

THE ORDER OF STARLIKENESS OF CERTAIN INTEGRAL OPERATORS

by

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1. Introduction. Let α be a real number, $\alpha < 1$, and denote by $S^*(\alpha)$ the class of functions $f(z) = z + a_2 \cdot z^2 + \dots$, which are regular in the unit disc $U = \{ |z| < 1 \}$ and satisfy

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > \alpha, \quad z \in U.$$

If $\alpha \in [0, 1]$, then $S^*(\alpha)$ is the well-known class of starlike functions of order α and $S^*(\alpha) \subset S^*(0) = S^*$, the class of starlike functions. We also are interested in the case $\alpha < 0$, and continue to call "starlike of order α " any function in $S^*(\alpha)$, although it is not necessarily starlike, not even univalent.

Let β and γ be real numbers and suppose $\beta > 0$, $\beta + \gamma > 0$ and $-\frac{\gamma}{\beta} \leq \alpha < 1$. From a more general result on Briot - Bouquet differential subordination [1, Theorem 1] it is easy to deduce that the integral operator $I_{\beta, \gamma}$ defined by $g = I_{\beta, \gamma}(f)$, where

$$(1) \quad g(z) = \left(\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(w) w^{\gamma-1} dw \right)^{1/\beta}, \quad z \in U, \quad f \in S^*(\alpha),$$

maps $S^*(\alpha)$ into $S^*(\alpha)$, i.e. $I_{\beta, \gamma} : S^*(\alpha) \rightarrow S^*(\alpha)$.

For a given $\alpha \in [-\frac{\gamma}{\beta}, 1)$ we define the order of starlikeness of the class $I_{\beta, \gamma}(S^*(\alpha))$ by the largest number $\delta = \delta(\alpha; \beta, \gamma)$ such that $I_{\beta, \gamma}(S^*(\alpha)) \subset S^*(\delta)$.

In this paper we find $\delta(\alpha; \beta, \gamma)$, for appropriate choices of α , β and γ , by using the sharp subordination result recently obtained in [1] and [3], as well as the useful lemma due to J. FENG and D.R. WILKEN [2]. Our general result includes some particular ones obtained by several authors [2, 4, 5, 6, 7, 8, 9].

2. Preliminaries. Let f and g be regular in U . We say that f is subordinate to g , written $f \prec g$, or $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

We will make use of the following lemmas to prove our main result.

Lemma A. Let $\beta > 0$, $\beta + \gamma > 0$ and $-\gamma/\beta \leq \alpha < 1$. Then the differential equation

$$q(z) + \frac{z q'(z)}{\beta q(z) + \gamma} = \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad q(0) = 1,$$

has a univalent solution in U , given by

$$(2) \quad q(z) = \frac{1}{\beta q(z)} - \frac{\gamma}{\beta}$$

where

$$(3) \quad q(z) = \int_0^1 \left(\frac{1-z}{1-tz} \right)^{2\beta(1-\alpha)} t^{\beta+\gamma-1} dt, \quad z \in U,$$

and $q(z) \prec [1 + (1 - 2\alpha)z]/(1 - z)$.

If $p(z) = 1 + p_1 z + \dots$ is regular in U and satisfies the differential subordination

$$(4) \quad p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} \prec \frac{1 + (1 - 2\alpha)z}{1 - z}$$

then $p(z) \prec q(z)$ and this subordination is sharp.

More general forms of this lemma may found in [1, Corollary 2.1] and [3, Corollary 1.1].

L e m m a B. Let $d\mu(t)$ be a positive measure on $[0, 1]$ and let $Q(z, t)$ be a complex-valued function defined on $U \times [0, 1]$ such that $Q(z, \cdot)$ is integrable on $[0, 1]$ for each $z \in U$. Suppose that $\operatorname{Re} Q(z, t) > 0$ in U , $Q(-r, t)$ is real and

$$\operatorname{Re} \frac{1}{Q(z, t)} \geq \frac{1}{Q(-r, t)} .$$

for $|z| \leq r < 1$ and $t \in [0, 1]$.

If

$$Q(z) = \int_0^1 Q(z, t) d\mu(t) ,$$

then

$$\operatorname{Re} \frac{1}{Q(z)} \geq \frac{1}{Q(-r)} , \text{ for } |z| \leq r.$$

This lemma is due to J. FENG and D.R. WILKEN [2, Lemma 2].

3. Main result.

THEOREM. Let $\beta > 0$, $\beta + \gamma > 0$ and consider the integral o-
perator $I_{\beta, \gamma}$ defined by (1). If $\alpha \in [-\gamma/\beta, 1)$ then the or-
der of starlikeness of the class $I_{\beta, \gamma}(S^*(\alpha))$ is given by

$$(5) \quad \delta(\alpha; \beta, \gamma) = \inf_{|z| < 1} \operatorname{Re} q(z).$$

Moreover if $\alpha \in [\alpha_0, 1)$, where

$$(6) \quad \alpha_0 = \max \left\{ \frac{\beta - \gamma - 1}{2\beta}, -\frac{\gamma}{\beta} \right\}$$

and $g = I_{\beta, \gamma}(f)$, for $f \in S^*(\alpha)$, then

$$(7) \quad \operatorname{Re} \frac{zg'(z)}{g(z)} \geq q(-r) = \frac{1}{\beta} \left[\frac{\beta + \gamma}{F(1, 2\beta(1-\alpha), \beta + \gamma + 1; r/(1+r))} - \gamma \right],$$

for $|z| \leq r < 1$ and

$$(8) \quad \delta(\alpha; \beta, \gamma) = q(-1) = \frac{1}{\beta} \left[\frac{\beta + \gamma}{F(1, 2\beta(1-\alpha), \beta + \gamma + 1; 1/2)} - \gamma \right],$$

where q is given by (2) and $F(a, b, c; z)$ is the hypergeometric function. The extremal function is given by $g = I_{\beta, \gamma}(k)$, where

$$k(z) = z(1-z)^{2(\alpha-1)}.$$

Proof. If we let $p(z) = \frac{zg'(z)}{g(z)}$ from (1) we obtain

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{zf'(z)}{f(z)}.$$

Since $f \in S^*(\alpha)$ is equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1-2\alpha)z}{1-z},$$

we deduce that $p(z)$ satisfies the differential subordination (4) and hence, by lemma A, $p(z) \prec q(z)$ which implies (5).

Next we shall use the following well-known formulas

$$(9) \quad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b, c; z),$$

$c > b > 0$, and

$$(10) \quad F(a, b, c; z) = (1-z)^{-a} F(a, c-b, c; \frac{z}{z-1}),$$

which hold for all $z \in \mathbb{C} \setminus (1, \infty)$.

Suppose $\alpha \in (\alpha_0, 1)$, where α_0 is given by (6) and denote $a = 2\beta(1-\alpha)$, $b = \beta + \gamma$ and $c = \beta + \gamma + 1 = b + 1$. Since $c > b > 0$, from (3), by using (9) and (10), we deduce

$$\begin{aligned} Q(z) &= (1-z)^a \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b, c; z) = \frac{1}{\beta + \gamma} F(a, c-b, c; \frac{z}{z-1}) = \\ &= \frac{1}{\beta + \gamma} F(1, a, c; \frac{z}{z-1}), \quad z \in U. \end{aligned}$$

Since $\alpha > \alpha_0$ implies $c > a$, by using again (9), we get

$$Q(z) = \int_0^1 \frac{1-z}{1-(1-t)z} d\mu(t), \quad z \in U,$$

where

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1} (1-t)^{c-a-1} dt.$$

If we let $Q(z, t) = (1-z)/(1-(1-t)z)$, then $\operatorname{Re} Q(z, t) > 0$, $Q(-r, t)$ is real and

$$\operatorname{Re} \frac{1}{Q(z, t)} = \operatorname{Re} \frac{1-(1-t)z}{1-z} \geq \frac{1+(1-t)r}{1+r} = \frac{1}{Q(-r, t)},$$

for $|z| \leq r < 1$, $0 \leq t \leq 1$. By lemma B we have

$$\operatorname{Re} \frac{1}{Q(z)} \geq \frac{1}{Q(-r)},$$

which implies (7). If we take $g = I_{\beta, \gamma}(k)$, then a simple computation yield $p = q$, which shows that (7) is sharp. In the case $\alpha = \alpha_0$, we obtain (7), by letting $\alpha \rightarrow \alpha_0^+$. From (7) we obtain (8), by letting $r \rightarrow 1^-$. This completes the proof of our theorem.

4. Particular cases. 1) Let $\beta > 0$ and $\beta + \gamma \geq 1$. In this case $\alpha_0 = (\beta - \gamma - 1)/2\beta$ and if we take $\alpha = \alpha_0$, i.e. $c = a$, by using $F(1, a, a; z) = (1 - z)^{-1}$, we easily get $q(z) = (\beta + \gamma z)/\beta(1 - z)$ and

$$\left(\frac{\beta - \gamma - 1}{2\beta}; \beta, \gamma\right) = \frac{\beta - \gamma}{2\beta},$$

which shows that, if $\beta \geq \gamma \geq 1 - \beta$, the integral operator $I_{\beta, \gamma}$ maps each starlike function of order $(\beta - \gamma - 1)/2\beta$ into a starlike (univalent) function. This result is sharp if $\beta = \gamma \geq 1/2$. For instance, if $\beta = \gamma = 1$, we deduce that the Libera integral operator maps starlike functions of order $-1/2$ into starlike functions.

2) Let $\beta > 0$ and $\beta + \gamma \leq 1$. In this case $\alpha_0 = -\gamma/\beta$ and we have

$$(-\gamma/\beta; \beta, \gamma) = \frac{1}{\beta\sqrt{\pi}} \frac{\Gamma(\beta + \gamma + 1/2)}{\Gamma(\beta + \gamma)} - \gamma/\beta.$$

If in this last formula we take $\gamma = 0$ and $\beta = 1/\alpha \leq 1$, we get

$$\delta(0; 1/\alpha, 0) = \frac{\Gamma(1/\alpha + 1/2)}{\sqrt{\pi} \Gamma(1/\alpha + 1)},$$

which is the order of starlikeness of α -convex functions, for $\alpha \geq 1$, [4]. We note that $\delta(0; 1/\alpha, 0) = 0 \neq q(-1)$, for $\alpha \in [0, 1)$.

3) Let $\beta > 0$ and $\beta + \gamma = 1$. In this case $\alpha_0 = 1 - 1/\beta$ and for $\alpha \in [1 - 1/\beta, 1)$ we have

$$\delta(\alpha; \beta, 1-\beta) = \begin{cases} \frac{1}{\beta} \left[\frac{1 - 2\beta(1-\alpha)}{2 - 2^{2\beta(1-\alpha)}} + \beta - 1 \right], & \alpha \neq 1 - 1/2\beta \\ \frac{1}{\beta} \left[\frac{1}{2 \ln 2} + \beta - 1 \right], & \alpha = 1 - 1/2\beta. \end{cases}$$

If in this last formula we take $\beta = 1$, we obtain the order of starlikeness of convex functions of order α , for $\alpha \in [0, 1)$, [2], [9]. If we take $\beta = 1/2$, then, for $\alpha \in [-1, 1)$, we have

$$\delta(\alpha; 1/2, 1/2) = \begin{cases} \alpha/(1 - 2^{-\alpha}) - 1, & \alpha \neq 0 \\ 1/(\ln 2) - 1, & \alpha = 0. \end{cases}$$

4) Let $\beta > 0$ and $\beta + \gamma = 2$. In this case $\alpha_0 = 1 - 3/(2\beta)$, and for $\alpha \in [1 - 3/(2\beta), 1)$ we obtain

$$\delta(\alpha; \beta, 2-\beta) = \begin{cases} \frac{1}{\beta} \left[\frac{(1-2\beta(1-\alpha))(1-\beta(1-\alpha))}{2^{2\beta(1-\alpha)} - 1} + \beta - 2 \right], & \alpha \neq 1 - \frac{1}{2\beta}, \neq 1 - \frac{1}{\beta}, \\ \frac{1}{\beta} \left[\frac{1}{2(1-\ln 2)} + \beta - 2 \right], & \alpha = 1 - 1/(2\beta), \\ \frac{1}{\beta} \left[\frac{1}{2(2 \ln 2 - 1)} + \beta - 2 \right], & \alpha = 1 - 1/\beta. \end{cases}$$

If we take $\beta = 1$, we obtain the following result.

C o r o l l a r y. If $\alpha \in [-1/2, 1)$ then the order of star-
likeness of the class $L(S^{\#}(\alpha))$, where L is the Libera inte-
gral operator defined by

$$L(f)(z) = \frac{z}{2} \int_0^z f(w) dw,$$

is given by

$$\delta(\alpha; 1, 1) = \begin{cases} \frac{\alpha(2\alpha-1)}{2(2^{-2\alpha} + \alpha - 1)} - 1, & \alpha \neq 1/2, \alpha \neq 0 \\ \frac{1}{2(1-\ln 2)} - 1, & \alpha = 1/2 \\ \frac{1}{2(2 \ln 2 - 1)} - 1, & \alpha = 0. \end{cases}$$

We note that $\delta(-1/2; 1, 1) = 0$, but $\delta(-1; 1, 1) = -1 \neq q(-1) = -1/4$.

Remarks. 1) If A_β is the operator defined by $A_\beta(f)(z) = z (f(z)/z)^\beta$, then we have $I_{\beta, \gamma} = A_{1/\beta} \circ I_{1, \gamma + \beta - 1} \circ A_\beta$. Since A_β maps bijectively $S^*(\alpha)$ onto $S^*(\alpha\beta + 1 - \beta)$, we easily deduce the following relationship

$$\beta \delta(\alpha; \beta, \gamma) = \delta(\alpha\beta + 1 - \beta; 1, \beta + \gamma - 1) + \beta - 1,$$

which shows that for the problem involved in this paper it is sufficient to consider integral operators of the form $I_{1, \gamma}$, with $\gamma + 1 > 0$.

2) For some particular cases, in [5], [6] and [7] elementary proofs of the fact that $\min_{|z|=1} \operatorname{Re} q(z) = q(-1)$ were given. We

note that in all those cases it was proved that $z = -1$ is the only point of minimum.

REFERENCES

- [1] E n i g e n b u r g , P. , M i l l e r , S. S. ,
M o c a n u , P. T. , R e a d e , M. O. , On a Briot-
Bouquet differential subordination (submitted)
- [2] F e n g , J. , W i l k e n , D. R. , A remark on convex and
starlike functions, J. London Math. Soc., (2),
21 (1980), 287 - 290.
- [3] M i l l e r , S. S. , M o c a n u , P. T. , Univalent so-
lutions of Briot-Bouquet differential equations
(in preparation).
- [4] M i l l e r , S. S. , M o c a n u , P. T. , R e a d e , M.O.
The order of starlikeness of alpha-convex func-

- tions, *Mathematica (Cluj)*, 20 (43), 1 (1978), 25 - 30.
- [5] Mocanu, P. T., Reade, M. O., Ripeanu, D. The order of starlikeness of a Libera integral operator, *Mathematica (Cluj)*, 19 (42), 1 (1977), 67 - 73.
- [6] Mocanu, P. T., Ripeanu, D., The order of starlikeness of the Libera transform of the class of starlike functions of order $1/2$, *Mathematica (Cluj)* (to appear).
- [7] Mocanu, P. T., Ripeanu, D., Şerb, I., On the order of starlikeness of convex functions of order α (submitted).
- [8] Ruscheweyh, S., Singh, V., On a Briot-Bouquet equation related to univalent functions, *Rev. Roum. Math. Pures et Appl.* 24 (1979), 285 - 290.
- [9] Zmorovich, V. A., Korobkova, I. K., On the order of starlikeness of convex functions of order α , *Dopov. Akad. Nauk Ukr. SSR, ser. A*, 7 (1977), 584 - 587 (Russian).

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