



Optimization: A Journal of Mathematical Programming and Operations Research

Publication details, including instructions for authors and subscription information:

<http://www.tandfonline.com/loi/gopt20>

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Published online: 27 Jun 2007.

To cite this article: A.B. Németh (1989) Between Pareto efficiency and Pareto ϵ -efficiency, Optimization: A Journal of Mathematical Programming and Operations Research, 20:5, 615-637, DOI: [10.1080/02331938908843483](https://doi.org/10.1080/02331938908843483)

To link to this article: <http://dx.doi.org/10.1080/02331938908843483>

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Between PARETO Efficiency and PARETO ϵ -Efficiency

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Summary: By using an efficiency notion placed between PARETO efficiency and PARETO ϵ -efficiency it is shown that semi Archimedean ordered vector spaces and regular ordered locally convex spaces can be characterized by existence results concerning efficient points of this type for lower bounded sets. The technique developed in the paper allows us to obtain necessary and sufficient conditions for the existence of PARETO efficient points, too.

AMS 1980 Subject Classification:

Primary: 90 C 31; Secondary: 49 A 27

Key words: Cone orderings, PARETO efficient points, approximative vector minimization.

0. Introduction

A possible subtitle of the paper could be: Some theoretical aspects of the existence of PARETO efficient and approximative PARETO efficient points of lower bounded sets in ordered vector spaces. The main problem within the paper is that the existence of points of this kind for each nonempty lower bounded set actually characterizes well known types of ordered vector spaces, respectively ordered locally convex spaces.

Existence results for efficient points have been obtained in the optimization literature recently. They are in fact sufficient conditions which work either for a relative large class of sets but for restricted classes of orderings (see e.g. [5], [23], [3], [18], [1], [7], [21], [8], [20]) or for sets with compactness type properties (see e.g. [6], [4], [1]). The common machinery of the existence proofs consists of exploiting explicitly or implicitly the convergence of some monotone sequences. Our main contribution is that the convergence properties of the ordered space assuring this approach are not purely technical auxiliaries in obtaining efficient points but they are indeed necessary for their existence. This idea, even not sufficiently emphasized, derives from our results in [14], [15] and [17], which focus on some related questions.

The basic tool in our proofs is the H -efficiency which takes place between PARETO efficiency and PARETO ϵ -efficiency. This notion extends formally that of ϵ -efficiency, when the element ϵ in the positive cone is replaced by a set H in it. H -efficiency links naturally to other standard terms of the ordered vector space

theory, and hence it will be a useful instrument in most proofs we are going to carry out.

After introducing some basic terms (Section 1) and going briefly over some well known facts concerning PARETO efficient points and PARETO ε -efficient points (Section 2), an algebraic theory of H -efficiency is developed in the Sections 3, 4 and 5. In Section 6 it is shown that the existence for every nonempty lower bounded set of ε -efficient points for arbitrary points ε in the positive cone, and also of H -efficient points for arbitrary finite sets H in this cone, characterizes the semi Archimedean ordered vector spaces. Topological notions are taken into consideration from Section 7 on, where some auxiliary results are surveyed on normal and regular ordered locally convex spaces. We prove in Section 8 that if every nonempty lower bounded set in an ordered locally convex space possesses H -efficient points for arbitrary sets H in the positive cone such that their complements are neighbourhoods of 0, then this space must be regular. This result furnishes in passing necessary and sufficient conditions for the existence of efficient points too (Section 9). Usage of a notion of regularity of sets allows a unified approach of various types of existence results (Section 10). In the Sections 11 and 12 we consider the problem of H -efficiency in spaces with two cones and its relation with the existence of H -efficient points for every nonempty lower bounded set when H is fixed.

1. Basic Terminology

Throughout this paper Y denotes a real vector space and K a cone therein, i.e., a set having the properties:

- (i) $K + K \subset K$;
- (ii) $tK \subset K$ for each non-negative real number t , and
- (iii) $K \cap (-K) = \{0\}$.

Putting $x \leq y$ whenever $y - x$ is in K , we obtain a reflexive, transitive and anti-symmetrical order relation in Y , which is translation-invariant and invariant with respect to the multiplication with non-negative reals. The resulting object will be called an *ordered vector space* (abbreviated, o.v.s.) with the *positive cone* K , and will be denoted by (Y, K) . Lower and upper bounds of sets, infima and suprema, monotonicity etc. are considered in the sense of this order relation as usual. We shall deal in the last sections with two cones and respectively two orderings. Hence, if necessary, we shall use terms as: K lower bound, K increasing etc., to emphasize that the respective notions are taken with respect to the ordering induced by K .

If H is a set in Y , we shall denote by cone H the intersection of all the cones containing H . (In the cases we shall consider cones with this property always exist.) By $\text{co } H$ we shall mean the convex hull of the set H .

In order to simplify the style we shall use the same terms for ordered vector spaces and their positive cones as well.

We shall suppose throughout that $K \neq \{0\}$.

2. Efficiency and ε -Efficiency

Let (Y, K) be an o.v.s. and let M be a set in Y . A point x_0 in M is said *efficient point* (or PARETO efficient point, PARETO minimum point, minimum point, K -minimum point) of M if there is no x in M such that $x \leq x_0$ and $x \neq x_0$. It is straightforward to see that x_0 is an efficient point of M if and only if

$$(x_0 - K) \cap M = \{x_0\}. \quad (2.1)$$

The set of the efficient points of M is denoted by $\text{eff } M$. It is well known and easy to verify that $\text{eff } M = \text{eff } (M + K)$.

Let ε be an element of K . Then $x_0 \in M$ is said to be an ε -efficient point (or PARETO ε -efficient point, or ε near to minimum point) of M if there is no $x \in M$ such that $x \leq x_0 - \varepsilon$ (see [11] or [14]). The point x_0 is an ε -efficient point of M if and only if

$$x_0 \in M \quad \text{and} \quad (x_0 - \varepsilon - K) \cap M = \emptyset. \quad (2.2)$$

The set of ε -efficient points of M is denoted by $\varepsilon\text{-eff } M$. The relation (2.2) implies that $\varepsilon\text{-eff } M \subset \varepsilon\text{-eff } (M + K)$. Simple examples show that the opposite inclusion does not always hold. From this definition we have also that in order to exist some M having ε -efficient points, it is necessary to ε be different from 0. Hence $\varepsilon \neq 0$ will be implicitly supposed in the following. Observe that the notion of the ε -efficient point does not include that of efficient point. The definition of ε -efficient points can be slightly modified (in the sense of [10]) to comprise as particular case efficient points, but this change is not advantageous for us.

A *section of the set M* is a nonempty set of form $(y - K) \cap M$ with some y in Y .

We shall see in Section 6 that except some particular o.v. spaces the existence of ε -efficient points for a set results from the existence of lower bounded sections of this set. This latter condition can be considered as a natural and rather mild requirement for a set. On the other hand, as it will be shown in Section 9, the condition that every set with a complete lower bounded section has efficient points implies strong restrictions on the order relation of the space. These conditions can be transferred to sets in the way it is presented in Section 10.

The above remarks explain the reduced interest for existence results on ε -efficient points, while efficiency from this point of view is well represented in the recent literature (see [18], [4], [1], [7], [8]). This circumstance brings up the natural question to search some intermediate notions which can characterize better the various types of o.v. spaces by minimality conditions. The only possibility of this kind seems to be the replacing of ε in the above definition by sets in the positive cone. One of the main results of our paper is that this way in fact no new prospective can be gained on the o.v. spaces and ordered locally convex spaces (see Section 6 and Sections 8 and 9, respectively). However, such a notion was a good implement in some related problems (see [15] and [17]) and is a useful tool in obtaining necessary conditions for the existence of the efficient points (see Section 9).

3. H -Efficiency: Definition, Basic Relations and Examples

Let (Y, K) be an o.v.s.

Definition 3.1: Let H be a nonempty subset of $K \setminus \{0\}$. Given a set M in Y , a point x_0 in M is an H -efficient point of M if there is no $x \in M$ such that $x \leq x_0 - h$ for some $h \in H$.

This means that x_0 is an H -efficient point of M if there is no x in M which is simultaneously in $x_0 - h - K$ for some h in H , that is if

$$x_0 \in M \quad \text{and} \quad (x_0 - H - K) \cap M = \emptyset. \quad (3.1)$$

The set of H -efficient points of M will be denoted by H -eff M .

When speaking about H -efficiency we always suppose that H is a nonempty subset in $K \setminus \{0\}$.

From the characterization (3.1) of H -efficient points there follow the relations:

$$H\text{-eff } M = (H + K)\text{-eff } M \subset H\text{-eff } (M + K) = (H + K)\text{-eff } (M + K). \quad (3.2)$$

When H reduces to the singleton $\{e\}$, then H -efficiency becomes e -efficiency. The following formula, which is straightforward from (3.1) and (2.2) establishes a further relation of these two notions:

$$H\text{-eff } M = \bigcap_{h \in H} (h\text{-eff } M). \quad (3.3)$$

This relation yields directly:

$$(H_1 \cup H_2)\text{-eff } M = (H_1\text{-eff } M) \cap (H_2\text{-eff } M). \quad (3.4)$$

From (3.3) it follows also the implication:

$$H_1 \subset H_2 \Rightarrow H_2\text{-eff } M \subset H_1\text{-eff } M \quad (3.5)$$

wherefrom

$$(H_1 \cap H_2)\text{-eff } M \supset (H_1\text{-eff } M) \cup (H_2\text{-eff } M). \quad (3.6)$$

We shall show by examples that it may happen that $H_1\text{-eff } M = H_2\text{-eff } M = \emptyset$ while $(H_1 \cap H_2)\text{-eff } M \neq \emptyset$.

Examples: 3.2. Let $Y = \mathbb{R}$ be ordered by $K = \mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$. Let (x_n) and (y_n) be two decreasing sequences in Y both converging to 1. Put $H_1 = \{x_n \mid n \in \mathbb{N}\}$ and $H_2 = \{y_n \mid n \in \mathbb{N}\}$. It is easy to see that for each lower bounded set M in $\mathbb{R}$ one has $H_1\text{-eff } M = H_2\text{-eff } M \neq \emptyset$. When M is not bounded from below, then $H_1\text{-eff } M = H_2\text{-eff } M = \emptyset$. Then we have that $H_1\text{-eff } M = H_2\text{-eff } M$ for each M in Y , whereas it can happen that $H_1 \cap H_2 = \emptyset$.

3.3. Let u and v be two linearly independent vectors in $\mathbb{R}^2$ and put $K = \text{cone } \{u, v\}$. Consider the set $M = \mathbb{R}^2 \setminus \text{int } (-K)$. (By $\text{int } A$ we mean the interior of the set $A \subset \mathbb{R}^2$ in the unique locally convex HAUSDORFF topology of this vector space.) The following relations hold:

(i) If $H = \text{int } K$, then $H\text{-eff } M = \partial M$, with ∂M the boundary of M .

From this property it follows in particular via (3.5) that for every $H_0 \neq \emptyset$ in $\text{int } K$ one has $H_0\text{-eff } M \supset \partial M$.

(ii) $u\text{-eff } M = \{tv + su \mid t < 0, s \in [0, 1]\}$.

$v\text{-eff } M = \{tu + sv \mid t < 0, s \in [0, 1]\}$.

(iii) If $H_1 = \{u, v\}$, then $H_1\text{-eff } M = \emptyset$.

(iv) Put $H_2 = \{u, 2v, u + v\}$, $H_3 = \{2u, v, u + v\}$. Then $H_2\text{-eff } M = H_3\text{-eff } M = \emptyset$ while $(H_2 \cap H_3)\text{-eff } M = (u + v)\text{-eff } M \supset \partial M \neq \emptyset$ by (i). This is an example which shows that in (3.6) the converse inclusion not always holds.

(v) If $H_4 = \{tu \mid t > 0\}$ and $H_5 = \{tv \mid t > 0\}$ then $H_4\text{-eff } M = \{sv \mid s < 0\}$ and $H_5\text{-eff } M = \{su \mid s < 0\}$. Thus $(H_4\text{-eff } M) \cup (H_5\text{-eff } M) = \partial M \setminus \{0\}$ and $(H_4 \cap H_5)\text{-eff } M$ is meaningless since $H_4 \cap H_5 = \emptyset$.

(vi) Put $H_6 = (\text{int } K) \cup H_4$ and $H_7 = (\text{int } K) \cup H_5$. Then $H_6 = H_4 + K$ and $H_7 = H_5 + K$, hence by (3.2) we have

$$H_6\text{-eff } M = H_4\text{-eff } M = \{sv \mid s < 0\},$$

$$H_7\text{-eff } M = H_5\text{-eff } M = \{su \mid s < 0\}.$$

Here $H_6 \cap H_7 = \text{int } K$, hence by (i), $(H_6 \cap H_7)\text{-eff } M = \partial M$ while $(H_6\text{-eff } M) \cup (H_7\text{-eff } M) = \partial M \setminus \{0\}$.

3.4. Consider the space $C[0, 1]$ of the continuous real valued functions on $[0, 1]$. Put $K = \text{cone } \{t^2, t^3, \dots\}$. Then K will be the set of all polynomials of order at least 2 with nonnegative coefficients completed with the zero polynomial.

If M is the set of all piecewise linear functions on $[0, 1]$, then it is dense in $C[0, 1]$ with respect to the uniform topology of this space. If we consider now $C[0, 1]$ to be ordered by K , then all the points in M are efficient points.

Put $K_1 = \text{cone } \{1, t, t^2, \dots\}$. If we consider $C[0, 1]$ to be ordered by K_1 , all the points in M , except those in the subspace spanned by 1 and t , are efficient points. If $\varepsilon = 1$ or $\varepsilon = t$, then the ε -efficient points of M coincide with the efficient points. This is the case also for the H -efficient points of M with $H = \{1, t\}$.

4. Comparison of H -Efficiencies

The relation (3.5) shows that the operation $H \rightarrow H\text{-eff } M$ is antitonic, i.e., the larger H implies the smaller $H\text{-eff } M$. That is, H -efficiency for a larger H is more restrictive. This justifies the following terminology.

We say that H_1 -efficiency is *coarser* than H_2 -efficiency if for every set M in Y it holds $H_1\text{-eff } M \supset H_2\text{-eff } M$. In this case we shall write $H_1\text{-eff} \preceq H_2\text{-eff}$. If H_1 -efficiency and H_2 -efficiency are coarser each other, then we say that they are *equivalent* and write $H_1\text{-eff} = H_2\text{-eff}$.

Relation (3.5) shows that $H_1 \subset H_2$ implies $H_1\text{-eff} \preceq H_2\text{-eff}$. Example 3.2 shows that the relation $H_1\text{-eff} = H_2\text{-eff}$ does not imply any inclusion relation for the sets H_1 and H_2 .

Being faced with two H -efficiencies, it is desirable to know the conditions when they are in some relation. Then we have the following result:

Proposition 4.1: *Let H_1 and H_2 be nonempty sets in $K \setminus \{0\}$. Then the following assertions are equivalent:*

- (a) $H_1\text{-eff} \subseteq H_2\text{-eff}$,
- (b) $H_1 \subseteq H_2 + K$,
- (c) *for every $x \in H_1$ it holds $(x - K) \cap H_2 \neq \emptyset$.*

Proof: Let us see first that (a) $\Leftrightarrow$ (b). According to (3.2) and (3.5) the relation (b) implies for an arbitrary set M in Y that

$$H_2\text{-eff } M = (H_2 + K)\text{-eff } M \subseteq H_1\text{-eff } M,$$

wherefrom we have (a).

To prove the converse inclusion let us assume that

$$x \in H_1 \setminus (H_2 + K) \tag{4.1}$$

and consider the set $M = \{0, x\}$. Then obviously

$$0 \in (x - H_1 - K) \cap M,$$

that is, $x \notin H_1\text{-eff } M$. Let us show that on the other hand

$$x \in H_2\text{-eff } M.$$

If this isn't true, it would hold either $0 \in x - H_2 - K$, or $x \in x - H_2 - K$. In the first case one would have $x \in H_2 + K$ contradicting (4.1). In the second case it would follow $0 \in H_2 + K$ which contradicts the condition $H_2 \subseteq K \setminus \{0\}$.

In consequence we have got a set M so that $H_2\text{-eff } M \not\subseteq H_1\text{-eff } M$ which shows that (a) cannot hold.

The equivalence (b) $\Leftrightarrow$ (c) is obvious. ■

Corollary 4.2: *We have*

$$H_1\text{-eff} = H_2\text{-eff}$$

if and only if it holds

$$H_1 + K = H_2 + K.$$

Proof: The second relation in the corollary implies that $H_1 \subseteq H_2 + K$ and $H_2 \subseteq H_1 + K$. ■

Corollary 4.3: *If $H_1\text{-eff} = H_2\text{-eff}$, then it follows that $\text{eff } H_1 = \text{eff } H_2$.*

Proof: By the remark after the definition of the efficient points in Section 2, it follows that $\text{eff } H_i = \text{eff } (H_i + K)$, $i = 1, 2$. Use now Corollary 4.2. to conclude the proof. ■

Proposition 4.4: *Suppose that $H_0 \subseteq H$. Then the following assertions are equivalent:*

- (a) $H_0\text{-eff} = H\text{-eff}$,
- (b) $H \subseteq H_0 + K$,

- (c) for every $x \in H$ one has $(x - K) \cap H_0 \neq \emptyset$,
 (d) there exists a set H_1 with $H_1 \cap (\text{eff } H + K) = \emptyset$, such that $H_0 \supset \text{eff } H \cup H_1$, where H_1 can be empty if $H \subset \text{eff } H + K$, whereas if the last inclusion does not hold, then H_1 is a set in $H \setminus (\text{eff } H + K)$ having the property that for each x in the last set $(x - K) \cap H_1$ is infinite.

Proof: The equivalence of (a), (b) and (c) is contained in Proposition 4.1.

Let us see that (b) and (c) imply (d). We have by (b) that $\text{eff } H \subset H_0 + K$, wherefrom using $H_0 \subset H$ and the characterization (2.1) of the points in $\text{eff } H$ we see that $\text{eff } H \subset H_0$. Hence the only thing which we have also to verify is that for $H \cap (\text{eff } H + K) \neq \emptyset$, H_0 contains a subset H_1 which is disjoint from $\text{eff } H + K$ such that it has the property at (d). (The condition that for $H \subset \text{eff } H + K$ the set H_1 can be empty will be important only in the proof of some converse implication.) Since $x \in H \setminus (\text{eff } H + K)$ implies that $(x - K) \cap \text{eff } H = \emptyset$, it suffices to verify the condition in (d) for H_0 itself. We have by (c) that $(x - K) \cap H_0 \neq \emptyset$. Let us show that this set is infinite. Assume the contrary. Then we have for some $y \in (x - K) \cap H_0$ that $(y - K) \cap H_0 = \{y\}$. If $y \notin \text{eff } H$, then $(y - K) \cap H$ must contain some element $z \in H$, $z \neq y$ for which obviously $(z - K) \cap H_0 = \emptyset$ which contradicts (c). If $y \in \text{eff } H$, then since $y \equiv x$ we should have that $x \in \text{eff } H + K$, contradicting the mode of definition of x . The obtained contradictions conclude the proof of (d).

Suppose now that (d) holds. If $H \subset \text{eff } H + K$, then the inclusion $\text{eff } H \subset H_0$ is sufficient to conclude (a) (that is, H_1 in (d) can be empty). Indeed, in this case we have $H_0 + K \subset H + K \subset \text{eff } H + K + K = \text{eff } H + K \subset H_0 + K$, that is $H_0 + K = H + K$ and using Corollary 4.2 we obtain that (a) holds.

Let x be in H and consider the intersection $(x - K) \cap H_0$. If $x \in H \setminus (\text{eff } H + K)$, then this set is obviously nonempty since $(x - K) \cap H_0 \supset (x - K) \cap H_1$ and the last set is nonempty (indeed infinite) by assumption. If $x \in \text{eff } H + K$ we have obviously $(x - K) \cap \text{eff } H \neq \emptyset$. Now, since $\text{eff } H \subset H_0$, it follows that $(x - K) \cap H_0 \neq \emptyset$, and we have the condition (c) fulfilled. ■

Observe that we can take in (d) $H_1 = H_0 \cap (H \setminus (\text{eff } H + K))$. If $H_1 \neq \emptyset$ then we can remove a finite set in H_1 without disturbing the conditions of (d). This implies the following results:

Corollary 4.5: *The following assertions are equivalent:*

- (a) $H \subset \text{eff } H + K$,
 (b) there exists a minimal subset H_0 of H such that $H_0\text{-eff} = H\text{-eff}$. In this case we must have $H_0 = \text{eff } H$.

Corollary 4.6: *If $\text{eff } H = \emptyset$ we can remove any finite set of H such to have for the remaining set H_0 the relation*

$$H_0\text{-eff} = H\text{-eff}. \quad (4.2)$$

If $H \not\subset \text{eff } H + K$ then we can remove some denumerable set of points in H such to have for the remaining set H_0 the relation (4.2). If for some finite set $H_1 \subset H$ one has $(H \setminus H_1)\text{-eff} \neq H\text{-eff}$, then necessarily $H_1 \cap \text{eff } H \neq \emptyset$.

5. The Existence of H -Efficient Points of Lower Bounded Sets

In obtaining general results on the existence of H -efficient points we must distinguish classes of sets in the space by some regularity conditions. It seems that the most natural ones are either the lower boundedness of sets or the existence of lower bounded sections for them. From the point of view of the existence of H -efficient points these two properties are similar. That is, if in an o.v.s. all the lower bounded sets have H -efficient points, then the same holds for all the sets with lower bounded sections. Hence, it follows that we shall consider the family of lower bounded sets with the remark that it can be replaced throughout by the family of sets having lower bounded sections.

Let (Y, K) be an o.v.s. The following technical result will be often used in our next proofs:

Proposition 5.1: *Let H be nonempty in $K \setminus \{0\}$. Then the following assertions are equivalent:*

- (a) *Every nonempty lower bounded set in Y possesses H -efficient points.*
- (b) *No set of the form $\left\{ \sum_{i=1}^n h_i \mid n \in \mathbb{N} \right\}$, with h_i (not necessarily distinct) elements in H , can be upper bounded.*

Proof: Let us show that (a) $\Rightarrow$ (b) by assuming that (b) does not hold. Then there exists some sequence (h_i) in H with $\left\{ \sum_{i=1}^n h_i \mid n \in \mathbb{N} \right\}$ upper bounded. Then $M = \left\{ - \sum_{i=1}^n h_i \mid n \in \mathbb{N} \right\}$ has a lower bound. Pick up $x \in M$ arbitrarily. Then $x = - \sum_{i=1}^m h_i$ for some m . We have

$$x - h_{m+1} = - \sum_{i=1}^{m+1} h_i \in M$$

and

$$x - h_{m+1} \in x - H \subset x - H - K.$$

Hence for arbitrary x in M we have the relation

$$(x - H - K) \cap M \neq \emptyset,$$

wherefrom H -eff $M = \emptyset$ and (a) cannot hold.

(b) $\Rightarrow$ (a): Assume that the set M is lower bounded and has no H -efficient points, i.e., that

$$(x - H - K) \cap M \neq \emptyset$$

for every x in M . Pick up $x_1 \in M$ arbitrarily and construct the sequence (x_n) by taking

$$x_{i+1} \in (x_i - H - K) \cap M, \quad i \in \mathbb{N}.$$

This relation implies that $x_{i+1} = x_i - h_i - k_i$ with some $h_i \in H$ and some $k_i \in K$. We have then that

$$h_i \leq x_i - x_{i+1}, \quad i \in \mathbb{N}.$$

By summing up one obtains

$$\sum_{i=1}^n h_i \leq x_1 - x_{n+1} \leq x_1 - y, \quad n \in \mathbb{N},$$

with y some lower bound of M . Then the obtained relations contradict condition (b) and the proof is complete. ■

Corollary 5.2: *Suppose that each nonempty lower bounded set in Y has H_j -efficient points for H_j nonempty. $H_j \subset K \setminus \{0\}$, $j = 1, \dots, m$. Then the same holds for H defined by*

$$H = \left\{ \lambda_1 h_1 + \dots + \lambda_m h_m \mid h_j \in H_j, \lambda_j \geq 0, j = 1, \dots, m; \sum_{j=1}^m \lambda_j = 1 \right\}.$$

Proof: Assume the contrary. Then by Proposition 5.1 it exists a sequence (y_i) in H such that $\left\{ \sum_{i=1}^n y_i \mid n \in \mathbb{N} \right\}$ is upper bounded. From the definition of the set H we have that $y_i = \lambda_1^{(i)} h_1^{(i)} + \dots + \lambda_m^{(i)} h_m^{(i)}$ with $h_j^{(i)} \in H_j$, $\lambda_j^{(i)} \geq 0$ and $\sum_{j=1}^m \lambda_j^{(i)} = 1$ for every i . Since for every i there is some j in $\{1, \dots, m\}$ such that $\lambda_j^{(i)} \geq 1/m$, it follows that there exists some subscript, say j_0 , and a subsequence $(h_{j_0}^{(i_k)})_{k=1}^\infty$ of $(h_{j_0}^{(i)})_{i=1}^\infty$ such that $\lambda_{j_0}^{(i_k)} \geq 1/m$ for each k . We have for this subsequence:

$$\frac{1}{m} \sum_{k=1}^n h_{j_0}^{(i_k)} \leq \sum_{i=1}^{i_n} y_i$$

for each n . From our hypothesis on the sequence (y_n) we deduce that the set $\left\{ \sum_{k=1}^n h_{j_0}^{(i_k)} \mid n \in \mathbb{N} \right\}$ is order bounded. This conclusion contradicts by Proposition 5.1 the assumption on H_{j_0} . ■

Corollary 5.3: *If every nonempty lower bounded set in Y has H_j -efficient points for $j = 1, \dots, m$, then such every set has $(H_1 \cup \dots \cup H_m)$ -efficient points too.*

Proof: From the fact that $H_1 \cup \dots \cup H_m \subset H$ with H defined in Corollary 5.2 and by using the relation (3.5) the proof follows from the cited corollary. ■

Corollary 5.2 particularly shows that if H is a finite set and if every lower bounded set in Y has H -efficient points, then the same is true for $(\text{co } H)$ -efficient points. Let us see that such an assertion does not hold for an arbitrary set H .

Example 5.4: Consider the vector space c of all the convergent real sequences, ordered by the cone c^+ of sequences with nonnegative terms. Put $h_n = (\delta_n^m)_{m=1}^\infty$ for $n = 1, 2$ (here δ_n^m stands for the KRONECKER symbol), and $h_n = (\delta_n^m 2^{k-1})_{m=1}^\infty$ for n satisfying $2^{k-1} < n \leq 2^k$ ($k \geq 2$). Then $H = \{h_n \mid n \in \mathbb{N}\}$ has obviously the property that no set of form $\left\{ \sum_{k=1}^n h_{i_k} \mid h_{i_k} \in H, n \in \mathbb{N}, i_k \text{ not necessarily distinct} \right\}$ can be upper bounded.

Consider now the elements

$$y_m = \frac{1}{2^m} \sum_{i=1}^{2^m} h_i, \quad m \in \mathbb{N}.$$

A direct calculation yields that every sequence of the form

$$y_1 + \dots + y_m$$

has its terms < 1 . Hence the set

$$\left\{ \sum_{i=1}^m y_i \mid m \in \mathbb{N} \right\}$$

is upper bounded by $y = (1, 1, \dots)$. From Proposition 5.1 we have that every lower bounded set in c has H -efficient points, while the same proposition together with the above construction show that this is not true referring to the $(\text{co } H)$ -efficient points.

The construction in the above example uses the fact that H is infinite dimensional. For finite dimensional sets H we have indeed the assertion:

Corollary 5.5: *Let H be a nonempty finite dimensional set in $K \setminus \{0\}$. If every nonempty lower bounded set in Y has H -efficient points, then such every set has also $(\text{co } H)$ -efficient points.*

Proof. Suppose that $\dim H = m$. Then by the theorem of CARATHEODORY (see Theorem 2.2.12 in [22]) for every $y \in \text{co } H$ there exist some elements $h_1, \dots, h_{m+1}$ in H such that y is their convex combination. Hence we can conclude the proof by using Corollary 5.2 with $m+1$ instead of m and $H_1 = \dots = H_{m+1} = H$. ■

6. Finite H -Efficiency and Semi Archimedian Property

The o.v.s. (Y, K) and its positive cone K are called almost Archimedian if $y = 0$ whenever $-tz \leq y \leq tz$ for some z in K and all the reals $t > 0$. They are called semi Archimedian if from $0 \leq ty \leq z$ for some z in K and all $t > 0$ it follows that $y = 0$. The almost Archimedian property of an o.v.s. implies the semi Archimedian property of it. Simple examples show that the converse implication does not always hold even for ordered vector spaces of dimension two [14]. It is straightforward that the condition in the definition of the semi Archimedian property can be rewritten in the form: from $0 \leq ny \leq z$ for some z in K and all $n \in \mathbb{N}$, it follows that $y = 0$.

According to the Propositions II. 1.29 and II. 1.31 in [19], if there exists some locally convex topology on Y such that the closure of K with respect to this topology is also a cone, then K is almost Archimedian. Hence this property and so much the more, the semi Archimedian property for an ordered vector space is very mild.

The semi Archimedian property links to H -efficiency by the following result which constitutes an extension of Proposition 2 in [14]:

Theorem 6.1: *The following assertions are equivalent:*

- (a) (Y, K) is semi Archimedian;
- (b) Every nonempty lower bounded set in Y has H -efficient points for each H with the property that there exists a finite set $H_0 \subset K \setminus \{0\}$ such that $H \subset \text{co } H_0 + K$;
- (c) Every nonempty lower bounded set in Y has ε -efficient points for every $\varepsilon \in K \setminus \{0\}$.

Proof: (a) $\Rightarrow$ (b). By the relations (3.2), (3.5) and the Corollary 5.2 it is obvious that it suffices to carry out the proof for the finite set $H_0 = \{h_1, \dots, h_m\}$. According to Proposition 5.1 we have to show that if (Y, K) is semi Archimedian, then no set of the form $\left\{ \sum_{k=1}^n h_{i_k} \mid i_k \in \{1, \dots, m\}, n \in \mathbb{N} \right\}$ can be upper bounded. If this is not true there would exist a subscript $j_0 \in \{1, \dots, m\}$ such that the set $\{nh_{j_0} \mid n \in \mathbb{N}\}$ would be upper bounded. Since (Y, K) is semi Archimedian this would imply that $h_{j_0} = 0$, contradicting the hypothesis on H_0 .

The implication (b) $\Rightarrow$ (c) is obvious.

Let us prove finally that (c) $\Rightarrow$ (a) by assuming that (Y, K) is not semi Archimedian, i.e., that there exist some x and y in K , $x \neq 0$, such that

$$0 \leq tx \leq y \quad (6.1)$$

for any $t > 0$. Consider the set M defined by

$$M = \{ry - sx \mid r, s > 0\}.$$

Then by (6.1), $M \subset K$ and hence it is lower bounded. Let z be an arbitrary element of M . Then $z = ry - sx$ for some positive r and s . We have obviously $z - x \in z - x - K$ and $z - x = ry - (s+1)x \in M$. Thus for each $z \in M$ it holds

$$(z - x - K) \cap M \neq \emptyset$$

which shows that condition (c) does not hold for $\varepsilon = x$. ■

Remarks 6.2: (a) The lower boundedness or the existence of lower bounded sections of a set are not necessary conditions in order to that set has H -efficient points. To illustrate this consult Example 3.3. The cited example shows that the sets without lower bounded sections which admit H -efficient points must be of a rather special form.

(b) The condition on H at the point (b) of the theorem can be still weakened; we have used the above form for the sake of simplicity.

7. Normal and Regular Cones

Let (Y, K) be an o.v.s. The set $[x, y] = \{z \in Y \mid x \leq z \leq y\}$ is called the *order interval* defined by x and y . The set M in Y is called *full* if for any x and y in M the order interval $[x, y]$ is contained in M .

If Y is endowed with a HAUSDORFF locally convex vector space topology and if K is closed with respect to this topology, then following [19] we call (Y, K) an *ordered locally convex space* (abbreviated o.l.c.s.).

The o.l.c.s. (Y, K) is called *normal* (or locally full) if Y possesses a base of neighbourhoods consisting of full sets.

A *decreasing* (*increasing*) *net* in (Y, K) is by definition a set which is totally ordered by the order relation of the space and is considered to be directed downward (upward) by it. (Note this usage of the notion of decreasing (increasing) net in

ordered vector space theory. By a difference, directed nets are sometimes called directed sets—see e.g. [19, Chapter 2, § 3].) Such a net is called **CAUCHY** or respectively *convergent* if the filter of its lower (upper) sections is **CAUCHY** or respectively convergent. Nets with a denumerable set of elements will be called *sequences*.

The o.l.c.s. (Y, K) is called *regular* (*fully regular*) if every decreasing lower bounded net (respectively every decreasing topologically bounded net) in Y is convergent. If we require to each net of above type be only **CAUCHY**, then (Y, K) is called *quasi regular* (*quasi fully regular*). Decreasing lower bounded nets in the above definitions can be obviously changed in upper bounded increasing nets. In the definition of (quasi) regularity we can only consider decreasing nets in K . When the space is metrizable we can use sequences in the above definitions. We shall see in this section that this can be done indeed for the general case too. Regular spaces are sometimes called **DANIELL** spaces (see e.g. [1]).

The above introduced terms will be also used for the positive cone of the space.

It is well known (see [9, Theorem 1.6]) that ordered regular **BANACH** spaces are normal. The same is true for **FRÉCHET** spaces (see [13, Theorem 3]). **MCARTHUR** has shown in [12] that if an ordered normal **FRÉCHET** space does not contain any subspace isomorphic with c_0 (the space of real sequences converging to 0 endowed with the uniform norm), then it is regular.

The completeness of an ordered regular **FRÉCHET** space is essential to its normality. Let us remember an example of a normed regular ordered vector space which is not normal [16]. It is an adaptation of an example due to **BRECKNER** and **ORBÁN** which was used for other purposes in [2, Remark 2.4.3].

Example 7.1: Let Y denote the vector space of all polynomials with real coefficients defined on $[-1, 0]$, equipped with the uniform norm. Let us consider the cone K in Y defined by $K = \{x \in Y \mid x^{(i)}(0) \geq 0 \text{ for all non-negative integers } i\}$. The cone K is regular since if (x_n) is a decreasing sequence in K , then it is necessarily a sequence of polynomials of a given finite order, say m , and according to the order induced by K , their coefficients form decreasing non-negative sequences of real numbers. Since all the coefficients of the terms of order $> m$ are 0, the sequence (x_n) tends uniformly to a polynomial with nonnegative coefficients, and the regularity of (Y, K) follows.

To prove that K is not normal, consider the sequences (x_n) and (y_n) defined by

$$x_n(t) = \frac{1}{2n} \sum_{i=1}^n \binom{2n}{2i} t^{2i} \quad \text{and} \quad y_n(t) = \frac{1}{2n} (1+t)^{2n}, \quad t \in [-1, 0].$$

One has $0 \leq x_n \leq y_n$ for each $n \in \mathbb{N}$. The sequence (y_n) tends uniformly to 0, but (x_n) does not. This contradicts a well-known criterion of normality (see e.g. [19, Proposition II. 1.3]).

The techniques we shall use in the next sections are based on the fact that in various types of regularity we can deal with sequences only. This allows us to use summation criteria. The following lemma is the basic tool in all the next proofs.

Lemma 7.2: *The following assertions are equivalent:*

- (a) *K is quasi regular;*
- (b) *Every decreasing sequence in K is CAUCHY;*
- (c) *For any sequence (x_n) in K the condition $x_n \notin U$ for every n and for some neighbourhood U of 0, implies that the set $\left\{ \sum_{i=1}^n x_i \mid n \in \mathbb{N} \right\}$ cannot be upper bounded.*

If the cone K is complete, the term “quasi regular” in (a) can be changed into “regular”.

Proof: The implication (a) $\Rightarrow$ (b) is obvious. Let us prove that (b) $\Rightarrow$ (a). Assume that (a) does not hold. Then there exists a decreasing net (x_μ) in K which is not CAUCHY. This means that there exists some neighbourhood U of 0 such that for any subscript μ there exist $\nu \geq \mu$ and $\eta \geq \mu$ such that $x_\nu - x_\eta \notin U$. Fix an arbitrary subscript and denote the corresponding element by x_1 . Then there exist some elements x_2 and x_3 of the net such that $x_2 \leq x_1$, $x_3 \leq x_2$ and $x_2 - x_3 \notin U$. Start now with x_3 in place of x_1 to get two elements x_4 and x_5 of the net with $x_5 \leq x_4 \leq x_3$ and $x_4 - x_5 \notin U$. Continuing this process we find a decreasing sequence (x_n) in K which is not CAUCHY. This contradicts (b) and we proved the implication.

If (c) isn't true, there would exist a sequence (y_n) in K and a neighbourhood U of 0 such that $y_n \notin U$ for any n , for which $\left(\sum_{i=1}^n y_i \right)_{n=1}^\infty$ would be an increasing upper bounded sequence. But such a sequence is obviously not CAUCHY, and (b) isn't true too. This proves (b) $\Rightarrow$ (c).

Let us show finally that (c) $\Rightarrow$ (b). If K is not quasi regular then we can contradict (c) as follows. The negation of (b) implies the existence of a decreasing sequence (z_n) in K which is not CAUCHY. This means that there exists some neighbourhood V of 0 such that for each subscript i_0 there exist $i \geq i_0$ and $j \geq i_0$ such that $z_i - z_j \notin V$. Take the balanced neighbourhood U of 0 so as to have $U + U \subset V$. If z_{i_1} is an arbitrary element of the above sequence then there is an $i > i_1$ for which $z_i - z_{i_1} \notin U$, for if this would not be true, then $z_i - z_{i_1} \in U$ and $z_j - z_{i_1} \in U$ for any i and $j > i_1$ and this would contradict the above assumption since it would imply that $z_j - z_i \in U + U \subset V$ for any such i and j . Hence we can construct a subsequence of (z_n) , denote it by (s_n) , such that $s_n - s_{n+1} \notin U$, $s_{n+1} \leq s_n$ for every n . Take $y_n = s_n - s_{n+1}$, $n \in \mathbb{N}$. Then (y_n) is a sequence in K which contradicts (c) and the proof is complete. ■

The method furnished by this lemma was used in our papers [14], [15] and [17]. In the last two papers in fact a more general assertion was considered which we shall state in Section 11.

8. Regularity and H -Efficiency

The advantage of the notion of H -efficiency derives from the fact that it links in a natural way with the regularity of the space. The ideas in this regard, originated in Propositions 3 and 4 in [14] when the ordered space is normed and in Proposi-

tions 13 and 14 in [15] for locally convex spaces, can be summarized and completed as follows:

Theorem 8.1: *Given an o.l.c.s. (Y, K) the following assertions are equivalent:*

- (a) *The space (Y, K) is quasi regular;*
- (b) *Every nonempty lower bounded set M in Y possesses H -efficient points for every nonempty H in K such that $Y \setminus H$ is a neighbourhood of 0;*
- (c) *Every nonempty denumerable lower bounded set M in Y possesses H -efficient points for every nonempty denumerable H in K such that $Y \setminus H$ is a neighbourhood of the origin.*

If K is complete, "quasi regular" in (a) can be changed into "regular".

If the space (Y, K) is normal, then (c) can be changed into:

- (c') *Every nonempty, denumerable, complete lower bounded set M in Y possesses H -efficient points for every nonempty denumerable H in K such that $Y \setminus H$ is a neighbourhood of 0.*

Proof: To prove the implication (a) $\Rightarrow$ (b) we assume that some nonempty lower bounded set M in Y does not have H -efficient points for some nonempty H in $K \setminus U$, with U a neighbourhood of 0. Then by Proposition 5.1 there exists some sequence (h_i) of (not necessarily distinct) elements in H so that the set $\left\{ \sum_{i=1}^n h_i \mid n \in \mathbb{N} \right\}$ is upper bounded. This implies by Lemma 7.2 that (a) does not hold.

The implication (b) $\Rightarrow$ (c) is obvious.

Let us prove next (c) $\Rightarrow$ (a) by assuming that (Y, K) is not quasi regular. Using Lemma 7.2 again we conclude that there exist a neighbourhood U of 0 and a sequence (x_i) with $x_i \in K \setminus U$, for each $i \in \mathbb{N}$, such that the sums $\sum_{i=1}^n x_i$, $n \in \mathbb{N}$ have an upper bound y . That is, the set

$$M = \left\{ - \sum_{i=1}^n x_i \mid n \in \mathbb{N} \right\}$$

is bounded from below by $-y$. Let x be an arbitrary element of M . Then $x = - \sum_{i=1}^m x_i$ for some m . We have further

$$x - x_{m+1} = - \sum_{i=1}^{m+1} x_i \in M \quad \text{and}$$

$$x - x_{m+1} \in x - H \subset x - H - K$$

with $H = \{x_i \mid i \in \mathbb{N}\}$. That is we have got that

$$(x - H - K) \cap M \neq \emptyset$$

for every x in M . Hence for the denumerable lower bounded set M in Y and the denumerable $H \subset K \setminus U$, the condition (c) does not hold.

If K is complete then (Y, K) is regular if and only if it is quasi regular and hence "quasi regular" in (a) can be obviously changed this case into "regular".

To prove the last assertion of the theorem it suffices to show that if (Y, K) is normal, then the set M constructed in the above proof of (c) $\Rightarrow$ (a) is in fact complete. We can assume in this case that the neighbourhood U considered there is full. Then from $u, v \in K \setminus U$ it follows that $u + v \in K \setminus U$. Hence we have that for the two arbitrary elements $-\sum_{i=1}^n x_i$ and $-\sum_{i=1}^{n+p} x_i$ of M , their difference $\sum_{i=1}^p x_{n+i}$ is not in U . Thus the set M is uniformly discrete and hence complete. ■

Remarks 8.2: (a) Theorem 8.1 together with Theorem 6.1 imply a theoretical conclusion about the notion of H -efficiency. Although intimately related to the order relation of the space, H -efficiency for the family of lower bounded sets and for sets H in K with $Y \setminus H$ a neighbourhood of the origin, does not give a fine characterization of this order relation by varying H . Indeed, the two extreme cases are: (i) when H is finite; then the existence of H -efficient points for lower bounded sets characterizes the semi Archimedean spaces (Theorem 6.1) and hence it takes obviously place for o.l.c. spaces (every closed cone is also semi Archimedean; see the observation at the beginning of Section 6); (ii) when H is infinite; then the existence of H -efficient points for lower bounded sets characterizes quasi regular o.l.c. spaces (Theorem 8.1).

(b) We shall see in the next section that condition (c') in Theorem 8.1 plays an important role. This condition uses the additional hypothesis of the normality of the space. To show the essentiality of this hypothesis we have to construct a regular not normal o.l.c.s. in which every complete denumerable lower bounded set has H -efficient points for any H as in the theorem, but there exist some such H and some denumerable not complete lower bounded sets without H -efficient points. Such a space, so far as it exists, is of a very special feature. However, we can see that the normality is essential in the method of deducing (c') $\Rightarrow$ (a) in the above proof. This will be done by considering the following example:

Example 8.3: Let K be a closed not normal cone in the BANACH space Y . Then by the equivalent definition of normal cones in [8], it follows that for every n there exist some u_n and v_n in K such that $\|u_n\| = \|v_n\| = 1$ and $\|u_n + v_n\| < 1/2^n$. Consider the series $\sum_{n=1}^{\infty} (u_n + v_n)$. It obviously converges to some $z \in K$. Denote $z_{2n} = -\sum_{i=1}^n (u_i + v_i)$ and $z_{2n-1} = z_{2n} + v_n$. Then (z_k) is obviously a decreasing sequence which is bounded from below by $-z$. If we consider $H = \{u_n, v_n \mid n \in \mathbb{N}\}$ then we have the construction in the proof of (c') $\Rightarrow$ (a), but here the set $\{z_k \mid k \in \mathbb{N}\}$ is not complete. Indeed (z_{2n}) converges to $-z$ which does not belong to $\{z_k \mid k \in \mathbb{N}\}$. This last assertion follows for instance from the fact that this set has no H -efficient point, while $-z$ is an infimum of it by Proposition II.3.1 in [19] and hence it is obviously an H -efficient point of the set $\{z_k \mid k \in \mathbb{N}\} \cup \{-z\}$.

Thus the absence of normality of (Y, K) makes inefficient the construction in the proof of $(c') \Rightarrow (a)$, construction which yields a contradiction with (c') in that proof.

Let us remark finally that the above reasoning is the same as that in the proof of Theorem 1.6 by KRASNOSEL'SKII in [9] of the fact that in a BANACH space regular cones are normal. Indeed, if K is not normal, then the above construction yields a decreasing lower bounded sequence in K which is not CAUCHY.

9. Efficient Points of Lower Bounded Sets

Theorem 8.1 allows us to deduce necessary and sufficient conditions for the existence of efficient points. We have in this regard the following result:

Theorem 9.1: *Let (Y, K) be a normal o.l.c.s. Then the following assertions are equivalent:*

- (a) (Y, K) is quasi regular;
- (b) Each nonempty complete lower bounded set possesses efficient points;
- (c) Each nonempty denumerable complete lower bounded set possesses efficient points.

If K is complete then "quasi regular" in (a) can be changed into "regular", and "complete" in (b) and (c) can be changed into "closed".

Proof: The implication $(a) \Rightarrow (b)$ is in fact (a) in Theorem 1 of [1]. The proof uses ZORN's lemma which turns out to be essential (see Section 3 of the cited paper).

$(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$: Since each complete lower bounded denumerable set in Y has efficient points, then it has also H -efficient points for every H in $K \setminus \{0\}$. Hence we can apply (c') of Theorem 8.1 to deduce that (Y, K) is quasi regular.

As to the last part of the theorem, let us note that if K is complete and quasi regular, then it is regular, and lower sections of closed sets are complete. ■

Remarks 9.2: (a) Concerning the hypothesis of normality of the space (Y, K) in Theorem 9.1 we refer to (b) in Remarks 8.2.

(b) An interesting question which is not touched by Theorem 9.1 is the relation of the condition $\text{eff } \bar{M} \neq \emptyset$ (where $\bar{M}$ is the closure of M) with the condition $H\text{-eff } M \neq \emptyset$. Example 8.3 shows that we can construct in a BANACH space ordered by a not normal closed cone, a decreasing lower bounded sequence (z_n) so as to have for $M = \{z_n \mid n \in \mathbb{N}\}$ and $H = \{z_n - z_{n+1} \mid n \in \mathbb{N}\} \subset K \setminus (\text{int } B)$ (with B the unit ball), that $H\text{-eff } M = \emptyset$, whereas $\text{eff } \bar{M} \neq \emptyset$. The possibility of the construction of some M with a similar property is not essentially related to the lack of normality of the space. To illustrate this, let us consider the following example:

Example 9.3: Let be $Y = \mathbb{R}^2$ and suppose that it is ordered by $K = \{(x, y) \mid x \geq 0, y \geq 0\}$. Consider the set $M = (K \setminus \{(0, 0)\}) \cup \{(x, y) \mid xy \leq -1\}$. If we take $H = \{(1, 0),$

$(0, 1)\}$, then $H\text{-eff } M = \emptyset$, while $\text{eff } \bar{M} = \{(0, 0)\}$. Note that M has no lower bounded sections.

When Y is a BANACH space, we can avoid the hypothesis of normality in Theorem 9.1. In this case we have:

Theorem 9.4: *Let Y be a BANACH space ordered by the closed cone K . Then the following assertions are equivalent:*

- (a) (Y, K) is regular;
- (b) Each closed nonempty lower bounded set in Y possesses efficient points;
- (c) Each closed nonempty denumerable lower bounded set in Y has efficient points.

Proof: The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) follow as in Theorem 9.1. We have only to show that (c) $\Rightarrow$ (a). To end this we prove first the following auxiliary result:

Lemma 9.5: *Let Y be a BANACH space and let K be a closed cone therein which is not normal. Then there exists a totally ordered, denumerable and closed lower bounded set in Y without efficient points.*

Proof: Let us repeat the construction at Example 8.3: Pick up u_n and v_n in K such that $\|u_n\| = \|v_n\| = 1$, $\|u_n + v_n\| < 1/2^n$ and put $z_{2n} = -\sum_{i=1}^n (u_i + v_i)$, $z_{2n-1} = z_{2n} + v_n$. Let $z = -\sum_{i=1}^{\infty} (u_i + v_i)$. Then (z_k) is decreasing and lower bounded by z . Since z is a limit point of the set $\{z_k \mid k \in \mathbb{N}\}$, one has by Proposition II. 3.1 in [19], that $z = \inf \{z_k \mid k \in \mathbb{N}\}$. We shall show that the set $\{z_{2n-1} \mid n \in \mathbb{N}\}$ cannot have any limit point. If it would have such a point w , then there would exist some subsequence $(z_{2n_k-1})_{k=1}^{\infty}$ which converges to w . Hence by passing to limit in

$$z_{2n_k} \leq z_{2n_k-1} \leq z_{2n_k-2}$$

with $k \rightarrow \infty$, we would get $z = w$. Thus, we should have $\lim_{k \rightarrow \infty} z_{2n_k} = \lim_{k \rightarrow \infty} z_{2n_k-1} = z$, which would contradict the condition that

$$\|z_{2n_k} - z_{2n_k-1}\| = \|v_{n_k}\| = 1.$$

It follows that (z_{2n-1}) is a decreasing sequence with the lower bound z and without limit points. Hence $\{z_{2n-1} \mid n \in \mathbb{N}\}$ is a totally ordered, closed, lower bounded set which cannot have any efficient point. ■

Let us conclude now the proof of (c) $\Rightarrow$ (a) in Theorem 9.4 as follows. Since each denumerable, closed, lower bounded set in Y has efficient points, by Lemma 9.5. (Y, K) must be normal. According to Theorem 9.1 in a normal o.l.c.s. (c) implies (a). ■

Remark 9.6: Observe that we cannot use in the proof KRASNOSEL'SKII's result on normality of regular cones, since we have in fact to prove regularity but for this we need the normality. The used procedure can perhaps be adapted to obtain a similar result for FRÉCHET spaces. But we cannot handle yet the general case.

10. Regular Sets in Normal o.l.c. Spaces

The existence results on efficient points exploit explicitly [1] or implicitly [7] the convergence of some monotone and order bounded sets. Introducing the notion of regular sets we can handle in a unified way some different aspects of this theory.

Definition 10.1: Let (Y, K) be an o.l.c.s. The set M in Y will be called (*quasi*) *regular* if every decreasing net in M converges to some element of M (respectively, is **CAUCHY**).

It is immediate from this definition that a complete quasi regular set is regular. If K is complete then every section of a closed set M is complete. Hence for complete K closed quasi regular sets are regular.

Following the way of the proof of Lemma 7.2 we can verify:

Lemma 10.2: Let (Y, K) be an o.l.c.s. Then the following assertions are equivalent:

- (a) The set M in Y is quasi regular;
- (b) Every decreasing sequence in M is **CAUCHY**;
- (c) We cannot have a sequence (y_i) with $y_i \in K \setminus U$ from a subscript i_0 on, where U is a neighbourhood of 0, so that the set $\left\{ - \sum_{i=1}^n y_i \mid n \in \mathbb{N} \right\}$ is in M .

If M is complete or if K is complete and M is closed then “quasi regular” at (a) can be replaced into “regular”, and “**CAUCHY**” at (b) can be changed into “convergent to some element of M ”.

Remarks 10.3: (a) The o.l.c.s. (Y, K) is quasi regular if and only if every lower bounded set in it is quasi regular. If K is complete, then (Y, K) is regular if and only if every closed lower bounded set in Y is regular.

(b) The reasoning in the proof of (c) in Theorem 1 of [1] shows that every compact set in an o.l.c.s. is regular.

By analogy with Theorem 8.1 we have

Theorem 10.4: Let (Y, K) be a normal o.l.c.s. and let M be a (complete) set in Y . Then the following assertions are equivalent:

- (a) M is quasi regular (regular);
- (b) Each subset of M which is nonempty has H -efficient points for every nonempty $H \subset K$ such that $Y \setminus H$ is a neighbourhood of 0;
- (c) Each closed nonempty denumerable subset of M has H -efficient points for every nonempty denumerable $H \subset K$ so that $Y \setminus H$ is a neighbourhood of the origin.

Proof: (a) $\Rightarrow$ (b): Assume that $M_0 \subset M$, $M_0 \neq \emptyset$, has no H -efficient point. Then we can construct by the way used in the proof of (a) $\Rightarrow$ (b) of Theorem 8.1 a decreasing sequence (x_i) in M such that $h_i \leq x_i - x_{i+1}$, $i \in \mathbb{N}$ for some $h_i \in H$. We can suppose that $H \subset K \setminus U$ with U some full neighbourhood of 0 since (Y, K) is normal. Because $h_i \notin U$ the last relation implies that

$$x_i - x_{i+1} \notin U, \quad i \in \mathbb{N}.$$

Put $y_i = x_i - x_{i+1}$ and observe that

$$x_{n+1} = - \sum_{i=1}^n y_i + x_1 \in M, \quad n \in \mathbb{N},$$

with $y_i \in K \setminus U$ and by Lemma 10.2 we see that M cannot be quasi regular (regular).

The proofs of $(b) \Rightarrow (c) \Rightarrow (a)$ run similar to the proofs of $(b) \Rightarrow (c') \Rightarrow (a)$ in Theorem 8.1. ■

Remark 10.5: As a difference of the proof of Theorem 8.1 here we cannot dispense with the hypothesis of normality at the proof of $(a) \Rightarrow (b)$. This condition is essential since we have to conclude from $h_i \leq x_i - x_{i+1}$ and $h_i \notin U$ that $x_i - x_{i+1} \notin U$ which may not hold if U is not full. The normality is also used in the proof of $(c) \Rightarrow (a)$ (see the proof of $(c') \Rightarrow (a)$ in Theorem 8.1 and the comments in Remark 8.2 (b)).

By the way we deduced Theorem 9.1 from Theorem 8.1, we can deduce from Theorem 10.4 the following result:

Theorem 10.6: Let (Y, K) be a normal o.l.c.s. and let M be a complete set in Y , or let K be complete and M be closed. Then the following assertions are equivalent:

- (a) The set M is regular;
- (b) Each closed subset of M which is nonempty, has efficient points.
- (c) Each nonempty totally ordered denumerable and closed subset of M possesses efficient points.

11. K Bound Regular Subcones of the Cone K

Let (Y, K) be an o.l.c.s.

Definition 11.1: Let K_0 be a cone contained in K . It is called (quasi) K bound regular subcone of K if every K_0 increasing K order bounded net in K_0 converges to an element of K_0 (respectively it is CAUCHY).

If K_0 is (quasi) K bound regular then it is (quasi) regular too; the converse of this assertion is not always true (see Example 11.2 (a) below).

If K_0 is a (quasi) K bound regular subcone of K , then it is (quasi) K_1 bound regular for every cone K_1 such that $K_0 \subset K_1 \subset K$. Every subcone of such a cone K_0 is itself quasi K bound regular, and every complete subcone of K_0 is K bound regular.

Examples 11.2: (a) Let c be the BANACH space of all the convergent sequences of real numbers endowed with the usual norm and let c_0 be its subspace consisting of sequences converging to 0. Let c^+ be the cone of sequences with non-negative terms in c , and put $c_0^+ = c_0 \cap c^+$. Then c^+ is normal and it is not regular. The cone c_0^+ is regular but it is not c^+ bound regular subcone of c^+ .

To see that c_0^+ is regular we consider the increasing sequence $(x^n)_{n=1}^\infty$ in c_0^+ with the upper bound x , where $x^n = (x_1^n, \dots, x_j^n, \dots)$; $x = (x_1, \dots, x_j, \dots)$, $n = 1, 2, \dots$. Let us

show that (x^n) is a CAUCHY sequence. Suppose that $\varepsilon > 0$ is arbitrary. Since $x \in c_0^+$, there exists a subscript m_0 such that for every $j > m_0$ one has $0 \leq x_j < \varepsilon/2$. From our monotonicity assumption it follows that $(x_j^n)_{n=1}^\infty$ is for each j an increasing sequence of non-negative real numbers bounded from above by x_j . Hence each sequence of this kind is a CAUCHY sequence. Let us choose an n_0 such that $|x_j^p - x_j^q| < \varepsilon$ whenever $p \geq n_0$ and $q \geq n_0$ and $1 \leq j \leq m_0$. We have for $j > m_0$ that $|x_j^p - x_j^q| \leq |x_j^p - x_j| + |x_j - x_j^q| = (x_j - x_j^p) + (x_j - x_j^q) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, since $0 \leq x_j - x_j^p \leq x_j < \varepsilon/2$ and $0 \leq x_j - x_j^q \leq x_j < \varepsilon/2$. Hence $|x_j^p - x_j^q| < \varepsilon$ for each j . This shows that $\|x^p - x^q\| < \varepsilon$ for any $p, q \geq n_0$, that is (x^n) is a CAUCHY sequence converging to some $x_0 \in c_0^+$ and the regularity of c_0^+ was deduced.

Consider the elements $y = (1, \dots, 1, \dots)$; $y^n = (1, \dots, 1, \underbrace{0, \dots}_{n \text{ times}})$. Then $y \in c^+$ and $y_n \in c_0^+$, $n = 1, 2, \dots$. The sequence (y^n) is c_0^+ increasing in c_0^+ and is c^+ upper bounded by y . Since this sequence is not CAUCHY, the cone c_0^+ is not c^+ bound regular subcone of c^+ .

Note that by the same way it follows that c_0^+ is not fully regular and that c^+ is not regular.

(b) Every fully regular subcone of a normal cone K is K bound regular. This follows from the fact that order bounded sets in normal spaces are topologically bounded. Accordingly K bound regular subcones can be constructed in the way it was done in Section 2 of [17].

The converse of the above assertion is not true: there exist K bound regular subcones of a normal cone K which are not fully regular. The simplest examples of this kind are regular cones in BANACH spaces which are not fully regular. For instance the cone c_0^+ considered at (a) is normal, regular but not fully regular. For other examples of this kind consult 6.3 in [9].

(c) Let us consider the space $Y = C^\infty[-1, 0]$ of infinitely differentiable real functions endowed with the uniform norm. Let K be the cone in Y of the functions x with $x^{(i)}(0) \geq 0$, for all the non-negative integers i . If $K_0 = K \cap Y_0$, where Y_0 is the subspace in Y of the polynomials, then we have seen at Example 7.1 that K_0 is regular but it is not normal. It is easy to show that K_0 is not K bound regular. If we consider the subcone K_1 of K defined by

$$K_1 = \{x \in K \mid x^{(i)}(0) > 0 \text{ only for a finite number of } i - s\}$$

then $K_0 \subset K_1 \subset K$, K_0 is K_1 bound regular subcone of K_1 . The cone K_1 is the maximal subcone in K with respect to which K_0 is K_1 bound regular.

By slight modifications of the proof of Lemma 7.2 we can verify the following auxiliary result (see also [15] and [17]):

Lemma 11.3: *Let (Y, K) be an o.l.c.s. The following assertions are equivalent:*

- (a) *The subcone K_0 in K is quasi K bound regular;*
- (b) *Every K_0 increasing K bounded sequence in K_0 is CAUCHY;*

(c) For any sequence (y_i) in K_0 the conditions $y_i \notin U$ for some neighbourhood U of 0 and for every i , imply that the set $\left\{ \sum_{i=1}^n y_i \mid n \in \mathbb{N} \right\}$ cannot be K order bounded.

Using this lemma in the mode we have done with Lemma 7.2 in the proof of Theorem 8.1, we obtain:

Theorem 11.4: Given an o.l.c.s. (Y, K) and a subcone K_0 of K , the following assertions are equivalent:

- (a) The subcone K_0 is quasi K bound regular;
- (b) Every nonempty K lower bounded set in Y possesses H -efficient points for every nonempty H in K_0 such that $Y \setminus H$ is a neighbourhood of 0.
- (c) Every nonempty denumerable K lower bounded set in Y possesses H -efficient points for every denumerable nonempty set H in K_0 such that $Y \setminus H$ is a neighbourhood of 0.

12. H -Efficiency for a Fixed H , and K Bound Regularity

After the universal results which link the existence of H -efficient points of every lower bounded set and each H with $Y \setminus H$ a neighbourhood of 0 on the one hand, and the regularity or the K bound regularity of the positive cone on the other (see Theorems 7.1 and 11.4), we consider at this point the situation when H is fixed. More precisely, we seek through the question about the conditions of the existence of H -efficient points for every lower bounded set when H is fixed. We have in this direction the following result which follows directly from Theorem 11.4:

Proposition 12.1: Consider the o.l.c.s. (Y, K) . If the nonempty set H in K has the property that $Y \setminus H$ is a neighbourhood of 0 and $K_0 = \text{cone } H$ is a quasi K bound regular subcone of K , then every nonempty K lower bounded set in Y possesses H -efficient points.

Remark 12.2: The converse of the assertion in Proposition 12.1 does not hold in general. That is, there exist sets H in K with the property that $Y \setminus H$ is a neighbourhood of 0 and that every lower bounded set in Y which is nonempty, possesses H -efficient points, but $K_0 = \text{cone } H$ is not K bound regular. This can be seen by a reinterpretation of Example 5.4, which says in fact much more.

Requiring some regularity conditions for H , the result in Proposition 12.1 can be strengthened. Let us consider some of the much natural conditions of this kind.

Proposition 12.3: Suppose that $H \subset K$ is nonempty topologically bounded, that $Y \setminus H$ is a neighbourhood of 0 and that for any $z \in \text{cone } H$ there exist some h in H and some $\lambda > 0$ such that $z = \lambda h$. Then every nonempty lower bounded set in Y possesses H -efficient points exactly when $K_0 = \text{cone } H$ is quasi K bound regular.

Proof : One part of this assertion is contained in Proposition 12.1. To prove the other part let us assume that K_0 is not quasi K bound regular. Then by Lemma 11.3. (c) there exists a sequence (y_n) in $K_0 \setminus V$, with V some neighbourhood of 0, such that

the set $\left\{ \sum_{i=1}^n y_i \mid n \in \mathbb{N} \right\}$ is K order bounded. By the hypothesis on H there exists some neighbourhood U of 0 such that $\lambda H \subset V \setminus U$ for some $\lambda > 0$. We have then for an arbitrary i some h_i in H such that $y_i \not\equiv \lambda h_i$. Hence it follows that the set $\left\{ \sum_{i=1}^n h_i \mid i \in \mathbb{N} \right\}$ is K order bounded, contradicting by Proposition 5.1 the hypothesis that each lower bounded set in Y has H -efficient points. ■

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Received February 1988, revised December 1988

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