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BEST BOUND FOR THE ARGUMENT OF CERTAIN ANALYTIC FUNCTIONS WITH POSITIVE REAL PART

by

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1. In a recent paper [3] the first author proved that if p is an analytic function in the unit disc U, which satisfies $p(0) = 1$ and

$$\text{(1)} \qquad \operatorname{Re}[zp'(z) + p(z)] > 0 \ , \quad z \in U \ ,$$

then

$$\text{(2)} \qquad \left|\arg p(z)\right| < \frac{\pi}{3} = 1.04719\ldots\ (60^{\circ}) \ , \quad z \in U \ .$$

By using onother result, recently obtained in [2, Theorem 5], the inequality (2) can by replaced by

$$\left|\arg p(z)\right| < 1.00263\ldots\ (57^{\circ}.446\ldots)$$

In the present paper we improve (2) by

$$\text{(3)} \qquad \left|\arg p(z)\right| < \varphi_0 = .91162190\ldots\ (52^{\circ}.232087\ldots)$$

where φ_0, given by a rather complcated equation, is the best bound of $|\arg p(z)|$, $z \in U$.

2. According to a well-known result due to Hallenbeck and Ruscheweyh [1], the inequality (1) implies the sharp inclusion $p(U) \subset q(U)$, where

$$\text{(4)} \qquad q(z) = \frac{2}{z}\log(1 + z) - 1 \qquad (\log 1 = 0).$$

Since the function q is convex, the value of φ_0 in (3) will be given by the argument of the tangent from the origin to the analytic curve $q(\partial U)$ (Figure 1)

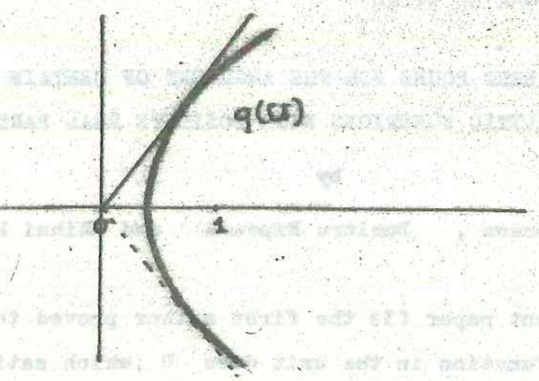

Figure 1.

3. PROPOSITION 1. Let

(5) $$\varphi(\theta) = \arg q(e^{i\theta}) \quad , \quad -\pi \leq \theta \leq \pi \quad ,$$

where q is given by (4). The behaviuor of φ is given by the following table

θ	$-\pi$		$-\theta_o$		0		θ_o		π
$\varphi(\theta)$	0	$\nearrow$	φ_o	$\searrow$	0	$\searrow$	$-\varphi_o$	$\nearrow$	0

Table 1

where $\varphi_o = \operatorname{arctg}(-\Phi(\theta_o))$, $(0 < \varphi_o < \pi/2)$, with

(6) $$\Phi(\theta) = \frac{\theta \cos\theta - 2\sin\theta \ln(2\cos\frac{\theta}{2})}{\theta\sin\theta - 1 + 2\cos\theta\ln(2\cos\frac{\theta}{2})}$$

and θ_o is the unique root in the interval $(0,\pi)$ of the function

(7) $$H_o(\theta) = 1 - \theta\, \mathrm{tg}\frac{\theta}{2}(2+\cos\theta) + \theta^2 - 2(1+\cos\theta)\ln(2\cos\frac{\theta}{2}) + 4\ln^2(2\cos\frac{\theta}{2}) \, ,$$

or, equivalently, of the function

(8) $$H_1(\theta) = 1 + \cos\theta - \sqrt{R(\theta)} - 4\ln(2\cos\frac{\theta}{2}) \quad ,$$

with

(9) $$R(\theta) = (2\theta - \sin\theta)[(3+\cos\theta)\,\mathrm{tg}\frac{\theta}{2} - 2\theta].$$

Moreover, we have

(10) $$1.9634953\ldots = \frac{5}{8}\pi < \theta_o < \frac{2}{3}\pi = 2.094395\ldots$$

Proof. From (4) and (5) we deduce

$$q(e^{i\theta}) = u(\theta) + i\,v(\theta) \quad ,$$

where

(11) $$u(\theta) = \theta\sin\theta - 1 + 2\cos\theta\ln(2\cos\frac{\theta}{2})$$
$$v(\theta) = \theta\cos\theta - 2\sin\theta\ln(2\cos\frac{\theta}{2}) \, .$$

and

$$\varphi(\theta) = \operatorname{arctg}\Phi(\theta) \quad ,$$

where $\Phi(\theta) = v(\theta)/u(\theta)$ is given by (6).

We have

$$\Phi'(\theta) = -\frac{H_o(\theta)}{u^2(\theta)}$$

with H_o given by (7). We also can write

(12) $$\Phi'(\theta) = \frac{1}{4u^2(\theta)}\left\{R(\theta) - [1+\cos\theta - 4\ln(2\cos\frac{\theta}{2})]^2\right\} = -\frac{H_1(\theta)H_2(\theta)}{4u^2(\theta)} \quad ,$$

where $H_2(\theta) = 1 + \cos\theta + \sqrt{R(\theta)} - 4\ln(2\cos\frac{\theta}{2})$ and H_1 and R are given by (8) and (9) respectively.

Since R can be written

(13) $$R(\theta) = (2\theta - \sin\theta)H_3(\theta) \quad , \quad H_3(\theta) = (3+\cos\theta)\mathrm{tg}\frac{\theta}{2} - 2\theta \, ,$$

we deduce

$$\operatorname{sign} R(\theta) = \operatorname{sign} H_3(\theta)$$

Since $H_3'(\theta) = -\cos\theta \operatorname{tg}^2 \frac{\theta}{2}$, we obtain the following table

θ	0		$\pi/2$		$3\pi/5$		θ_1		π
$H_3'(\theta)$	0	$-$	0	$+$		$+$			
$H_3(\theta)$	0	$\searrow$	$3-\pi$		$\nearrow$		0	$\nearrow$	

Table 2

which shows that $H_3(\theta) < 0$,for $0<\theta<\theta_1$ and from (12) and (13) we deduce $\Phi'(\theta) < 0$, for $0<\theta<\theta_1$.

If we write $H_2(\theta) = H_4(\theta) + \sqrt{R(\theta)}$, with

$$H_4(\theta) = 1 + \cos\theta - 4\ln(2\cos\frac{\theta}{2}) ,$$

we have

$$(14) \qquad H_4'(\theta) = \frac{2\sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}} \geq 0 .$$

Since $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$, $\operatorname{tg}\frac{3\pi}{10} = \sqrt{\frac{2+\sqrt{5}}{\sqrt{5}}}$,we have

$$H_3(\tfrac{3\pi}{5}) = \frac{1}{4}\sqrt{\tfrac{2}{5}(305 + 109\sqrt{5})} - \frac{6\pi}{5} < \frac{1}{4}\sqrt{\tfrac{2}{5}(305 + 109\times 2.24} - \frac{6}{5}\times 3.14 =$$

$$= \frac{1}{4}\sqrt{219.664} - 3.768 < \frac{1}{4}14.83 - 3.768 = -.0605 < 0 ,$$

and in Table 2 we have

$$(15) \qquad \theta_1 > \frac{3}{5}\pi .$$

We also have

$$H_4(\tfrac{3\pi}{5}) = 2(\frac{5-\sqrt{5}}{8} - \ln\frac{5-\sqrt{5}}{2}) = .04\ldots > 0$$

and by using (14) and (15) we deduce

$$H_4(\theta) > 0 , \quad \text{for } \theta_1 < \theta < \pi ,$$

so that

$$(16) \qquad H_2(\theta) > 0 , \quad \text{for } \theta_1 < \theta < \pi .$$

On the other hand we have

$$H_1'(\theta) = \frac{\operatorname{tg}\frac{\theta}{2}}{R(\theta)} H_5(\theta) ,$$

with

$$H_5(\theta) = (1-\cos\theta)\sqrt{R(\theta)} - [3-\cos^2\theta + \frac{2\theta}{\sin\theta}(-1-\cos\theta+\cos^2\theta)] .$$

Let

$$H_6(\theta) = (1-\cos\theta)^2 R(\theta) - [3-\cos^2\theta + \frac{2\theta}{\sin\theta}(-1-\cos\theta+\cos^2\theta)]^2 =$$

$$= \left(\frac{2}{\sin\theta}\right)^2 (2-\cos^2\theta)(\theta - \sin\theta) H_7(\theta) ,$$

where

$$(17) \qquad H_7(\theta) = \frac{3 - 2\cos\theta}{2 - \cos^2\theta}\sin\theta - \theta .$$

Since

$$H_7'(\theta) = -2(2-\cos\theta)\left(\frac{\sin\frac{\theta}{2}\cos\theta}{2-\cos^2\theta}\right)^2 \leq 0 , \text{for } 0 \leq \theta \leq \pi ,$$

we deduce $H_7(\theta) \leq H_7(0) = 0$, hence $H_6(\theta) < 0$, for $0<\theta<\pi$. and we have $H_5(\theta) \neq 0$ and $H_1'(\theta) \neq 0$,for $\theta_1<\theta<\pi$. Since $\lim_{\theta\to\pi} H_5(\theta) = -\infty$,we deduce $H_5(\theta)<0$,hence $H_1'(\theta)<0$, for $\theta_1<\theta<\pi$,which gives the following table

θ	θ_1		θ_0		π
$H_1(\theta)$	$H_1(\theta_1) > 0$	$\searrow$	0	$\searrow$	$-\infty$

Table 3

From (12),(14),(16) and Table 3 we obtain the following table

θ	0		$\frac{3\pi}{5}$		θ_0		π
$\Phi'(\theta)$	-1	$-$		$-$	0	$+$	
$\Phi(\theta)$	0		$\searrow$		$\Phi(\theta_0)$	$\nearrow$	0

Table 4

Since $\varphi(-\theta) = -\varphi(\theta)$, Table 4 gives Table 1. Since

$$H_3(\tfrac{5\pi}{8}) = 2(\sqrt{2}-1) + \tfrac{1}{2}\sqrt{82-31\sqrt{2}} - \tfrac{5\pi}{4} = -.009\ldots < 0 ,$$

in Table 2 we have $\theta_1 > \frac{5\pi}{8}$, and from Table 3 we deduce $\theta_o > \theta_1 > \frac{5\pi}{8}$.

On the other hand we have

$$H_1(\tfrac{2\pi}{3}) = \frac{8}{3}\,\frac{P(\pi)}{3+\sqrt{(8\pi-3\sqrt{3})(15\sqrt{3}-8\pi)}} ,$$

where $P(x) = 9 - 9\sqrt{3}\,x + 4x^2$. Since the polynomial P has the roots $3(3\sqrt{3}-\sqrt{11})/8 < 1$ and $3(3\sqrt{3}+\sqrt{11})/8 = 3.19\ldots > \pi$,we have $P(\pi) < 0$,hence $H_1(\frac{2\pi}{3}) < 0$ and from Table 3 we deduce $\theta_o < \frac{2}{3}\pi$. Therefore θ_o satisfies (10) and this completes the proof of Proposition 1.

3. PROPOSITION 2. Let θ_2 and θ_3 be given,such that $\pi/2 < \theta_1 < \theta_2 < \theta_o < \theta_3 < \pi$, where θ_o and θ_1 are the unique roots in the interval $(0,\pi)$ of $H_1(\theta)$ given by (8) and $H_3(\theta)$ given by (13) respectively. Then in Table 1 we have

$$(18)\qquad \operatorname{arctg}(-H_8(\theta_3)) < \varphi_o < \operatorname{arctg}(-H_8(\theta_2)) ,$$

where

$$H_8(\theta) = -2\,\frac{\theta - \sin\theta}{1-\cos\theta + \sqrt{R(\theta)}}$$

and R is given by (9).

Proof. Since $H_1(\theta_o) = 0$,from (6) and (8) we deduce $\Phi(\theta_o) = H_8(\theta_o)$. We also have

$$(19)\qquad H_8'(\theta) = \frac{H_9(\theta)}{2H_{1o}^2(\theta)\sqrt{R(\theta)}}$$

with

$$(20)\qquad H_9(\theta) = \frac{1}{1+\cos\theta}[-(1-\cos\theta)(3-3\cos\theta-2\cos^2\theta) + \frac{\theta\sin\theta}{1+\cos\theta}(5-3\cos\theta-7\cos^2\theta-\cos^3\theta) + 2\theta^2(-1+\cos\theta+\cos^2\theta)] + [(1-\cos\theta)\frac{1+2\cos\theta}{1+\cos\theta} - \theta\sin\theta]\sqrt{R(\theta)}$$

and

$$H_{1o}(\theta) = \operatorname{tg}\frac{\theta}{2} - \theta .$$

Since

$$H_{11}(\theta) = \Big\{\frac{1}{1+\cos\theta}[-(1-\cos\theta)(3-3\cos\theta-2\cos^2\theta) + \frac{\theta\sin\theta}{1+\cos\theta}(5-3\cos\theta-7\cos^2\theta-\cos^3\theta) + 2\theta^2(-1+\cos\theta+\cos^2\theta)]\Big\}^2 - \Big\{[(1-\cos\theta)\frac{1+2\cos\theta}{1+\cos\theta} - \theta\sin\theta]\sqrt{R(\theta)}\Big\}^2 = -4\,\frac{2-\cos^2\theta}{(1+\cos\theta)^2}(\theta-\sin\theta)H_{1o}^2(\theta)H_7(\theta) ,$$

where H_7 is given by (17) ,we deduce $H_{11}(\theta) > 0$, for $\theta_1 < \theta < \pi$. Since $\lim_{\theta\to\pi}(1+\cos\theta)H_9(\theta) = +\infty$, we conclude that $H_8'(\theta) > 0$, for $\theta_1 < \theta < \pi$ and we obtain the following table

θ	θ_1	θ_2	θ_o	θ_3	π
$-H_8(\theta)$	$-H_8(\theta_1) \searrow$	$-H_8(\theta_2) \searrow$	$-\Phi(\theta_o) \searrow$	$-H_8(\theta_3) \searrow$	0

Table 5

which yields (18). This completes the proof of Proposition 2.

4. Using a computer CORAL from ITC ,Cluj-Napoca,we obtained

$\theta_o = 2.000215348\ldots$ (in Proposition 1)

and

$\theta_1 = 1.97475141\ldots$ (in Table 2).

If in Proposition 2 we let

$$\theta_2 = 2.000215347 \quad \text{and} \quad \theta_3 = 2.000215349 ,$$

we obtain

$$\operatorname{arctg}(-H_8(\theta_3)) = .911621904...$$

and

$$\operatorname{arctg}(-H_8(\theta_2)) = .911621906... .$$

Therefore in Proposition 1 we have

$$.911621904 < \varphi_0 < .911621907$$

and we obtain the estimation(3).

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