

Hahn decompositions of finitely additive measures

By

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Let S be a set and $\mathcal{A}$ an algebra of subsets of S . Denote by $B(S, \mathcal{A})$ the Banach space of all real-valued functions, defined on S , which are uniform limits of sequences of $\mathcal{A}$ -measurable step functions with the sup-norm, and by $\text{ba}(S, \mathcal{A})$ its conjugate, i.e. the Banach space of all real-valued, finitely additive, bounded measures on $\mathcal{A}$ the norm of a measure being its total variation (see [1], IV.5). We denote by $|\mu|$ the total variation of the measure μ and by $\|\mu\|$ its norm, i.e. $\|\mu\| = |\mu|(S)$. A *Hahn decomposition* of $(S, \mathcal{A}, \mu)$ is a measurable dissection (A, B) of S (i.e. $S = A \cup B$, $A \cap B = \emptyset$ and $A, B \in \mathcal{A}$) such that $\mu \geq 0$ on A and $\mu \leq 0$ on B . It is well known that if $\mathcal{A}$ is a σ -algebra and μ is a countably additive measure on $\mathcal{A}$, then $(S, \mathcal{A}, \mu)$ admits a Hahn decomposition. In the case of finitely additive measure we have:

Theorem. *The measure space $(S, \mathcal{A}, \mu)$ admits a Hahn decomposition if and only if μ attains its norm on the unit ball of $B(S, \mathcal{A})$.*

Proof. If (A, B) is a Hahn decomposition of $(S, \mathcal{A}, \mu)$ then

$$\int_S x \cdot d\mu = \|\mu\|, \quad \text{where } x = \chi_A - \chi_B.$$

Conversely, suppose $x \in B(S, \mathcal{A})$ is such that $\|x\| = 1$ and $\int_S x \cdot d\mu = \|\mu\|$.

Define the measure $x \cdot \mu \in \text{ba}(S, \mathcal{A})$ by $x \cdot \mu(A) = \int_A x \cdot d\mu$, for all $A \in \mathcal{A}$.

The measure $x \cdot \mu$ has the following properties:

- (i) $|x \cdot \mu| = |x| \cdot |\mu|$ (see [1], III.2, Theorem 20(a)).
- (ii) $x \cdot \mu = |\mu|$.

Indeed, $\|x\| = 1$ implies $x \cdot \mu \leq |\mu|$. If there would be a set $A \in \mathcal{A}$ such that $(x \cdot \mu)(A) < |\mu|(A)$, then

$$\begin{aligned} \|\mu\| &= \int_S x \cdot d\mu = (x \cdot \mu)(S) = \\ &= (x \cdot \mu)(A) + (x \cdot \mu)(S \setminus A) < |\mu|(A) + |\mu|(S \setminus A) = \|\mu\|, \end{aligned}$$

which is a contradiction.

- (iii) $|x| = 1$, μ -a.e.

Indeed, by (i) and (ii) $\int_S (1 - |x|) \cdot d\mu = 0$, so that by [1], III.2, Theorem 20(d), $1 - |x| = 0$, μ -a.e.

Choose now an $\mathcal{A}$ -measurable step function y , such that $\|x - y\| < 2^{-1}$. Put

$$\begin{aligned} N &= \{s \in S: |x(s)| < 1\}, \\ M &= \{s \in S: y(s) > 0\}, \\ M_+ &= \{s \in S: x(s) = 1\} \quad \text{and} \\ M_- &= \{s \in S: x(s) = -1\}. \end{aligned}$$

Then

$$(iv) \quad M = M_+ \cup N \quad \text{and} \quad M = M_- \cup N.$$

Since $|\mu|^*(M) = 0$, for every $\varepsilon > 0$ there exists a set $N_\varepsilon \in \mathcal{A}$, such that $N_\varepsilon \supseteq N$ and $|\mu|(N_\varepsilon) < \varepsilon$. Then for each $A \in \mathcal{A}$

$$\mu(A \cap M) = \mu(A \cap M \cap N_\varepsilon) + \mu(A \cap M \cap CN_\varepsilon)$$

and

$$\mu(A \cap CM) = \mu(A \cap CM \cap N_\varepsilon) + \mu(A \cap CM \cap CN_\varepsilon).$$

But $|\mu|(A \cap M \cap N_\varepsilon) < \varepsilon$ and $|\mu|(A \cap CM \cap N_\varepsilon) < \varepsilon$. On the other hand, by (iv) and (ii)

$$\mu(A \cap M \cap CN_\varepsilon) = (x \cdot \mu)(A \cap M \cap CN_\varepsilon) = |\mu|(A \cap M \cap CN_\varepsilon) \geq 0,$$

and

$$\begin{aligned} \mu(A \cap CM \cap CN_\varepsilon) &= (-x \cdot \mu)(A \cap CM \cap CN_\varepsilon) = \\ &= -|\mu|(A \cap CM \cap CN_\varepsilon) \leq 0. \end{aligned}$$

For $\varepsilon \rightarrow 0$, one obtains $\mu(A \cap M) \geq 0$ and $\mu(A \cap CM) \leq 0$, so that (M, CM) is a Hahn decomposition of $(S, \mathcal{A}, \mu)$.

Remarks. 1) R. R. Phelps [2] has observed that $\mu \in \text{ba}(\mathbb{N})$ attains its norm on a step function from the unit ball of $l_\infty(\mathbb{N})$ if and only if μ admits a Hahn decomposition.

2) The author wishes to thank the referee for suggesting this short proof of the theorem.

References

- [1] N. DUNFORD and J. T. SCHWARTZ, Linear Operators, I. General Theory. New York-London 1958.
- [2] R. R. PHELPS, Some subreflexive Banach spaces. Arch. Math. **10**, 162–169 (1959).

Eingegangen am 26. 6. 1975*)

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*) Eine Neufassung ging am 18. 10. 1975 ein.