

Strictly positive linear functionals and Pareto subgradients of minimal range

By

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A mapping from a vector space X to an ordered vector space Y is called *convex*, if it satisfies the convexity inequality with respect to the ordering in Y . The extension of the results of the classical convex analysis to mappings of this kind, especially of those concerning the existence of subgradients for convex functionals, is possible only in the presence of some rather restrictive conditions on the ordering in Y (see e.g. [4] and [1]). This justifies the investigation of some weaker form of subdifferentiability even for convex mappings.

A linear operator A is a Pareto subgradient of the mapping F from X to Y if the subgradient inequality either holds, or the respective terms are not comparable in Y . (See condition (2) for C being the positive cone in Y without 0.)

The existence of Pareto subgradients of range of dimension ≤ 1 for a large class of mappings which includes the convex mappings depends on the existence of the *strictly positive linear functionals* on the ordered vector space Y , i.e., linear functionals being positive on all the positive cone in Y , except 0. Krein and Rutman have proved ([3], Theorem 2.1) that for any separable normed vector space ordered by a closed cone K with the property $K \cap (-K) = \{0\}$, there exist strictly positive linear functionals. For the Banach space $C(Q)$ of the continuous real valued functions defined on the compact Hausdorff space Q , ordered by the cone of non-negative functions, there exist strictly positive linear functionals if and only if Q is metrizable.

The presentation of the main result of the note in the general context in 1. was suggested by the referee's criticism (for which we express our thanks).

1. All the vector spaces which we shall consider are supposed to be over the reals.

Let Y denote a vector space. The subset C in Y will be called *weakly connected* if the set $\langle y', C \rangle = \{\langle y', y \rangle : y \in C\}$ is connected in R for any y' in the algebraic dual Y^* of Y .

Let X be a vector space, U a subset in X and let F be a mapping from U to the vector space Y . Then F is said to be *weakly subdifferentiable* at z in U with respect to the subset W in Y^* , if each functional $f = y'F$ with y' in W is *subdifferentiable* at z , i.e., there exists for each y' in W an element x' in X^* such that

$$(1) \quad f(z+x) - f(z) \geq \langle x', x \rangle$$

for any x in $U - z$. The element x' with this property will be said to be a *subgradient* of f at z .

If F is a linear operator from X to Y , then it is obviously weakly subdifferentiable at any point of X and with respect to any W in Y^* . In this case the relation (1) with $f = y'F$ ($y' \in W$) holds with equality for $x' = F'y'$, where F' denotes the adjoint of the linear operator F .

Let F be a mapping from the subset U in X to Y and let C be a given subset of Y . The linear operator $A: X \rightarrow Y$ is said to be a *Pareto subgradient* of F at z in U with respect to C (abbreviated, a *PC-subgradient*), if

$$(2) \quad Ax - G(x) \notin C$$

for any x in $U - z$, where $G(x) = F(z + x) - F(z)$.

Set $C^+ = \{y' \in Y^* : \langle y', y \rangle \geq 0 \text{ for any } y \text{ in } C\}$.

Proposition 1. Assume Y is a given vector space and C is a nonempty subset of it which is weakly connected. Then each mapping F from an arbitrary subset U of an arbitrary vector space X , which is weakly subdifferentiable with respect to C^* at z , has a *PC-subgradient* A at z with range of dimension ≤ 1 , if and only if there exists an y' in C^+ such that

$$(3) \quad \langle y', y \rangle > 0 \text{ for any } y \text{ in } C.$$

Proof. To prove the necessity, assume that F is the identity mapping of Y and $z = 0$. F is weakly subdifferentiable at 0 because it is linear. If A is a *PC-subgradient* of F at 0 with range of dimension ≤ 1 , then it has the representation $Ay = \langle y', y \rangle y_0$ with y_0 an element in Y , y' an element in Y^* . If $\langle y', y \rangle = 0$, then $A(-y) = 0$ and from (2) it follows that $y \notin C$. Therefore $\langle y', y \rangle \neq 0$ for any y in C . Because C is weakly connected it follows that y' or $-y'$ has the property (3). This proves the necessity.

In order to prove the sufficiency, assume that y' satisfies (3). Then y' is in C^+ , hence $y'F$ is subdifferentiable at z , i.e., there exists z' in X^* with the property that

$$\langle y', G(x) \rangle \geq \langle z', x \rangle$$

for any x in $U - z$. Choose v in Y such that $\langle y', v \rangle = 1$, which is possible by (3), and set

$$Ax = \langle z', x \rangle v.$$

Then for any x in $U - z$,

$$\langle y', Ax \rangle = \langle z', x \rangle \langle y', v \rangle = \langle z', x \rangle \leq \langle y', G(x) \rangle,$$

i.e., $\langle y', Ax - G(x) \rangle \leq 0$ and therefore $Ax - G(x) \notin C$.

2. Proposition 1 admits generalizations to both necessary and sufficient sense, but they cannot be related with the classical problem of the existence of strictly positive linear functionals in the theory of ordered spaces (see the comments at the beginning

of the note concerning strictly positive linear functionals on the ordered vector space Y , and compare with Proposition 1 for $C = K \setminus \{0\}$, where K is the positive cone in Y). Using the idea of the above proof it can be checked the

Proposition 2. *Let C be a nonempty convex set of the vector space Y and let $y'_1, \dots, y'_n$ be n linearly independent elements of Y^* . Then each mapping F from an arbitrary subset U of an arbitrary vector space X , which is weakly subdifferentiable at z in U with respect to the set $y'_1, \dots, y'_n$ has a PC -subgradient A at z with range of dimension $\leq n$, if and only if*

$$\left(\bigcap_{i=1}^n \ker y'_i \right) \cap C = \emptyset.$$

3. In order to avoid the explicit requirement of the weak subdifferentiability of the mapping F in the above propositions, we can impose some conditions about F and its domain of definition, which imply this property. The mapping F from the convex subset U of the vector space X to the vector space Y is said to be *weakly convex with respect to the set W* in Y^* , if each functional of form $y'F$ with y' in W is convex. We have then the

Corollary 1. *Let Y be a given vector space and C a weakly connected, nonempty subset of it. Then each mapping F from the convex subset U of the vector space X , which is weakly convex with respect to C^+ has a PC -subgradient in any point of the intrinsic core of U , having range of dimension ≤ 1 if and only if there exists y' in C^+ which is strictly positive on C .*

We observe that the conditions in the corollary are so chosen to ensure the weak subdifferentiability of F in the points of the intrinsic core of its domain (this follows for instance as particular case from [1]).

If F is a convex mapping with values in the ordered space Y , then F is weakly convex with respect to K^+ , where K is the positive cone in Y .

4. If we change in all the definitions vector spaces in topological vector spaces, algebraic duals in topological ones and linear operators in linear and continuous operators, then we have the results stated by Proposition 1 and 2 also for this case. The result similar to that contained in Corollary 1 is the following:

Corollary 2. *Let Y be a given topological vector space, C a nonempty weakly connected subset of Y . Then any mapping F which is continuous in the C^+ -topology of Y and is weakly convex with respect to C^+ , from the convex subset U of the topological vector space X , has a PC -subgradient in each interior point of U , having range of dimension ≤ 1 if and only if there exists y' in C^+ which is strictly positive on C .*

Via the results in [2] the conditions in the corollary imply the weak subdifferentiability of F with respect to C^+ in the interior points of U .

References

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