957), 444-450.
- [13] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Editura Academiei and Springer Verlag, Bucharest - Berlin 1970.
- [14] S. B. Stechkin, *Approximative properties of sets in linear normed spaces*, Rev. Math. Pures Appl. (Bucharest) **8** (1963), 5-18 (in Russian).

ȘTEFAN COBZAȘ, BABEȘ-BOLYAI UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, 400
084 CLUJ-NAPOCA, ROMANIA

E-mail address: scobzas@math.ubbcluj.ro