

Monotonicity of metric projections onto positive cones of ordered Euclidean spaces

By

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0. Introduction. If X is a Hilbert space and $D \subset X$ a nonempty closed convex set, then it is well known (see e.g. [5], Theorem 2.1.2) that for any $u \in X$ there exists a unique element $\bar{u} \in D$ satisfying

$$\|\bar{u} - u\| = \min \{\|x - u\| : x \in D\}.$$

The element $\bar{u}$ is called the metric projection of u onto D and we denote $\bar{u} = \pi_D(u)$. A laborious study of π_D is the subject of the paper of Zarantanello [14]. In some problems in optimal control theory (see e.g. [12]) and in nonlinear complementarity problem (see e.g. [6], [2], [3] and [1]) a special interest is focused on π_K in the case when K is a closed convex cone. However, as far as we know, the literature lacks a deep study of the operator π_K . One of the first steps in this direction would be the investigation of monotonicity properties of π_K . This is the aim of the present note in the case of the n -dimensional real Euclidean space $\mathbb{R}^n$. Consider $\mathbb{R}^n$ to be ordered by the closed convex cone K . The metric projection π onto K will be said *isotone* if $u \leq v$ ($u, v \in \mathbb{R}^n$) implies $\pi(u) \leq \pi(v)$. We are interested in characterizing K in order to π be isotone. Surprisingly this condition imposes rather strong restrictions on K : it must have a special structure which implies in particular its miniedrality (or equivalently, the latticiality of the induced order). This circumstance as well as the feature of the proofs relate our note to the very classical results on finite dimensional vector lattices due to F. Riesz [9], A. Youdine [13] and B. Szökefalvi Nagy [11].

1. Prerequisites and the main result. Let E be a subset of a convex set A in the vector space $\mathbb{R}^n$. E is *extremal* subset of A if $x, y \in A$ and $tx + (1 - t)y \in E$ for some t ($0 < t < 1$) entails $x, y \in E$. An *extreme ray* of A is an extremal half line; we write the set of all extreme rays of A as $\text{rext}(A)$.

The subset E of A is called *exposed* if it is the intersection of A with a closed half space which contains no point of A in its interior. An exposed set is extremal. An *exposed ray* of A is an exposed half line in A .

The subset K of $\mathbb{R}^n$ is a *convex cone* if (i) $K + K \subset K$ and (ii) $tK \subset K$ whenever $t \in \mathbb{R}_+$. If we have in plus $K \cap (-K) = \{0\}$ then K is called *proper* (or *acute*). K is said *generating* if $K - K = \mathbb{R}^n$.

If K is a closed convex cone in $\mathbb{R}^n$ which is proper, then $K = \text{co}(\text{ext}(K))$, i.e., it is the convex hull of its extreme rays (see e.g. [4], I. §8).

If M is a subset of $\mathbb{R}^n$ we denote by $\text{cone}(M)$ the cone engendered by M , i.e., the set $\{t_1 x_1 + \dots + t_m x_m : x_1, \dots, x_m \in M, m \in \mathbb{N}, t_1, \dots, t_m \in \mathbb{R}_+\}$. We shall refer to $\text{cone}(M)$ as to the *cone hull* of M .

The cone K induces a reflexive and transitive order relation in $\mathbb{R}^n$ if we put $u \leq v$ whenever $v - u \in K$. The relation is translation invariant and invariant with respect to multiplication with scalars in $\mathbb{R}_+$. The order relation is antisymmetric if and only if K is proper. If K induces an order relation in $\mathbb{R}^n$, then it is called the *positive cone* of this *ordered vector space*.

The convex cone $K \subset \mathbb{R}^n$ is said *miniedral* when

$$K = \text{cone}\{x_1, \dots, x_n\}$$

with $x_1, \dots, x_n$ linearly independent elements in $\mathbb{R}^n$.

The convex cone K in $\mathbb{R}^n$ is miniedral if and only if the order relation induced by it in $\mathbb{R}^n$ is a latticial one (see [13] or [11]), that is, if every pair of elements in $\mathbb{R}^n$ has a greatest lower bound. If K is miniedral then it is proper and generating.

Denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$. The *polar* of the cone K is by definition the set

$$K^0 = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 0, \forall x \in K\}.$$

The set K^0 is a cone too. K is proper and generating if and only if K^0 has the same properties. If K is closed, then $(K^0)^0 = K$ (see e.g. [10] Theorem IV.1.5). If K is miniedral then K^0 is miniedral too. (This follows from [9] and [13]; see also [7].)

Definition. The closed proper and generating cone K in $\mathbb{R}^n$ will be called *thin* if for any two vectors u, v on two different rays of K^0 one has $\langle u, v \rangle \leq 0$.

Using the introduced notion our main result can be formulated now as follows:

Theorem. *The metric projection onto the closed proper and generating positive cone of an ordered Euclidean space is isotone if and only if the cone is thin.*

The main tool in the proofs is the theorem of Moreau which asserts [8]:

Let P and Q be two convex cones in $\mathbb{R}^n$ each being the polar cone of the other. Then the following properties (i) and (ii) are equivalent:

- (i) $z = x + y, \quad x \in P, \quad y \in Q, \quad \langle x, y \rangle = 0,$
- (ii) $x = \pi_P(z), \quad y = \pi_Q(z).$

Observe finally that the condition on K to be proper and generating is not an essential one. A result similar to that in above Theorem could be obtained for a more general case. We assumed a priori in definition to thin cones be of this kind since this offers some technical facilities in terminology and in proofs.

2. Preparatory results. To simplify the terminology in the proofs we shall use the term *edge* of the cone K for a vector with unity norm on an extreme ray of K .

We consider first a technical result. Although it is rather intuitive and likely known, we did not find it in the literature.

Lemma 1. Let $a_1, \dots, a_n$ be linearly independent elements of $\mathbb{R}^n$ satisfying $\langle a_i, a_j \rangle \leq 0$, $i \neq j$, $i, j = 1, \dots, n$. Then there cannot exist $b_1, b_2 \in \mathbb{R}^n$ with the properties $\langle b_1, b_2 \rangle < 0$ and $\langle a_i, b_j \rangle \leq 0$ for $i = 1, \dots, n$, $j = 1, 2$.

Proof. We shall use the following algebraic result: If the matrix $A = (\alpha_{ij})_{i,j=1,\dots,n}$ has the properties that $\alpha_{ii} > 0$, and $\alpha_{ij} \leq 0$ for $i \neq j$, $i, j = 1, \dots, n$, then the system of inequalities

$$Ax \geq 0$$

may have only solutions $x = (x_1, \dots, x_n)$ with $x_i \geq 0$, $i = 1, \dots, n$. This assertion can be checked easily by induction on n .

We prove the lemma by contradiction. Suppose that we have found b_1 and b_2 satisfying the conditions in the lemma. We see first that if

$$(1) \quad b_1 = \alpha_1 a_1 + \dots + \alpha_n a_n$$

then $\alpha_i \leq 0$, $i = 1, \dots, n$. Indeed, one has

$$-\langle a_i, b_1 \rangle = -\alpha_1 \langle a_i, a_1 \rangle - \dots - \alpha_n \langle a_i, a_n \rangle \geq 0$$

$i = 1, \dots, n$. Put $A = (\langle a_i, a_j \rangle)_{i,j=1,\dots,n}$ and use the above algebraic result to get the required conclusion.

Let us consider $K = \text{cone}\{a_1, \dots, a_n\}$. Then K is proper, convex and generating cone in $\mathbb{R}^n$. It follows that K^0 possesses similar properties and hence $\text{int } K^0 \neq \emptyset$. Now $b_1 \in K^0$ and we can move it easily into $\text{int } K^0$ preserving the conditions in the lemma. Thus we can assume that $b_1 \in \text{int } K^0$. Hence $\langle a_i, b_1 \rangle < 0$ for $i = 1, \dots, n$, wherefrom it follows that the coefficients α_i in the representation (1) are all negative. We shall show that

$$(2) \quad \text{cone}\{a_1, \dots, a_n, b_1\} = \mathbb{R}^n.$$

Indeed, let $x \in \mathbb{R}^n$ be arbitrary. Consider its representation $x = \sum_{i=1}^n \beta_i a_i$. Let λ be a positive number satisfying

$$\lambda \geq \frac{\max\{|\beta_i| : \beta_i < 0\}}{\min\{|\alpha_i| : i = 1, \dots, n\}}.$$

Then $\gamma_i \geq 0$ can be chosen so as to have $\gamma_i + \lambda \alpha_i = \beta_i$, $i = 1, \dots, n$. Hence

$$\begin{aligned} \gamma_1 a_1 + \dots + \gamma_n a_n + \lambda b_1 &= (\gamma_1 + \lambda \alpha_1) a_1 + \dots + (\gamma_n + \lambda \alpha_n) a_n \\ &= \beta_1 a_1 + \dots + \beta_n a_n = x \end{aligned}$$

wherefrom (2) follows. We have by (2) for some $\lambda_i \geq 0$:

$$b_2 = \lambda_1 a_1 + \dots + \lambda_n a_n + \lambda_{n+1} b_1.$$

Consider now the scalar product of this relation with b_2 to get a contradiction.

We summarize some properties of thin cones in the following lemma:

Lemma 2. Let K be a thin cone in $\mathbb{R}^n$. Then

- (i) K is miniedral;
- (ii) $-K \subset K^0$ or equivalently $\langle x, y \rangle \geq 0$ for each pair x, y of elements in K ;

- (iii) If $e_1, \dots, e_n$ denote the edges of K and $u_1, \dots, u_n$ denote the edges of K^0 then by an appropriate reindexation we can realise that $\langle e_i, u_j \rangle = 0$ if $i \neq j$, $i, j = 1, \dots, n$.
- (iv) Every face of a thin cone K is a thin cone in the subspace spanned by this face. If K_1 is a face, if $H = \text{sp } K_1$, then the polar K_1^0 in H of K_1 can be obtained as $p(K^0)$, where p is the orthogonal projection onto the subspace H .

Proof. (i) Since K^0 is the cone hull of its edges and since it is generator, it must contain n linearly independent edges, say $u_1, \dots, u_n$. Let u be an arbitrary edge of K^0 . If $u \notin \{u_1, \dots, u_n\}$ one would have $\langle u, u_j \rangle \leq 0$, $j = 1, \dots, n$, $\langle u_i, u_j \rangle \leq 0$, $i \neq j$, $i, j = 1, \dots, n$ since K is thin. Consider $a \in \text{int } K$. Then $\langle u, a \rangle < 0$ and $\langle a, u_j \rangle < 0$, $j = 1, \dots, n$. This contradicts Lemma 1. In conclusion,

$$(3) \quad K^0 = \text{cone } \{u_1, \dots, u_n\},$$

that is, K^0 is miniedral. Combine this with the results in [13], [11], and [9] to conclude that K is also miniedral and hence has the representation

$$(4) \quad K = \text{cone } \{e_1, \dots, e_n\}$$

with $e_1, \dots, e_n$ linearly independent edges.

(ii) The assertion follows directly from (i) and from Lemma 1.

(iii) Let us consider the representations (3) and (4). We have to prove indeed that for an arbitrary i there exists a single j such that $\langle e_i, u_j \rangle \neq 0$. This follows from the fact that

$$K^0 = \bigcap_{i=1}^n H_i^-$$

with $H_i^- = \{x \in \mathbb{R}^n: \langle e_i, x \rangle \leq 0\}$. Hence $K \cap H_i$ with $H_i = \{x \in \mathbb{R}^n: \langle e_i, x \rangle = 0\}$ is an $(n-1)$ -dimensional face of K^0 and hence must contain exactly $n-1$ edges of K .

(iv) We shall verify the assertion for $(n-1)$ -dimensional faces of K only, wherefrom the general case will follow inductively using standard properties of projection operators. Let us consider the representations (3) and (4) which satisfy the conditions in (iii). Denote by K_1 the face cone $\{e_1, \dots, e_{n-1}\}$. Then $\text{sp } K_1 = H = \{x \in \mathbb{R}^n: \langle u_n, x \rangle = 0\}$. Let p be the orthogonal projection onto H . Then we have

$$v_j = p(u_j) = u_j - \langle u_j, u_n \rangle u_n, \quad j = 1, \dots, n-1.$$

Observe that $\langle e_i, v_j \rangle = 0$, $\langle e_i, v_i \rangle < 0$ and $\langle v_i, v_j \rangle \leq 0$ for $i \neq j$, $i, j = 1, \dots, n-1$. In conclusion

$$K_1^0 = \text{cone } \{v_1, \dots, v_{n-1}\},$$

K_1 is a thin cone in H and $K_1^0 = p(K^0)$.

3. Proof of the necessity. (a) Let K be a closed generating cone in $\mathbb{R}^n$ and let K^0 be its polar. We show that if b is an exposed edge of K^0 (i.e., a unit vector on an exposed ray), then there exists an element a of K such that $\langle a, b \rangle = 0$ and $\langle a, x \rangle < 0$ for every x in K^0 which is not a multiple of b .

Consider a supporting hyperplane H of K^0 which contains the single edge b of K^0 . Let H be given by

$$H = \{x \in \mathbb{R}^n: \langle a, x \rangle = 0, \|a\| = 1\}$$

and choose the sign of a so as to have $\langle a, x \rangle \leq 0$ for all x in K^0 . Then $a \in (K^0)^0 = K$ will satisfy the requirement.

(b) We see next that if b_1, b_2 are two different exposed edges in K^0 and a_i is chosen in K to satisfy requirement (a) with respect to b_i ($i = 1, 2$), then $\lambda a_2 - a_1 \notin K$ for any positive λ . Assume the contrary: $z = \lambda a_2 - a_1 \in K$ for some $\lambda > 0$. Then by the definition of a_2

$$\langle z, b_2 \rangle = \langle \lambda a_2 - a_1, b_2 \rangle = -\langle a_1, b_2 \rangle.$$

Since $z, a_1 \in K$, $b_2 \in K^0$ we have $\langle z, b_2 \rangle \leq 0$, $\langle a_1, b_2 \rangle \leq 0$. Hence $\langle a_1, b_2 \rangle = 0$ in contradiction with the mode of definition of a_1 .

(c) Assume that there are two edges of K^0 with positive scalar product. By Straszewicz's result in [15] they can be arbitrarily approximated by exposed edges, hence we can suppose that there are two exposed edges b_1, b_2 in K^0 with $\langle b_1, b_2 \rangle > 0$.

Let a_i be the element of K with the property stated at (a) with respect to b_i ($i = 1, 2$). We shall show that $\lambda > 0$ and $\mu > 0$ can be chosen so as to have

$$(5) \quad \lambda a_2 - a_1 - \mu b_1 \in K.$$

Consider the set $B = \partial K^0 \cap S$ where $S = \{x \in \mathbb{R}^n: \|x\| = 1\}$. B is obviously compact. To verify our assertion it suffices to see that $\lambda > 0$ and $\mu > 0$ can be chosen such that

$$\langle \lambda a_2 - a_1 - \mu b_1, b \rangle \leq 0, \quad \forall b \in B.$$

Put $\varepsilon = \frac{1}{2}\langle b_1, b_2 \rangle$. Consider the set

$$(6) \quad B_2 = \{b \in B: \langle b_1, b \rangle > \varepsilon\}.$$

Then B_2 is an open neighbourhood of b_2 in B . Thus $B_1 = B \setminus B_2$ is compact. We have by the definition of a_2 that $\langle a_2, b \rangle < 0$ whenever $b \in B_1$. Hence there exists a $\delta > 0$ such that

$$(7) \quad \langle a_2, b \rangle \leq -\delta \quad \text{for each } b \in B_1.$$

Denote $\alpha = \max \{-\langle a_1, b \rangle: b \in B\}$, choose $\mu > 0$ such that

$$(8) \quad \mu\varepsilon \geq \alpha$$

and determine $\lambda > 0$ so as to have

$$(9) \quad \lambda\delta \geq \alpha + \mu.$$

Suppose that $b \in B_1$. We have by (7) and (9):

$$\lambda\langle a_2, b \rangle - \langle a_1, b \rangle - \mu\langle b_1, b \rangle \leq -\lambda\delta + \alpha + \mu \leq 0.$$

If $b \in B_2$, according to (6) and (8) one has:

$$\begin{aligned} \lambda\langle a_2, b \rangle - \langle a_1, b \rangle - \mu\langle b_1, b \rangle &\leq -\langle a_1, b \rangle - \mu\langle b_1, b \rangle \\ &\leq \alpha - \mu\varepsilon \leq 0. \end{aligned}$$

In conclusion,

$$\langle \lambda a_2 - a_1 - \mu b_1, b \rangle = \lambda \langle a_2, b \rangle - \langle a_1, b \rangle - \mu \langle b_1, b \rangle \leq 0$$

for each b in B . This proves (5).

Put $z_2 = \lambda a_2$ and $z_1 = a_1 + \mu b_1$. We have according to the above conclusion that $z_2 - z_1 \in K$. Further, by the theorem of Moreau,

$$\pi(a_1 + \mu b_1) = a_1$$

with π the metric projection onto K . Hence

$$\pi(z_2) - \pi(z_1) = \lambda a_2 - a_1.$$

But $\lambda a_2 - a_1 \notin K$ according to the assertion at (b). Thus the metric projection onto K is not isotone and the necessity is proved.

4. Proof of the sufficiency. We proceed by induction with respect to the dimension n .

For $n = 1$ we have nothing to prove. Suppose the isotony of the metric projections onto thin cones holds for $\mathbb{R}^k$ with $k < n$.

Let K be a thin cone in $\mathbb{R}^n$. According (iii) in Lemma 2 we can write

$$K = \text{cone}\{e_1, \dots, e_n\}; \quad K^0 = \text{cone}\{u_1, \dots, u_n\}$$

where $e_1, \dots, e_n; u_1, \dots, u_n$ are edges (vectors of unity norms on extreme rays) with $\langle e_i, u_j \rangle = 0$ if $i \neq j, i, j = 1, \dots, n$.

The proof of the induction step will be done by contradiction. Assume that z_1 and z_2 are elements of $\mathbb{R}^n$ with the properties:

$$(10) \quad z_2 - z_1 \in K, \quad \pi(z_2) - \pi(z_1) \notin K,$$

where π denotes the metric projection onto K . According to the theorem of Moreau, one has

$$(11) \quad z_i = a_i + b_i \quad \text{with} \quad a_i = \pi(z_i), \quad b_i \in K^0, \quad \langle a_i, b_i \rangle = 0, \quad i = 1, 2.$$

That is, we have

$$z_2 - z_1 = a_2 - a_1 + b_2 - b_1.$$

(a) Consider first the case $b_2 = 0$. Because

$$a_2 - a_1 = \pi(z_2) - \pi(z_1) \notin K,$$

there must exist an index, say $j_0, 1 \leq j_0 \leq n$, such that

$$(12) \quad \langle a_2 - a_1, u_{j_0} \rangle > 0.$$

Since $z_2 - z_1 \in K$, it holds

$$\langle a_2 - a_1 - b_1, u_{j_0} \rangle = \langle a_2 - a_1, u_{j_0} \rangle - \langle b_1, u_{j_0} \rangle \leq 0.$$

Combine this relation with (12) to deduce

$$\langle b_1, u_{j_0} \rangle > 0.$$

Consider the representation of b_1 in cone $\{u_1, \dots, u_n\}$ given by

$$b_1 = \beta_1 u_1 + \dots + \beta_n u_n,$$

where $\beta_i \geq 0$, $i = 1, \dots, n$. From the above deduced inequality we must have $\beta_{j_0} > 0$, since $\langle u_i, u_{j_0} \rangle \leq 0$ if $i \neq j_0$ by hypothesis. Let us consider the representation of a_1 in cone $\{e_1, \dots, e_n\}$ given by

$$a_1 = \alpha_1 e_1 + \dots + \alpha_n e_n,$$

with $\alpha_i \geq 0$, $i = 1, \dots, n$. From the orthogonality $\langle a_1, b_1 \rangle = 0$ and the mode of indexing the edges e_i and u_j it follows then that $\alpha_{j_0} = 0$. Thus we have $\langle a_1, u_{j_0} \rangle = 0$. Since $\langle a_2, u_{j_0} \rangle \leq 0$, we get a contradiction with (12).

(b) Suppose now that $b_2 \neq 0$. We shall show that from the conditions on z_1 and z_2 it follows that there exists an index j_0 , $0 \leq j_0 \leq n$ such that

$$(13) \quad \langle a_1, u_{j_0} \rangle = \langle a_2, u_{j_0} \rangle = 0.$$

Assume the contrary. Then a_1 and a_2 can lie in the same $(n-1)$ -dimensional face cone $\{e_1, \dots, e_{j_0-1}, e_{j_0+1}, \dots, e_n\}$ for no $j_0 \in \{1, \dots, n\}$. Let us consider the representation in cone $\{e_1, \dots, e_n\}$ of a_i given by

$$a_i = \alpha_1^i e_1 + \dots + \alpha_n^i e_n, \quad (i = 1, 2).$$

Then, according our assumption, $\alpha_j^1 + \alpha_j^2 > 0$ for $j = 1, \dots, n$. Consider the representation of b_i in cone $\{u_1, \dots, u_n\}$ given by

$$b_i = \beta_1^i u_1 + \dots + \beta_n^i u_n, \quad (i = 1, 2).$$

By the orthogonality conditions $\langle a_i, b_i \rangle = 0$, $i = 1, 2$ and the mode of the indexing the edges e_i and u_j , taking into account the relation on α_j^1 and α_j^2 ($j = 1, \dots, n$), we must have $\beta_j^1 \beta_j^2 = 0$ for $j = 1, \dots, n$. (If $\beta_j^1 \neq 0$ and $\beta_j^2 \neq 0$ one would have $\alpha_j^1 = 0$, $\alpha_j^2 = 0$.) Therefore

$$(14) \quad \langle b_1, b_2 \rangle = \sum_{i,j=1}^n \beta_i^1 \beta_j^2 \langle u_i, u_j \rangle \leq 0$$

since K is thin. Because $a_2 - a_1 + b_2 - b_1 \in K$ we have

$$\langle a_2 - a_1 + b_2 - b_1, b_2 \rangle \leq 0.$$

From the other hand, using the orthogonality of a_2 and b_2 , the hypothesis that $b_2 \neq 0$ and the relation (14) one deduces

$$\langle a_2 - a_1 + b_2 - b_1, b_2 \rangle = -\langle a_1, b_2 \rangle + \langle b_2, b_2 \rangle - \langle b_1, b_2 \rangle > 0.$$

The obtained contradiction proves (13).

(c) If z_1 and z_2 satisfies the conditions (10) and $b_2 \neq 0$ then there exist the elements z'_1 and z'_2 which satisfies the properties of type (10), having representations of type (11) such that for some index j_0 , $1 \leq j_0 \leq n$ there hold

$$\langle a'_1, u_{j_0} \rangle = \langle a'_2, u_{j_0} \rangle = 0$$

and

$$\langle b'_1, u_{j_0} \rangle = \langle b'_2, u_{j_0} \rangle.$$

To verify this assertion we observe that according (b) we can assume that (13) holds for some j_0 , say $j_0 = n$. Since $z_2 - z_1 \in K$ one has

$$\langle a_2 - a_1 + b_2 - b_1, u_n \rangle = \langle b_2, u_n \rangle - \langle b_1, u_n \rangle \leq 0.$$

If $\langle b_2, u_n \rangle = \langle b_1, u_n \rangle$ we are done. Let $\langle b_2, u_n \rangle < \langle b_1, u_n \rangle$ and let us consider the element

$$(15) \quad b'_2 = b_2 + \beta u_n$$

with $\beta = \langle b_1, u_n \rangle - \langle b_2, u_n \rangle > 0$. Then $b'_2 \in K^0$, and $\langle b'_2, a_2 \rangle = 0$ since $\langle b_2, a_2 \rangle = 0$ and (13) (remind that we have put $j_0 = n$). We have also

$$\begin{aligned} \langle b'_2, u_n \rangle &= \langle b_2, u_n \rangle + \beta \langle u_n, u_n \rangle = \langle b_2, u_n \rangle + \beta \\ &= \langle b_2, u_n \rangle + \langle b_1, u_n \rangle - \langle b_2, u_n \rangle = \langle b_1, u_n \rangle. \end{aligned}$$

Put $a'_1 = a_1$, $a'_2 = a_2$, $b'_1 = b_1$ and let b'_2 given by (15). Denote $z'_i = a'_i + b'_i$, $i = 1, 2$. We have $a'_i \in K$, $b'_i \in K^0$, $i = 1, 2$. Further,

$$\langle z'_2 - z'_1, u_n \rangle = 0,$$

and for arbitrary $j \leq n - 1$,

$$\langle z'_2 - z'_1, u_j \rangle = \langle z_2 - z_1, u_j \rangle + \beta \langle u_n, u_j \rangle \leq 0$$

since $z_2 - z_1 \in K$, $\beta > 0$ and $\langle u_n, u_j \rangle \leq 0$. Accordingly one hold

$$z'_2 - z'_1 \in K,$$

$$\langle a'_1, u_n \rangle = \langle a'_2, u_n \rangle = 0,$$

and

$$\langle b'_2, u_n \rangle = \langle b'_1, u_n \rangle.$$

Since $\pi(z'_2) - \pi(z'_1) = a'_2 - a'_1 = a_2 - a_1 \notin K$, the assertion (c) was proved.

(d) Suppose now that z_1, z_2 are elements in $\mathbb{R}^n$ with the properties (10) for which according assertion (c) we can suppose that hold

$$\langle a_1, u_n \rangle = \langle a_2, u_n \rangle = 0, \quad \langle b_1, u_n \rangle = \langle b_2, u_n \rangle = \gamma.$$

Denote $H = \{x \in \mathbb{R}^n: \langle u_n, x \rangle = 0\}$ and $K_1 = K \cap H = \text{cone}\{e_1, \dots, e_{n-1}\}$. Let p stand for the orthogonal projection onto H . Consider the elements

$$z'_i = p(z_i) = a_i + b_i - \langle b_i, u_n \rangle u_n = a_i + b_i - \gamma u_n$$

and put $z'_i = a'_i + b'_i$ with

$$a'_i = a_i, \quad b'_i = b_i - \gamma u_n, \quad i = 1, 2.$$

Since $b'_i = p(b_i)$, we have according (iv) of Lemma 2, $b'_i \in K_1^0$, $i = 1, 2$, where K_1^0 is the polar of K_1 in the subspace H . We have further

$$\langle a'_i, b'_i \rangle = \langle a_i, b_i - \gamma u_n \rangle = \langle a_i, b_i \rangle - \gamma \langle a_i, u_n \rangle = 0, \quad i = 1, 2.$$

Denote the metric projection onto K_1 in H by π' . Then from the above conclusions using the theorem of Moreau one has $\pi'(z'_i) = a'_i = a_i$, $i = 1, 2$. Hence $\pi'(z'_2) - \pi'(z'_1) = a_2 - a_1 \notin K_1$. But $z'_2 - z'_1 = z_2 - z_1 \in K$ and since $z'_2 - z'_1$ is in H , we must have $z'_2 - z'_1 \in K_1$.

We have got two elements z'_1 and z'_2 in H with $z'_2 - z'_1 \in K_1$ and $\pi'(z'_2) - \pi'(z'_1) \notin K_1$. This contradicts the induction hypothesis since K_1 is thin cone in the $(n - 1)$ -dimensional space H according (iv) in Lemma 2.

This completes the proof of sufficiency.

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