

Equally Spaced Points for Families of Compact Sets in Euclidean Spaces

By

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0. Introduction. The distance function in finite dimensional spaces constitutes the object of various investigations. Some of these, which are more connected to the problems considered by us in the sequel are for example those considered in [1] and [2], which concerns some analytical properties of this function.

In the notes [5] and [6] we have studied the problem of the existence and the unicity of a point equally spaced from the members of some families of convex and compact sets in Euclidean and respectively, in Minkowski spaces. In the method used there the convexity of sets has an essential role. The present note considers the same problem without the hypothesis of convexity.

1. Definitions and results. Denote by $\mathbb{R}^m$ the real m -dimensional Euclidean space and by $d(x, y)$ the metric in this space. We introduce also the notations

$$\delta(M', M'') = \min \{d(p', p'') : p' \in M', p'' \in M''\}$$

and

$$\varrho(M', M'') = \max \{d(p', p'') : p' \in M', p'' \in M''\},$$

where M' and M'' are compact subsets in $\mathbb{R}^m$.

Definition 1. The family $\mathcal{M}$ of sets in $\mathbb{R}^n$ will be said to be *independent*, if for any k pairwise distinct members $M_1, \dots, M_k$ of $\mathcal{M}$, $k \leq n + 1$, any set of points $x_1, \dots, x_k$, where $x_i \in M_i$, $i = 1, \dots, k$, determines a simplex of dimension $k - 1$, or equivalently, the vectors $x_2 - x_1, \dots, x_k - x_1$ are linearly independent.

Definition 2. A set M is called *nonseparable by hyperplanes* if there is no hyperplane H such that $H \cap M = \emptyset$, and M contains points in both the open half-spaces determined by H . The sets which are nonseparable by hyperplanes are sometimes called *convexly connected* (see [3] and [7], p. 174).

Theorem. Let $M_1, \dots, M_{n+1}$ be compact sets in $\mathbb{R}^n$ with the property that their convex hulls, $\text{Co}(M_1), \dots, \text{Co}(M_{n+1})$, are independent. Then there exists at least one point $q \in \mathbb{R}^n$ with the property that

$$(1) \quad \delta(q, M_1) = \dots = \delta(q, M_k) = \varrho(q, M_{k+1}) = \dots = \varrho(q, M_{n+1}),$$

where k is an integer $0 \leq k \leq n + 1$ (the case $k = 0$ or $k = n + 1$ means that in (1)

appears only ρ and respectively, only δ). In plus if each set M_i , $i = 1, \dots, n+1$ is nonseparable by hyperplanes, then the point q with the property (1) is uniquely determined.

In the sequel we shall consider only the case $n \geq 2$. For $n = 1$ the theorem is trivial.

2. Auxiliary results. Let $x_1, \dots, x_{n+1}$ be in general position in the sense that the sets $M_i = \{x_i\}$, $i = 1, \dots, n+1$ are independent. We define the mapping $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ by

$$(2) \quad \varphi: x \mapsto (d^2(x, x_1), \dots, d^2(x, x_{n+1})).$$

Consider in $\mathbb{R}^{n+1}$ the hyperplane

$$H = \{u: u^1 + u^2 + \dots + u^{n+1} = 0\},$$

where $u = (u^1, \dots, u^{n+1})$ denotes a vector in $\mathbb{R}^{n+1}$. Suppose that χ is the orthogonal projection of $\mathbb{R}^{n+1}$ onto H . We shall prove:

Lemma 1. *The mapping $\chi\varphi$ is a homeomorphism of $\mathbb{R}^n$ onto the hyperplane H .*

Proof. $\chi\varphi$ is obviously continuous. We shall show that it is a surjective mapping. The orthogonal projection of the vector

$$\varphi(x) = ((x - x_1, x - x_1), \dots, (x - x_{n+1}, x - x_{n+1}))$$

has the form

$$\begin{aligned} & ((x - x_1, x - x_1), \dots, (x - x_{n+1}, x - x_{n+1})) - \\ & - \left(\sum_{j=1}^{n+1} (x - x_j, x - x_j) \right) (1/(n+1), \dots, 1/(n+1)). \end{aligned}$$

Let now $v = (v^1, \dots, v^{n+1})$ be a vector in H . We shall show that there is an x in $\mathbb{R}^n$, such that $\chi\varphi(x) = v$, that is

$$(3) \quad (x - x_i, x - x_i) - \frac{1}{n+1} \sum_{j=1}^{n+1} (x - x_j, x - x_j) = v^i, \quad i = 1, \dots, n+1.$$

By subtraction of the first equation from the others we get

$$(3') \quad 2(x, x_1 - x_i) = v^i - v^1 + (x_1, x_1) - (x_i, x_i), \quad i = 2, \dots, n+1.$$

But for the given values $v^i - v^1$, $i = 2, \dots, n+1$, this system of linear equations has a unique solution x , because $x_1 - x_i$, $i = 2, \dots, n+1$ are linear independent vectors, by hypothesis. As the systems (3) and (3') are equivalent for $v \in H$, there follows the existence and also the unicity of x with $\chi\varphi(x) = v$. This, together with Brouwer's theorem of the invariance of the domain proves the lemma.

Lemma 2. *Suppose that S_0^{n-1} is a geometrical sphere in $\mathbb{R}^n$ with the centre in the origin. Let be $x_1, \dots, x_{n+1} \in S_0^{n-1}$, points in general position and let φ be the mapping defined by (2). Suppose that S^{n-1} is any other geometrical sphere with the centre in the origin. Denote $\xi = \varphi|S^{n-1}$, the restriction of φ to S^{n-1} . Then ξ is an essential mapping*

of S^{n-1} in $\mathbf{R}^{n+1} \setminus \Delta$ (i.e., it is not homotopic with a constant mapping in this space), where

$$\Delta = \{u \in \mathbf{R}^{n+1} : u^1 = \dots = u^{n+1}\}.$$

Proof. We observe first that $H \setminus \{0\}$, where 0 is the origin of $\mathbf{R}^{n+1}$, is a strong deformation retract ([4], p. 16) of $\mathbf{R}^{n+1} \setminus \Delta$. This retraction may be realized for example by the function $F : (\mathbf{R}^{n+1} \setminus \Delta) \times I \rightarrow \mathbf{R}^{n+1} \setminus \Delta$, $I = [0, 1]$, given by

$$F : (u, t) \mapsto u - t(u, w)w,$$

where w is the normal vector to H . Moreover, we have $F(\cdot, 1) = \chi$, where χ is the orthogonal projection of $\mathbf{R}^{n+1} \setminus \Delta$ onto $H \setminus \{0\}$. Therefore ξ and $\chi\xi$ are homotopic mappings in $\mathbf{R}^{n+1} \setminus \Delta$, and it follows also that ξ is an essential mapping in $\mathbf{R}^{n+1} \setminus \Delta$ if and only if $\chi\xi$ is essential in $H \setminus \{0\}$.

We have obviously $\chi\varphi(0) = 0$, where 0 on the left side is the origin in $\mathbf{R}^n$ and that on the right side is the origin in $\mathbf{R}^{n+1}$. Observe also that $\chi\xi(S^{n-1})$ is a strong deformation retract of $H \setminus \{0\} = \chi\varphi(\mathbf{R}^n) \setminus \{0\}$, by the deformation retraction $G : (\chi\varphi(\mathbf{R}^n) \setminus \{0\}) \times I \rightarrow \chi\varphi(\mathbf{R}^n) \setminus \{0\}$ of the form

$$G : (\chi\varphi(x), t) \mapsto \chi\varphi((1-t)x + trx/\|x\|),$$

where r is the radius of S^{n-1} . The mapping G is well defined and continuous by Lemma 1. This means that the set $H \setminus \{0\}$ is deformable into $\chi\xi(S^{n-1})$. Now, if $\chi\xi$ would be inessential in $H \setminus \{0\}$, it would follow that $H \setminus \{0\}$ is deformable into a point, i.e., $H \setminus \{0\}$ is contractible to a point. But this is absurd.

Lemma 3. Let M be a compact set in $\mathbf{R}^n$, $p \in \mathbf{R}^n$, and let be

$$M_t = p(1-t) + tM, \quad t \in I = [0, 1],$$

where sM denotes the set obtained by multiplying each element of M by the scalar s . Then $\delta(x, M_t)$ and $\varrho(x, M_t)$ are continuous functions from $\mathbf{R}^n \times I$ in $\mathbf{R}$.

This lemma can be verified by standard reasonings from the functional analysis.

Consider now the compact sets $M_1, \dots, M_{n+1}$ in $\mathbf{R}^n$ such that their convex hulls $\text{Co}(M_1), \dots, \text{Co}(M_{n+1})$ are independent in the sense of Definition 1. Because $\text{Co}(M_i)$, $i = 1, \dots, n+1$ are compact and independent sets, it follows then that the centres of the spheres determined by the points $x_1, \dots, x_{n+1}$ form a compact set Ω in $\mathbf{R}^n$, when x_i runs through $\text{Co}(M_i)$, $i = 1, \dots, n+1$.

Suppose that $p_i \in M_i$, $i = 1, \dots, n+1$ and let us consider the centre 0 of the sphere determined by these points and let 0 be the origin in $\mathbf{R}^n$. Let S^{n-1} be a geometrical sphere with the centre in 0 such that Ω is contained in the interior of the ball D^n having S^{n-1} as boundary.

We define the mapping $\psi : \mathbf{R}^n \rightarrow \mathbf{R}^{n+1}$ by

$$\psi : x \mapsto (\delta^2(x, M_1), \dots, \delta^2(x, M_k), \varrho^2(x, M_{k+1}), \dots, \varrho^2(x, M_{n+1})).$$

If $\eta = \psi|_{S^{n-1}}$ is the restriction of ψ to S^{n-1} , then from the definition of S^{n-1} , it follows that $\eta(S^{n-1}) \subset \mathbf{R}^{n+1} \setminus \Delta$. Otherwise there would be a point $x \in S^{n-1}$ such that $\psi(x) \in \Delta$, but this means that $x \in \Omega$, which is absurd.

Let φ denote the mapping defined by the formula (2) with $x_i = p_i, i = 1, \dots, n+1$. Then we have

Lemma 4. *Let us denote $\xi = \varphi|S^{n-1}, \eta = \psi|S^{n-1}$. Then the mappings ξ and η of S^{n-1} in $\mathbf{R}^{n+1} \setminus \Delta$ are in the same homotopy class.*

Proof. We define the mapping $F: S^{n-1} \times I \rightarrow \mathbf{R}^{n+1}$, by

$$F: (x, t) \mapsto (\delta^2(x, M_{1,t}), \dots, \delta^2(x, M_{k,t}), \varrho^2(x, M_{k+1,t}), \dots, \varrho^2(x, M_{n+1,t}))$$

where

$$M_{i,t} = p_i(1-t) + tM_i, \quad i = 1, \dots, n+1,$$

and $p_i \in M_i, i = 1, \dots, n+1$ are the above chosen points. F is by Lemma 3 a continuous function on $S^{n-1} \times I$, and we have for $x \in S^{n-1}$

$$F(x, 0) = \xi(x) \quad \text{and} \quad F(x, 1) = \eta(x).$$

To verify the lemma it is sufficient to prove that we have $\Delta \cap F(S^{n-1} \times I) = \emptyset$. Suppose that there exists $(x, t) \in S^{n-1} \times I$ such that $F(x, t) \in \Delta$. This means that

$$\delta^2(x, M_{1,t}) = \dots = \delta^2(x, M_{k,t}) = \varrho^2(x, M_{k+1,t}) = \dots = \varrho^2(x, M_{n+1,t}).$$

But this means that x is a point which is equally spaced to some points $x_i \in \text{Co}(M_i), i = 1, \dots, n+1$, i.e. x is contained in Ω , which contradicts the definition of S^{n-1} and proves the lemma.

3. Proof of the Theorem. Consider the geometrical sphere S^{n-1} which we have used in Lemma 4, and let D^n be the closed ball having S^{n-1} as boundary. We shall show that

$$\psi(D^n) \cap \Delta \neq \emptyset.$$

Suppose the contrary. Then it would follow that the mapping $\eta = \psi|S^{n-1}$ is homotopic with a constant mapping in the set of mappings of S^{n-1} in $\mathbf{R}^{n+1} \setminus \Delta$. But according to Lemma 4, $\xi = \varphi|S^{n-1}$ would have the same property, and this contradicts Lemma 2. Let q be a point in D^n such that $\psi(q) \in \Delta$. This means that

$$\delta^2(q, M_1) = \dots = \delta^2(q, M_k) = \varrho^2(q, M_{k+1}) = \dots = \varrho^2(q, M_{n+1}),$$

i.e. q is a point with the property desired by the first part of the theorem.

To complete the proof of the theorem we have to show the unicity of q in the case when $M_1, \dots, M_{n+1}$ are nonseparable by hyperplanes. Suppose that q_1 and q_2 satisfy (1). Denote by r_1 and respectively by r_2 the common value of the entities appearing in (1) for q_1 and respectively for q_2 . Let be $S_1 = S(q_1, r_1)$ and $S_2 = S(q_2, r_2)$, where $S(q, r)$ denotes the sphere in $\mathbf{R}^n$ with the centre q and the radius r . Let H be a hyperplane which satisfies the relation $H \cap S_1 = H \cap S_2 = S_1 \cap S_2$ and separates q_1 and q_2 if $S_1 \cap S_2 = \emptyset$. If $k < n+1$, then $S_1 \cap S_2 \neq \emptyset$ and H is uniquely determined. Let H_1 and H_2 be the two closed halfspaces determined by H , which are denoted in the following way. If H separates the points q_1 and q_2 let be H_1 the closed halfspace containing q_1 and H_2 the other. If H doesn't separate q_1 and q_2 we can suppose $r_1 < r_2$. Denote then by H_2 the closed halfspace containing both points q_1 and q_2 and by H_1 the other closed halfspace.

Suppose first $i \leq k$. Then we have $M_i \cap S_1 \subset H_1$ and $M_i \cap S_2 \subset H_2$. Suppose $M_i \cap H = \emptyset$. But then M_i contains points in H_1 and H_2 and contradicts the hypothesis that M_i is nonseparable by hyperplanes. Hence $M_i \cap H \neq \emptyset$, $i = 1, \dots, k$. If $i \geq k+1$, then $M_i \cap S_1 \subset H_2$ and $M_i \cap S_2 \subset H_1$. By the same reasoning as above follows $M_i \cap H \neq \emptyset$, $i = k+1, \dots, n+1$. Thus we have shown that $M_i \cap H \neq \emptyset$, $i = 1, \dots, n+1$, which is in contradiction with the independence of the sets $\text{Co}(M_1), \dots, \text{Co}(M_{n+1})$ and completes the proof of the theorem.

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