

In the note we prove that in a Banach space  $c$  there exists a closed bounded symmetric convex division ring  $V_1$  such that for any  $x \in c \setminus V_1$ ,  $P_{V_1}(x) = \emptyset$ , where  $P_{V_1}$  is the metric projection onto  $V_1$ .

Let  $X$  be a normed linear space and  $M$  be a nonempty subset of it. For  $x \in X$  we denote by  $P_M(x)$  the set of all points  $y \in M$  satisfying the condition  $\|x - y\| = d(x, M)$ , where  $d(x, M) = \inf \{\|x - z\| : z \in M\}$ . Set  $M$  is called antiproximal (see [1]) if  $P_M(x) = \emptyset$  ( $\emptyset$  is the empty set) for any  $x \in X \setminus M$ . Space  $X$  is called a space of type  $N_2$  (see [2]) if a closed bounded antiproximal set exists in  $X$ . According to one result of Edelstein [3], a separable adjoint Banach space of type  $N_2$  does not exist. (Banach space  $X$  is called adjoint if a normed linear space  $Y$  exists such that  $X = Y^*$ .) Edelstein and Thompson [4] proved that  $c_0$  is a space of type  $N_2$ . More precisely, a closed bounded symmetric antiproximal convex division ring exists in  $c_0$ . (A convex set is called a convex division ring if its interior is nonempty.)

As usual, by  $c_0$  we denote the space of all sequences, converging to zero, of real numbers with norm

$$\|x\| = \max \{|x(i)| : i \in \mathbb{N}\}, \quad x \in c_0. \quad (1)$$

In this note we prove that the Banach space of all convergent sequences of real numbers with norm

$$\|x\| = \sup \{|x(i)| : i \in \mathbb{N}\}, \quad x \in c, \quad (2)$$

is a space of type  $N_2$ . More precisely, there holds

**THEOREM 1.** A closed bounded symmetric antiproximal convex division ring exists in the Banach space  $c$ .

It is very well known (see [5]) that the spaces  $c_0$  and  $c$  are isomorphic, so that this result has an obvious connection with the following problem [4]: is  $c_0$  a space of type  $N_2$  for any equivalent renumbering?

Let us recall certain notions and notation which we are going to use subsequently. Let  $X$  be a Banach space,  $X^*$  be its adjoint space. For  $M \subseteq X$  we denote by  $\text{Sp}(M)$  the linear subspace generated by set  $M$ , by  $\overline{M}$  its closure, and by  $M^\circ$  its interior. The word isomorphism is used in the sense of a linear topological isomorphism, i.e., a linear homeomorphic mapping of one space onto another.

Let  $M$  be a convex subset of  $X$ . A functional  $f \in X^*$  is called a support functional of set  $M$  if a point  $x \in M$  exists such that  $f(x) = \sup f(M) = \sup \{f(y) : y \in M\}$ . By  $J(M)$  we denote the set of all support functionals of set  $M$ .

Let  $X, Y$  be Banach spaces and let  $A : X \rightarrow Y$  be an isomorphism. Then the adjoint operator  $A^* : Y^* \rightarrow X^*$  also is an isomorphism and  $(A^*)^{-1} = (A^{-1})^*$  (Lemma VI.3.7 in [6]).

The following simple remark will be used several times in what follows.

**LEMMA 1.** Let  $X, Y$  be Banach spaces,  $M$  be a convex subset of  $X$ , and  $A : X \rightarrow Y$  be an isomorphism. Then  $f \in J(M)$  if and only if  $(A^*)^{-1}f \in J(A(M))$  (in other words,  $J(M) = A^*(J(A(M)))$ ).

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Proof. We have

$$\begin{aligned} f \in J(M) &\Leftrightarrow \exists x \in M, \quad f(x) = \sup f(M) \Leftrightarrow \exists x \in M, \\ f(A^{-1}Ax) &= \sup f(A^{-1}A(M)) \Leftrightarrow \exists x \in M, \end{aligned}$$

$$(A^{-1})^* f(Ax) = \sup (A^{-1})^* f(A(M)) \Leftrightarrow$$

$$\Leftrightarrow \exists y \in A(M), (A^*)^{-1}f(y) = \sup (A^*)^{-1}f(A(M)) \Leftrightarrow (A^*)^{-1}f \in J(A(M)),$$

where  $y = Ax \in A(M)$ .

Let  $X$  be a Banach space. The sequence  $\{x_k: k \in N\} \subseteq X$  is called complete if  $\overline{\text{Sp}\{x_k: k \in N\}} = X$ . The sequence  $\{x_k: k \in N\}$  is called a base if for any  $x \in X$  there exists a unique series  $x = \sum_{k=1}^{\infty} a_k x_k$  converging to  $x$ . The sequence  $\{x_k: k \in N\} \subseteq X$  is called minimal if a system  $\{f_k: k \in N\} \subseteq X^*$  exists such that

$$f_k(x_i) = \delta_{ki} = \begin{cases} 1, & k = i, \\ 0, & k \neq i. \end{cases}$$

The system  $\{f_k: k \in N\}$  is said to be adjoint to system  $\{x_k: k \in N\}$ . A base is always a minimal system. See [7] concerning base and minimal sets.

LEMMA 2. Let  $X, Y$  be Banach space,  $\{x_k: k \in N\} \subseteq X$  be a minimal sequence (respectively, a base) in space  $X$  with adjoint  $\{f_k: k \in N\} \subseteq X^*$ , and  $A: X \rightarrow Y$  be an isomorphism. Then  $\{Ax_k: k \in N\}$  is a minimal system (respectively, a base) in space  $Y$  with adjoint  $\{(A^*)^{-1}f_k: k \in N\}$ .

Proof. It is obvious that

$$(A^*)^{-1}f_k(Ax_i) = (A^{-1})^*f_k(Ax_i) = f_k(A^{-1}Ax_i) = f_k(x_i) = \delta_{ki},$$

whence it follows that the systems  $\{Ax_k: k \in N\}$  and  $\{(A^*)^{-1}f_k: k \in N\}$  are adjoint. The remaining assertions of the lemma are trivial.

We now pass onto the space  $c$ . Let  $\omega$  be the first infinite ordinal number and let  $[1, \omega] = \{\alpha: \alpha \text{ is an ordinal number and } 1 \leq \alpha \leq \omega\}$ . Then in the order topology  $[1, \omega]$  is a compact Hausdorff space and  $c = C[1, \omega]$ , where  $C[1, \omega]$  denotes the space of all real continuous functions, defined on  $[1, \omega]$ , with norm

$$\|x\| = \max \{|x(i)|: 1 \leq i \leq \omega\}. \quad (3)$$

Its adjoint is the space  $M[1, \omega]$  of all Radon measures on  $[1, \omega]$  (see [6]). For each Radon measure  $\mu \in M[1, \omega]$  we have

$$\|\mu\| = \text{var}(\mu, [1, \omega]) = \sum_{1 \leq i \leq \omega} |\mu(i)| < \infty.$$

By  $S(\mu)$  we denote the support of measure  $\mu$ , i.e., the closed set complementary to all open subsets of compactum  $[1, \omega]$ , on each of which the variation of  $\mu$  equals zero. In our case  $S(\mu) = \overline{\{i \in [1, \omega]: \mu(i) \neq 0\}}$ . Every Radon measure  $\mu$  can be represented uniquely in the form

$$\mu = \mu^+ - \mu^-,$$

where  $\mu^+$  and  $\mu^-$  are nonnegative Radon measures (the Jordan expansion [6]). Let

$$U = \{x \in C[1, \omega]: \|x\| \leq 1\}. \quad (4)$$

According to [8],  $\mu \in J(U)$  if and only if  $S(\mu^+) \cap S(\mu^-) = \emptyset$ . Keeping in mind that  $\omega$  belongs to each infinite closed subset of  $[1, \omega]$ , that  $S(\mu^+) = \overline{\{i \in [1, \omega]: \mu(i) > 0\}}$ , and that  $S(\mu^-) = \overline{\{i \in [1, \omega]: \mu(i) < 0\}}$ , we obtain

LEMMA 3. Let  $\mu \in M[1, \omega]$ . Then  $\mu \in J(U)$  if and only if one of the following conditions is fulfilled:

a)  $S(\mu)$  is a finite set,

b<sub>1</sub>)  $S(\mu^+)$  is finite,  $S(\mu^-)$  is infinite, and  $\mu(\omega) \leq 0$ ,

$b_2)$   $S(\mu^+)$  is infinite,  $S(\mu^-)$  is finite, and  $\mu(\omega) \geq 0$ .

In what follows we shall identify the set  $N$  of all positive integers with the set  $\{a \in [1, \omega] : 1 \leq a < \omega\}$

We define the functions  $\delta_i, g_i \in M[1, \omega]$  by the equalities

$$\delta_i(x) = x(i), \quad x \in C[1, \omega], \quad 1 \leq i \leq \omega,$$

respectively,

$$g_i(x) = \delta_i(x) + \sum_{j=1}^{3 \cdot 2^{i-1}-1} (-1)^j \frac{x(j)}{2^{j+2}} + \frac{1}{2^{i+1}} \sum_{n=1}^{\infty} (-1)^n \frac{x(k_n^i)}{2^n}$$

for  $x \in C[1, \omega]$ ,  $1 \leq i \leq \omega$ , where

$$k_n^i = 2^{i-1}(2n+1), \quad n = 1, 2, \dots, \quad 1 \leq i < \omega,$$

and

$$g_\omega(x) = \delta_\omega(x) + \sum_{1 \leq j < \omega} (-1)^j \frac{x(j)}{2^{j+2}}, \quad x \in C[1, \omega].$$

Since both sets  $S(g_i^+)$  and  $S(g_i^-)$  are infinite, according to Lemma 3,  $g_i \notin J(U)$ ,  $1 \leq i \leq \omega$ . This statement is true for any nonzero finite linear combination of elements of  $g_i$ .

Denoting

$$X = \text{Sp} \{g_i : 1 \leq i \leq \omega\}, \quad (5)$$

we get that

$$X \cap J(U) = \{0\}. \quad (6)$$

The proof of the next lemma repeats the proof of the fact that the operator  $A$  constructed in [4] is an isomorphism of spaces  $c_0$  and  $c$  (see [4]). Therefore, we do not present it.

**LEMMA 4.** The linear operator defined by the equalities

$$Ax(i) = g_i(x), \quad 1 \leq i \leq \omega, \quad x \in C[1, \omega],$$

is an isomorphism of the space  $C[1, \omega]$  onto  $C[1, \omega]$ . Its adjoint operator satisfies the relations

$$A^* \delta_i = g_i, \quad 1 \leq i \leq \omega. \quad (7)$$

Let  $e_i \in C[1, \omega]$  be defined by the equalities

$$\begin{aligned} e_i(j) &= \delta_{ij}, \quad 1 \leq j \leq \omega, \quad 1 \leq i < \omega, \\ e_\omega(j) &= 1, \quad 1 \leq j \leq \omega. \end{aligned} \quad (8)$$

Since any element  $x \in C[1, \omega]$  can be represented uniquely in the form  $x = x(\omega) e_\omega + \sum_{1 \leq i < \omega} [x(i) - x(\omega)] e_i$ , the sequence  $\{e_i : 1 \leq i \leq \omega\}$  is a base of space  $C[1, \omega]$ . Its adjoint is the sequence

$$\{\delta_\omega\} \cup \{\delta_i - \delta_\omega : 1 \leq i < \omega\}. \quad (9)$$

According to Lemma 2 the sequence

$$y_i = A^{-1} e_i, \quad 1 \leq i \leq \omega, \quad (10)$$

also is a base of space  $C[1, \omega]$  with adjoint

$$\begin{aligned} u_i &= A^* (\delta_i - \delta_\omega) = g_i - g_\omega, \quad 1 \leq i < \omega, \\ u_\omega &= A^* \delta_\omega. \end{aligned} \quad (11)$$

Let

$$V = \{x \in C[1, \omega] : |\delta_\omega(x)| \leq 1, |(\delta_i - \delta_\omega)(x)| \leq 1, 1 \leq i < \omega\}. \quad (12)$$

It is easy to see that  $V$  is a closed bounded symmetric convex division ring. Since  $A$  is an isomorphism, the set

$$V_1 = A^{-1}(V) \quad (13)$$

has the same property.

The spaces  $C[1, \omega]$  and  $c_0$  are isomorphic. Let

$$H: C[1, \omega] \rightarrow c_0 \quad (14)$$

be the isomorphism of space  $C[1, \omega]$  onto  $c_0$ . We define a new norm  $\|\cdot\|_1$  on  $c_0$ , equivalent to norm (1), by

$$\|x\|_1 = \|H^{-1}x\|, \quad x \in c_0. \quad (15)$$

Then  $H$  will be an isometric isomorphism of the space  $(C[1, \omega], \|\cdot\|)$  onto  $(c_0, \|\cdot\|_1)$ . Applying Lemma 2 once again, we get that the sequence

$$x_i = Hy_i, \quad 1 \leq i \leq \omega, \quad (16)$$

is a base of space  $c_0$  with adjoint

$$h_i = (H^*)^{-1}u_i, \quad 1 \leq i \leq \omega. \quad (17)$$

Let

$$V_2 = H(V_1) = HA^{-1}(V). \quad (18)$$

**LEMMA 5.** Set  $V_2$  is antiproximal in space  $(c_0, \|\cdot\|_1)$ .

**Proof.** We introduce the following notation:

$$\begin{aligned} \bar{e}_\omega &= (\delta_{1j})_{j \in N} \in c_0; \quad \bar{e}_i = (\delta_{i+1,j})_{j \in N} \in c_0, \quad 1 \leq i < \omega; \\ Z &= \text{Sp}\{\bar{e}_i : 1 \leq i \leq \omega\}; \quad Y = \text{Sp}\{x_i : 1 \leq i \leq \omega\}. \end{aligned} \quad (19)$$

Since  $V_2$  is a closed bounded symmetric convex division ring, its Minkowski functional  $\|\cdot\|_2$  is a norm on  $c_0$ , equivalent to norm (1), and

$$V_2 = \{x \in c_0 : \|x\|_2 \leq 1\}. \quad (20)$$

Taking (18), (12) ([6], VI.3.7), (17), and (11) into account, we obtain

$$\begin{aligned} \|x\|_2 &= \inf\{\lambda > 0 : x \in \lambda V_2\} = \inf\{\lambda > 0 : \lambda^{-1}x \in V_2\} = \\ &= \inf\{\lambda > 0 : \lambda^{-1}x \in HA^{-1}(V)\} = \inf\{\lambda > 0 : AH^{-1}(\lambda^{-1}x) \in V\} = \inf\{\lambda > 0 : |\delta_\omega(AH^{-1}(\lambda^{-1}x))| \leq 1, \\ &\quad |(\delta_i - \delta_\omega)(AH^{-1}(\lambda^{-1}x))| \leq 1, 1 \leq i < \omega\} = \inf\{\lambda > 0 : |(H^*)^{-1}A^*\delta_\omega(\lambda^{-1}x)| \leq 1, \\ &\quad |(H^*)^{-1}A^*(\delta_i - \delta_\omega)(\lambda^{-1}x)| \leq 1, 1 \leq i < \omega\} = \inf\{\lambda > \\ &> 0 : |h_i(\lambda^{-1}x)| \leq 1, 1 \leq i \leq \omega\} = \inf\{\lambda > 0 : |h_i(x)| \leq \lambda, 1 \leq i \leq \omega\} = \sup\{|h_i(x)| : 1 \leq i \leq \omega\}. \end{aligned} \quad (21)$$

Let  $T: Y \rightarrow Z$  be a linear operator defined by the equalities

$$Tx_i = \bar{e}_i, \quad 1 \leq i \leq \omega. \quad (22)$$

Taking (16), (17), and (21) into account, for any  $y = \sum_{i=1}^n a_i x_{\alpha_i} \in Y$  we obtain:

$$\begin{aligned} \|Ty\| &= \left\| \sum_{i=1}^n a_i \bar{e}_{\alpha_i} \right\| = \max\{|a_1|, \dots, |a_n|\} = \\ &= \max\{|h_{\alpha_1}(y)|, \dots, |h_{\alpha_n}(y)|\} = \sup\{|h_k(y)| : 1 \leq k \leq \omega\} = \|y\|_2. \end{aligned}$$

Thus,  $T$  is an isometric isomorphism of the space  $(Y, \|\cdot\|_2)$  onto  $(Z, \|\cdot\|)$ . Since the sequence  $\{x_k : 1 \leq k \leq \omega\}$  is a base of space  $c_0$ ,  $\bar{Y} = c_0$  and  $T$  can be continuously extended (and, moreover, uniquely) up to the linear isometry of space  $(c_0, \|\cdot\|_2)$  into the space  $(c_0, \|\cdot\|)$ . Further, since the norms  $\|\cdot\|_2$  and  $\|\cdot\|$  are equivalent,  $\alpha, \beta > 0$  exist such that  $\alpha \|x\| \leq \|x\|_2 \leq \beta \|x\|$ . Thus,  $\|Tx\| = \|x\|_2 \geq \alpha \|x\|$ ,  $x \in c_0$ , whence it follows that the range of operator  $T$  is closed. But then,  $T(c_0) = \overline{T(c_0)} \cong \overline{T(Y)} = \bar{Z} = c_0$ . Thus, we have obtained that  $T$  is an isometric isomorphism of space  $(c_0, \|\cdot\|_2)$  onto  $(c_0, \|\cdot\|)$ . Its adjoint operator  $T^*$  is the isometric isomorphism of  $(l_1, \|\cdot\|)$  onto  $(l_1, \|\cdot\|_2)$  (by  $\|\cdot\|$  and  $\|\cdot\|_2$  we have denoted norms on  $l_1$ , adjoint, respectively, to norm (1) and to  $\|\cdot\|_2$  on  $c_0$ ). Let  $\bar{\delta}_i \in l_1$  be defined by the equalities

$$\begin{aligned} \bar{\delta}_i(x) &= x(i+1), \quad x \in c_0, \quad 1 \leq i < \omega, \\ \bar{\delta}_\omega(x) &= x(1), \quad x \in c_0. \end{aligned} \quad (23)$$

Then

$$T^* \bar{\delta}_i(x_k) = \delta_i(Tx_k) = \delta_i(\bar{e}_k) = \delta_{ik} = h_i(x_k), \quad 1 \leq k \leq \omega.$$

Since  $\{x_k : 1 \leq k \leq \omega\}$  is a base of space  $c_0$ ,

$$T^* \bar{\delta}_i = h_i, \quad 1 \leq i \leq \omega. \quad (24)$$

Let us show that  $V_2$  is an antiproximal set in  $(c_0, \|\cdot\|_1)$ . For this it is enough to prove that  $J(V_2) \cap J(B_1) = \{0\}$  ([4], Prop. 1 (iii)), where

$$B_1 = \{x \in c_0 : \|x\|_1 \leq 1\}. \quad (25)$$

Let  $B$  be the unit ball in space  $(c_0, \|\cdot\|)$  and  $U$  be the unit ball in space  $(C[1, \omega], \|\cdot\|)$ . Since  $T$  is an isometric isomorphism of space  $(c_0, \|\cdot\|_2)$  onto  $(c_0, \|\cdot\|)$ , taking (20), (15), and (4) into account, we obtain

$$B = T(V_2), \quad B_1 = H(U). \quad (26)$$

From Lemma 1, (26), (24), (17), (5), and (11) it follows that

$$\begin{aligned} f \in J(V_2) &\Leftrightarrow (T^*)^{-1}f \in J(T(V_2)) \Leftrightarrow (T^*)^{-1}f \in J(B) \Leftrightarrow \\ &\Leftrightarrow \exists \alpha_1, \dots, \alpha_n \in [1, \omega], \exists a_1, \dots, a_n \in \mathbb{R}, (T^*)^{-1}f = \\ &= \sum_{i=1}^n a_i \bar{\delta}_{\alpha_i} = \sum_{i=1}^n a_i (T^*)^{-1}h_{\alpha_i} = (T^*)^{-1} \left( \sum_{i=1}^n a_i h_{\alpha_i} \right) \Leftrightarrow \\ &\Leftrightarrow f = \sum_{i=1}^n a_i h_{x_i} = \sum_{i=1}^n a_i (H^*)^{-1}u_{\alpha_i} = (H^*)^{-1} \left( \sum_{i=1}^n a_i u_{\alpha_i} \right) \Leftrightarrow f \in (H^*)^{-1}(X). \end{aligned}$$

On the other hand, according to Lemma 1 and (26),

$$f \in J(B_1) \Leftrightarrow H^*f \in J(U) \Leftrightarrow f \in (H^*)^{-1}(J(U)).$$

Further, taking (6) into account, we obtain

$$f \in (H^*)^{-1}(X) \cap (H^*)^{-1}(J(U)) = (H^*)^{-1}(X \cap J(U)) = (H^*)^{-1}(0) = \{0\}$$

Thus,  $J(B_1) \cap J(V_2) = \{0\}$ . The lemma is proved.

**Proof of Theorem 1.** Since  $H: (C[1, \omega], \|\cdot\|) \rightarrow (c_0, \|\cdot\|_1)$  is an isometric isomorphism and set  $V_2$  is an antiproximal in  $(c_0, \|\cdot\|_1)$ ,  $V_1 = H^{-1}(V_2)$  is a closed bounded symmetric antiproximal convex division ring in  $(C[1, \omega], \|\cdot\|)$ .

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