

FEM and/or BEM for numerical simulation of Marangoni flow

C. I. GHEORGHIU

Computer Center, Polytechnic Institute of Cluj Napoca, R-3400, Cluj Napoca, Romania

INTRODUCTION

The aim of this work is a mathematical study of the Marangoni effect using comparatively three numerical methods.

The first is the FEM (finite element method) and this is exposed in some of our previous works^{1,2}. The second is the BEM (boundary element method) and the third a modified variant of this, MBEM.

THE MARANGONI FLOW THROUGH A LIQUID LAYER OF TRIANGULAR SECTION SITUATED ON AN INCLINED PLANE

The study of the Marangoni effect is made in the framework of physicochemical hydrodynamics. In two of our previous papers^{1,2} we have given more details about this phenomenon from a hydrodynamic and physicochemical point of view.

It is actually a viscous incompressible fluid in laminar flow and for simplicity a steady one due to simultaneous action of a surface tension gradient and gravity. The flow is considered in a canal of a triangular transverse section situated on an inclined plane (see Fig. 1(a)). At the ends of this canal a surface tension gradient is instantaneously created. This produces a flow that is opposite to the gravitational one.

The flow along the canal is essential, so that only the corresponding component of velocity is assumed nonzero. Supposing a constant pressure in a thin liquid layer (admissible from a physical point of view), the Navier-Stokes equations lead to the following mixed boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega \quad (1)$$

$$u = g_1 \quad \text{on } S \quad (2)$$

$$\frac{\partial u}{\partial n} = g_2 \quad \text{on } L \quad (3)$$

The domain Ω , the solid walls of the canal S , and the liquid gas - interface L are given in Fig. 1(b). The right member of equation (1) contains the external volume force. The function g_1 is identical zero, i.e., the condition for the viscous fluid to adhere to the solid walls, and g_2 contains the surface tension gradient.

Details on f and g_2 are to be found in some of our previous papers¹⁻³. In all the cases considered both are

constants. However, if on the liquid gas-interface L only the surface density is neglected but not the shear and dilatational viscosity, the previous problem becomes more complicated, viz. Dirichlet-Robin.

NUMERICAL ANALYSIS FOR THE MARANGONI AND THE GRAVITY FLOW IN A THIN LIQUID LAYER

Finite element method

Using this method, in the works already mentioned, a minute analysis of the Marangoni effect was made. The

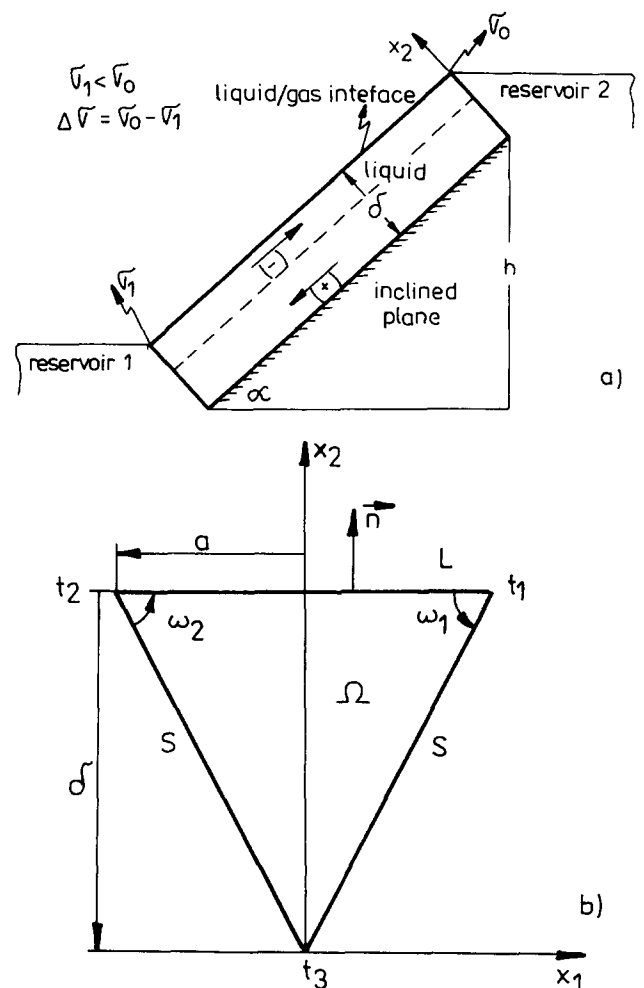


Fig. 1. (a) Longitudinal section of the thin liquid layer in which the Marangoni flow (—), is in competition with the gravity flow (+). (b) Transversal section of canal flow assumed in numerical calculus

Accepted October 1987. Discussion closes May 1989.

starting points were equations (1)–(3) to which the classical FEM was applied. The details of the mathematical procedure are given in Chifu *et al.*¹. No importance was given to the fact that the Ω domain contains corners and collision points. The numerical results were still satisfactory for engineering practice. The analysis was made for a large set of surface tension gradients and for different slopes of the plane on which the canal is situated.

In Chifu *et al.*², the surface tension gradient is induced by temperature gradients and in Chifu and Gheorghiu³ the flow was considered unsteady and shear and dilatational viscosity were taken into account.

In all these papers besides the velocity field the rates of flow due to the Marangoni effect and gravitation were also determined. These were the quantities that were actually compared to the experimental ones. In most of the cases the computed rates of flow were close to the experimental ones.

Considering the necessity of high precision results the mesh size was reduced but still insufficient. This led to the idea of using a boundary type method. Its well known advantage is that of finding the unknown in any interior points of the domain we want without any additional effort.

Boundary element method

Below we will employ, without special stipulation, the results and terminology of Costabel and Stephan⁴.

So, for $u \in H^1(\Omega, \Delta, p)$, $p > 1$, satisfying equation (1), we have the following representation:

$$u(z) = \frac{-1}{2\pi} \int_{\Omega} f(x) \log|z-x| dx + \frac{1}{2\pi} \int_{\Gamma} u(\rho) \frac{\partial}{\partial n_{\rho}} \log|z-\rho| ds_{\rho} - \frac{1}{2\pi} \int_{\Gamma} \frac{\partial u(\rho)}{\partial n} \log|z-\rho| ds_{\rho}, \quad z \in \Omega \quad (4)$$

Here ds_{ρ} is the measure on $\Gamma = \text{SUK}$ defined by the arc length.

BEM in its naive form has been applied to equation (4) making z to be on Γ . Using the continuity properties of single and double layer potentials we obtain the following integral equation on Γ :

$$u(z) = \frac{1}{\pi} \int_{\Gamma} u(\rho) \frac{\partial}{\partial n_{\rho}} \log|z-\rho| ds_{\rho} - \frac{1}{\pi} \int_{\Gamma} \frac{\partial u(\rho)}{\partial n} \log|z-\rho| ds_{\rho} - \frac{1}{\pi} \int_{\Omega} f(x) \log|z-x| dx, \quad z \in \Gamma \quad (5)$$

To integrate this equation constant boundary elements were used to avoid difficulties caused by the existence of corners.

Boundary integrals from equation (5) were computed exactly using the work of Ingham and Kelmanson⁵. The quadrature formulae were used only for domain integral (see Piessens *et al.*⁶).

In Fig. 2 the velocity field is drawn (for the sake of symmetry only the right half of the domain) corresponding to some given values for the physical and geometrical parameters. Here, of course, $\Delta\sigma$ is surface tension gradient, ρ is the liquid density, μ is the liquid viscosity and g is the acceleration of gravity. So the Marangoni effect is underlined.

For other results regarding this problem and obtained through this method, see Petrila and Gheorghiu⁷.

Modified boundary element method

Using the same work of Costabel and Stephan⁴, the weak solution $u \in H^1(\Omega)$ of the problem represented by equations (1)–(3) is supposed to be:

$$u = \sum_{k=1}^r c_k l_k u_k + u_0 \quad (6)$$

The regular part of u, u_0 , belongs to $H^{s+1/2}(\Omega)$, $s > 1/2$, and $u_k, k = 1, 2, \dots, r$ are singular functions not depending on g_1 and g_2 but only on the geometry of domain Ω . These are described below. The $l_k, k = 1, 2, \dots, r$ are C_0^∞ – cut off functions whose support are small parts of the domain Ω in the neighbourhood of t_k . More exactly l_k is 1 near t_k . The quantities $c_k, k = 1, 2, \dots, r$ are constant.

Considering that g_1 and g_2 are constant, corresponding to a degree of smoothness presumed for u_0 and taking into account the values of $\omega_j, j = 1, 2, 3, r$ has to be found.

So, for $s = 5/2$ ($u_0 \in \dot{H}^3(\Omega) \rightarrow C^1(\Omega)$) and $\pi/3 < \omega_1 = \omega_2 < \pi/2$, $r = 2$ and equation (6) implies:

$$u = \sum_{k=1}^2 c_k l_k u_k + u_0 \quad (7)$$

Considering that Ox_2 is the symmetric axis for the problem represented by equations (1)–(3) the constants c_1 and c_2 are considered equal and will be both noted with c . In order to determine c we need another equation. This is the necessary condition of the solution's existence for equations (1)–(3):

$$\int_{\Omega} f dx + \int_{\Gamma} \frac{\partial u}{\partial n} ds = 0 \quad (8)$$

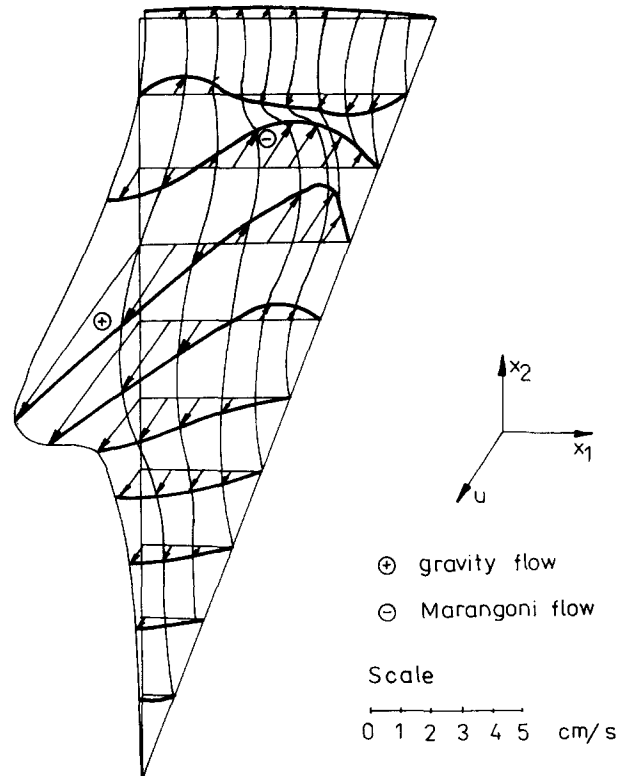


Fig. 2. The velocity field corresponding to: $\Delta\sigma = 20.4$, $\mu = 0.00656$, $\rho = 0.9952$, $h = 0.6$ cm, $f = \rho g \sin \alpha / \mu$, $g_2 = -\Delta\sigma \sin \alpha / h$, $g = 980.7$, $\alpha = \pi/9$, $a = 0.021$ cm and $\delta = 0.0239$ cm

For the functions u_k , we have:

$$u_1 = \rho_1^{\alpha_1} \cos(\alpha_1 \phi_1) \quad u_2 = \rho_2^{\alpha_2} \sin(\alpha_2 \phi_2) \quad (9)$$

where $\alpha_1 = \alpha_2 = (1 - 1/2) \pi / \omega_j$, $j = 1, 2$. Here (ρ_j, ϕ_j) are positively oriented local polar coordinates at vertex t_j so that $\phi_1 = 0$ on L and $\phi_1 = \omega_1$ on S and $\phi_2 = 0$ on S and $\phi_2 = \omega_2$ on L .

The system of integral equations (5), (8) was solved using for u_0 and $\partial u_0 / \partial n$ constant elements⁸. Numerical test were made for various shapes of the Ω domain and for some cut off functions. The choice of these cut off functions as well as their influence on the accuracy of the results is still open.

The velocity field obtained through this method is very close to the one obtained by using BEM, so that a new representation of it is useless. One notices that in the neighbourhood of the collision points (t_1 and t_2 on Fig. 1(a)) the flow is 'quiet' enough so that determining c becomes of a purely theoretical interest.

CONCLUSIONS

For the degree of smoothness presumed, the corner t_3 has no influence. The collision points are the only points that disturb the regular part u_0 .

In the cited work of Costabel and Stephan⁴, an extremely detailed analysis of this MBEM is made.

In some of Professor Wendland's works⁹, the same analysis of this method is to be found. His works also offer the possibilities of compiling this method to the classical FEM. As we are also interested in this possibility of coupling we will consider it in future research.

In order to make a global analysis of the Marangoni effect the problem of hydrodynamic instability is still

open. As it is well known the spontaneous generation of motion in a system previously at rest is an instability problem of great interest to science and engineering. This is an eigenvalue problem for the system of equations, and its solution represents a great task.

We are also interested in making numerical analysis of this eigenvalue problem.

Finally we will underline that BEM is superior to FEM from at least two points of view. As we have already mentioned BEM makes possible the estimation of the velocity in any part of the domain. BEM also leads to a local study of the solution's behaviour.

REFERENCES

- 1 Chifu, E., Gheorghiu, C. and Stan, I. Surface Mobility of Surfactant Solutions. XI. Numerical Analysis for the Marangoni and the Gravity Flow in a Thin Liquid Layer on Triangular Section, *Revue Roumaine de Chimie*, 1984, **29**, 31-42
- 2 Chifu, E., Albu, I., Gheorghiu, C. I., Gavrilă, E. and Sălăjan, M. Marangoni Flow-Induced by Temperature Gradients-Against Gravity Forces, *Revue Roumaine de Chimie*, 1986, **31**, 105-112
- 3 Chifu, E. and Gheorghiu, C. I. Surface Mobility of Surfactant Solutions. XII. Remarks Concerning the Marangoni Flow Through an Inclined Surface Canal, *Revue Roumaine de Chimie*, 1987, inpress
- 4 Costabel, M. and Stephan, E. Boundary Integral Equations for Mixed Boundary Value Problems in Polygonal Domains and Galerkin Approximation, THD-Preprint Nr. 593, 1981
- 5 Ingham, B. D. and Kelmanson, M. A. Boundary Integral Equation Analysis of Singular, Potential and Biharmonic Problems, Springer-Verlag, 1984
- 6 Piessens, R., de Donker-Kapenga, E., Überhuber, C. W. and Kahaner, D. K. QUADPACK A Subroutine Package for Automatic Integration, Springer-Verlag, 1983
- 7 Petrila, T. and Gheorghiu, C. I. 1987 Metode element finit și aplicații, Editura Academiei, București, 1987
- 8 Brebbia, C. A. The Boundary Element Method for Engineers, Pentech Press, UK
- 9 Wendland, W. L. On Asymptotic Error Estimates for Combined BEM and FEM, CISM-Courses, 1986