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ON A CLASS OF APPROXIMATION OPERATORS

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Abstract

In this note we construct a class of approximation operators using polynomial sequences of binomial type. We compute the expressions of these operators on the test functions and we give a convergence theorem. We give also quantitative evaluations of the order of approximation by using the moduli of continuity of first and second order and we establish an integral representation of the remainder term in the case when the approximation formula has the degree of exactness one. Some special cases, almost all of them being studied previously by D.D. Stancu, are mentioned.

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1 Introduction

In this note we introduce a class of linear and positive approximation operators using polynomial sequences of binomial type.

Definition 1 A sequence of polynomials $(p_m(x))_{m \geq 0}$ is called a sequence of binomial type if $\deg p_m = m$, $\forall m \in \mathbb{N}$ and it satisfies the relations

$$p_m(x+y) = \sum_{k=0}^m \binom{m}{k} p_k(x) p_{m-k}(y)$$

for every real numbers x and y and every positive integer m .

In the following we will consider linear operators defined on the algebra of polynomials.

A linear operator T is a shift invariant operator if $E^a T = T E^a$, for every a , where E^a is the shift operator defined by $E^a p(x) = p(x+a)$.

A linear operator Q is called a delta operator if Q is shift invariant and $Qx = \text{const.} \neq 0$.

Definition 2 We say that a sequence of polynomials $(p_m(x))_{m \geq 0}$ is a basic sequence for the delta operator Q if:

- i) $p_0(x) = 1$,
- ii) $p_m(0) = 0, \forall m \geq 1$,
- iii) $Qp_m = mp_{m-1}, \forall m \geq 1$.

It is known that every delta operator has a unique basic sequence (see [19]).

Proposition 3 [19] If $(p_m(x))_{m \geq 0}$ is the basic sequence for a delta operator then it is a sequence of binomial type; if $(p_m(x))_{m \geq 0}$ is a sequence of binomial type then there exists a delta operator for which $(p_m(x))_{m \geq 0}$ is the basic sequence.

Some examples of delta operators and their basic sequences are:

- i) The derivative D has basic sequence $e_m(x) = x^m$;
- ii) The operator $\frac{\Delta}{\alpha}$, where $\Delta_\alpha = E^\alpha - I$ is the forward difference operator has basic sequence the lower shifted factorials $x^{[m, \alpha]} = x(x - \alpha) \dots (x - (m - 1)\alpha)$;
- iii) The operator $\frac{\nabla}{\alpha}$, where $\nabla_\alpha = I - E^{-\alpha}$ is the backward difference operator has basic sequence the upper shifted factorials $x^{[m, -\alpha]} = x(x + \alpha) \dots (x + (m - 1)\alpha)$;
- iv) The Touchard operator $T = \ln(I + D) = D - \frac{1}{2}D^2 + \frac{1}{3}D^3 - \frac{1}{4}D^4 + \dots$, has basic sequence the exponential polynomials $(\varphi_m(x))_{m \geq 0}$ defined by $\varphi_0(x) = 1$ and $\varphi_m(x) = \sum_{k=1}^m S(m, k)x^k$ for $m \geq 1$, where $S(m, k)$ are the Stirling numbers of second kind;
- v) The Laguerre operator $L = \frac{D}{I+D} = D - D^2 + D^3 - D^4 + \dots$, has basic sequence $(l_m(x))_{m \geq 0}$ defined by $l_0(x) = 1$ and $l_m(x) = \sum_{k=1}^m \frac{m!}{k!} \binom{m-1}{k-1} x^k, m \geq 1$.
- vi) The Abel operator $DE^{-\alpha}$ has basic sequence $a_m(x) = x(x + m\alpha)^{m-1}$.

Definition 4 If T is a linear operator, then its Pincherle derivative T' is defined by $T' = TX - XT$, where the linear operator X is defined by $(Xp)(x) = xp(x)$ for all x and all polynomials p .

A basic sequence satisfies Rodrigues formula [19]

$$(1) \quad p_m(x) = x(Q')^{-1} p_{m-1}(x)$$

Definition 5 A linear operator θ is called the umbral shift operator for the sequence $(p_m(x))_{m \geq 0}$ if $\theta p_m = p_{m+1}$ for every positive integer m .

If $(p_m(x))_{m \geq 0}$ is a basic sequence for the delta operator Q then from (1) results that $\theta = X(Q')^{-1}$.

We mention that Umbral calculus allows a unified and simple study of sequences of binomial type. More details about these sequences can be found in [6], [7], [15], [18], [19].

Due to their nice properties polynomial sequences of binomial type have applications in various fields of mathematics. T.Popoviciu proposed the use of binomial sequences in order to construct approximation operators of the form [16],

$$(2) \quad (T_m^Q f)(x) = \frac{1}{p_m(1)} \sum_{k=0}^m \binom{m}{k} p_k(x) p_{m-k}(1-x) f\left(\frac{k}{m}\right).$$

These kind of operators and their generalizations and variants were studied in [1], [3], [4], [5], [10], [11], [12], [13], [14], [20], [27], [30], [31], [32], [33], [35], [34].

2 Definition and convergence of the operator

$$L_{m,p,r}^Q$$

Let Q be a delta operator with the basic sequence $(p_k(x))_{k \geq 0}$, which satisfy $p_m(1) \neq 0$ and $p'_m(0) \geq 0$ for every positive integer m . Let p and r be two nonnegative integer parameters such that $2r \leq m$.

For every function $f : [0, 1 + \frac{p}{m}] \rightarrow \mathbb{R}$ we introduce an approximation operator of the form

$$(3) \quad (L_{m,p,r}^Q f)(x) = \sum_{k=0}^{m+p-r} \binom{m+p-r}{k} \frac{p_k(x) p_{m+p-r-k+1}(1-x)}{p_{m+p-r+1}(1)} f\left(\frac{k}{m}\right) \\ + \sum_{k=r}^{m+p} \binom{m+p-r}{k-r} \frac{p_{k-r+1}(x) p_{m+p-k}(1-x)}{p_{m+p-r+1}(1)} f\left(\frac{k}{m}\right).$$

If $p'_m(0) \geq 0, \forall m \in \mathbb{N}$ then $p_m(x) \geq 0, \forall m \in \mathbb{N}$ and $\forall x \in [0, 1]$ so the operator $L_{m,p,r}^Q$ is a positive linear operator. It is easy to observe that if we apply this operator to a function $f : [0, 1 + \frac{p}{m}] \rightarrow \mathbb{R}$ we obtain an approximating polynomial of degree at most $m + p - r + 1$.

In the following we want to find a simple formula for this operator.

If in the second sum in the relation (3) we replace k with $k + r$ we obtain

$$(L_{m,p,r}^Q f)(x) = \frac{1}{p_{m+p-r+1}(1)} \sum_{k=0}^{m+p-r} \binom{m+p-r}{k} [p_k(x) p_{m+p-r-k+1}(1-x) f\left(\frac{k}{m}\right) \\ + p_{k+1}(x) p_{m+p-r-k}(1-x) f\left(\frac{k+r}{m}\right)].$$

It is easy to see that the last relation may be written as

$$(4) \quad (L_{m,p,r}^Q f)(x) = \frac{1}{p_{m+p-r+1}(1)} \sum_{k=0}^{m+p-r} \binom{m+p-r}{k} \left\{ \left[f\left(\frac{k}{m}\right) \theta_y + f\left(\frac{k+r}{m}\right) \theta_x \right] (p_k(x) p_{m+p-r-k}(y)) \right\} \Bigg|_{y=1-x},$$

where θ_x is the umbral shift operator for the binomial sequence $(p_n(x))$.

If we cut the previous relation in two parts and in the first we use $\binom{m+p-r}{k} = \frac{m+p-r+1-k}{m+p-r+1} \binom{m+p-r+1}{k}$, and in the second $\binom{m+p-r}{k} = \frac{k+1}{m+p-r+1} \binom{m+p-r+1}{k+1}$ we arrive at

$$(L_{m,p,r}^Q f)(x) = \frac{1}{m+p-r+1} \left[\sum_{k=0}^{m+p-r} (m+p-r+1-k) w_{m+p-r+1,k}^Q(x) f\left(\frac{k}{m}\right) + \sum_{k=0}^{m+p-r} (k+1) w_{m+p-r+1,k+1}^Q(x) f\left(\frac{k+r}{m}\right) \right],$$

$$\text{where } w_{n,k}^Q(x) = \binom{n}{k} \frac{p_k(x) p_{n-k}(1-x)}{p_n(1)}$$

If in the second sum we replace k with $k-1$ we obtain a simpler formula for the operator $L_{m,p,r}^Q f$

$$(5) \quad (L_{m,p,r}^Q f)(x) = \frac{1}{m+p-r+1} \sum_{k=0}^{m+p-r+1} w_{m+p-r+1,k}^Q(x) \left[(m+p-r+1-k) f\left(\frac{k}{m}\right) + k f\left(\frac{k+r-1}{m}\right) \right]$$

We remark that for $p=0$ and $r=1$ the approximation operator $L_{m,0,1}^Q$ is in fact the binomial operator of T. Popoviciu type T_m^Q .

It is easy to verify that

$$(L_{m,p,r}^Q f)(0) = 0 \text{ and } (L_{m,p,r}^Q f)(1) = f\left(1 + \frac{p}{m}\right).$$

The expression (5) for the operator $L_{m,p,r}^Q$ is useful for the computation of the values of this operator on the test functions $e_n(x) = x^n$ for $n=0, 1, 2$. In order to do this we need the expressions of the binomial operator of T. Popoviciu type T_m^Q on the test functions obtained by C. Manole [12], [13],

$$(6) \quad T_m^Q e_i = e_i, \text{ for } i=0, 1 \text{ and } (T_m^Q e_2)(x) = x^2 + x(1-x) d_m^Q,$$

where

$$(7) \quad d_m^Q = 1 - \frac{m-1}{m} \frac{(Q')^{-2} p_{m-2}(1)}{p_m(1)}$$

and Q' is the Pincherle derivative of delta operator Q .

We mention that later Sablonnière [20] have computed also the values of the operator T_m^Q on the test functions using the generating function of a binomial sequence.

Now we can start to compute the values of the operator $L_{m,p,r}^Q$ on the test functions $e_n(x) = x^n$, for $n = 0, 1, 2$.

For e_0 we have

$$\begin{aligned} (L_{m,p,r}^Q e_0)(x) &= \frac{1}{m+p-r+1} \sum_{k=0}^{m+p-r+1} w_{m+p-r+1,k}^Q(x) (m+p-r+1-k+k) \\ &= \sum_{k=0}^{m+p-r+1} w_{m+p-r+1,k}^Q(x) = 1 = e_0(x), \end{aligned}$$

so this operator preserve the constants.

For e_1 we can write

$$\begin{aligned} (L_{m,p,r}^Q e_1)(x) &= \frac{1}{m+p-r+1} \sum_{k=0}^{m+p-r+1} w_{m+p-r+1,k}^Q(x) \left[(m+p-r+1-k) \frac{k}{m} + \frac{k(k+r-1)}{m} \right] \\ &= \frac{1}{m(m+p-r+1)} \sum_{k=0}^{m+p-r+1} w_{m+p-r+1,k}^Q(x) k(m+p) \\ &= \frac{m+p}{m} \sum_{k=0}^{m+p-r+1} w_{m+p-r+1,k}^Q(x) \frac{k}{m+p-r+1} \\ &= \frac{m+p}{m} (T_{m+p-r+1}^Q e_1)(x) = \frac{m+p}{m} x, \end{aligned}$$

so this operator preserve the polynomials of degree one only when $p = 0$.

Finally, for e_2 we have

$$\begin{aligned} (L_{m,p,r}^Q e_2)(x) &= \frac{1}{m+p-r+1} \sum_{k=0}^{m+p-r+1} w_{m+p-r+1,k}^Q(x) \left[(m+p-r+1-k) \frac{k^2}{m^2} + \frac{k(k+r-1)^2}{m^2} \right] \\ &= \frac{1}{m^2(m+p-r+1)} \sum_{k=0}^{m+p-r+1} w_{m+p-r+1,k}^Q(x) k \left[k(m+p+r-1) + (r-1)^2 \right] \\ &= \frac{1}{m^2} \left[(m+p+r-1)(m+p-r+1) (T_{m+p-r+1}^Q e_2)(x) + (r-1)^2 (T_{m+p-r+1}^Q e_1)(x) \right] \\ &= \frac{1}{m^2} \left\{ (m+p+r-1)(m+p-r+1) \left[x^2 + x(1-x) d_{m+p-r+1}^Q \right] + (r-1)^2 x \right\}. \end{aligned}$$

Hence, we have proved the following

Lemma 6 If $L_{m,p,r}^Q f$ is the approximation operator defined by (3) then we have the following relations:

$$\begin{aligned} (L_{m,p,r}^Q e_0)(x) &= e_0(x) \\ (L_{m,p,r}^Q e_1)(x) &= \left(1 + \frac{p}{m}\right)x \\ (L_{m,p,r}^Q e_2)(x) &= \frac{1}{m^2} \left\{ (m+p+r-1)(m+p-r+1) \left[x^2 + x(1-x)d_{m+p-r+1}^Q \right] + (r-1)^2 x \right\} \\ &= x^2 \left(1 + \frac{p}{m}\right)^2 + \frac{x(1-x)}{m^2} \left[(m+p)^2 d_{m+p-r+1}^Q + (r-1)^2 (1 - d_{m+p-r+1}^Q) \right] \end{aligned}$$

where d_m^Q is defined by (7).

From this lemma it results that

$$(8) \quad L_{m,p,r}^Q \left((t-x)^2; x \right) = \frac{x}{m^2} \left\{ xp^2 + (1-x) \left[(m+p)^2 d_{m+p-r+1}^Q + (r-1)^2 (1 - d_{m+p-r+1}^Q) \right] \right\}$$

Using the Bohman-Korovkin convergence criterion and the above lemma we can state the following convergence theorem.

Theorem 7 If $f \in C \left[0, 1 + \frac{p}{m}\right]$ and $d_m^Q \rightarrow 0$ when $m \rightarrow \infty$ then the sequence of operators $L_{m,p,r}^Q f$ converges uniformly to the function f in the interval $\left[0, 1 + \frac{p}{m}\right]$.

3 Special cases

1. If $Q = D$ the operator defined by (3) reduces to

$$(L_{m,p,r}^D f)(x) = \sum_{k=0}^{m+p-r} \binom{m+p-r}{k} x^k (1-x)^{m+p-r-k} \left\{ (1-x) f\left(\frac{k}{m}\right) + x f\left(\frac{k+r}{m}\right) \right\}$$

Because the Pincherle derivative of D is the identity operator one obtains $d_m^D = \frac{1}{m}$, so from Lemma 6 results that $(L_{m,p,r}^D e_2)(x) = x^2 \left(1 + \frac{p}{m}\right)^2 + \frac{x(1-x)}{m^2} [m+p+r(r-1)]$

1.1. For $Q = D$ and $r = 0$ one obtains the Bernstein-Schurer operator [21]

$$(L_{m,p,0}^D f)(x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} f\left(\frac{k}{m}\right).$$

1.2. If $Q = D$ and $p = 0$ the corresponding operator was studied by D.D. Stancu [24]

$$(L_{m,0,r}^D f)(x) = \sum_{k=0}^{m-r} \binom{m-r}{k} x^k (1-x)^{m-r-k} \left\{ (1-x) f\left(\frac{k}{m}\right) + x f\left(\frac{k+r}{m}\right) \right\}$$

2. For $Q = \frac{\nabla_\alpha}{\alpha}$ the operator $L_{m,p,r}^{\frac{\nabla_\alpha}{\alpha}} f$ was also introduced and studied by D.D. Stancu 1996 [29]

$$\left(L_{m,p,r}^{\frac{\nabla_\alpha}{\alpha}} f \right) (x) = \sum_{k=0}^{m+p-r} \binom{m+p-r}{k} \frac{x^{[k,-\alpha]} (1-x)^{[m+p-r-k,-\alpha]}}{(1+\alpha)^{[m+p-r,-\alpha]}} \{ [1-x+(m+p-r-k)\alpha] f\left(\frac{k}{m}\right) + (x+k\alpha) f\left(\frac{k+r}{m}\right) \}$$

In this case $d_m^{\frac{\nabla_\alpha}{\alpha}} = \frac{1+\alpha m}{(1+\alpha)m}$ so from Lemma 6 one obtains the relation

$$\left(L_{m,p,r}^{\frac{\nabla_\alpha}{\alpha}} e_2 \right) (x) = \frac{x}{m^2(1+\alpha)} \left[x(m+p-r)(m+p+r-1) + \alpha(m+p)^2 + r^2 + m + p - r \right]$$

- 2.1. If we take $Q = \frac{\nabla_\alpha}{\alpha}$ and $p = 0$ we obtain an operator considered by D. D. Stancu in 1984 [25]

$$\left(L_{m,0,r}^{\frac{\nabla_\alpha}{\alpha}} f \right) (x) = \sum_{k=0}^{m-r} \binom{m-r}{k} \frac{x^{[k,-\alpha]} (1-x)^{[m-r-k,-\alpha]}}{(1+\alpha)^{[m-r,-\alpha]}} \{ [1-x+(m-r-k)\alpha] f\left(\frac{k}{m}\right) + (x+k\alpha) f\left(\frac{k+r}{m}\right) \}$$

and the expression of this operator on e_2 is

$$\left(L_{m,0,r}^{\frac{\nabla_\alpha}{\alpha}} e_2 \right) (x) = \frac{1}{1+\alpha} \left\{ x(x+\alpha) + \frac{x(1-x)}{m} \left[1 + \frac{r(r-1)}{m} \right] \right\}.$$

The bivariate analog of this operator, defined on the square $[0, 1] \times [0, 1]$, was studied by G. Vlaic in [36].

- 2.1.1 If we set $r = 0$ the above operator reduces to the well-known Stancu operator, introduced in 1968 [23]

$$\left(L_{m,0,0}^{\frac{\nabla_\alpha}{\alpha}} f \right) (x) = \sum_{k=0}^m \binom{m}{k} \frac{x^{[k,-\alpha]} (1-x)^{[m-k,-\alpha]}}{1^{[m,-\alpha]}} f\left(\frac{k}{m}\right)$$

3. If we take as delta operator the Touchard operator $T = \ln(I + D)$, which has the basic sequence the sequence of exponential polynomials defined by $\varphi_m(x) = \sum_{k=1}^m S(m, k) x^k$, where $S(m, k)$ are the Stirling numbers of second kind we have $d_m^T = \frac{1}{m} + \frac{m-1}{m} \frac{\varphi_{m-1}(1)}{\varphi_m(1)}$. It was proved that $\frac{\varphi_{m-1}(1)}{\varphi_m(1)} \sim \frac{1}{m} \left(\frac{m-1}{m} \right)^{m-1}$ ([20]) so d_m^T converges to 0 when $m \rightarrow \infty$ and consequently from Theorem 7 the operator $(L_{m,p,r}^T f)(x)$ converges to the function f . For the test function e_2 we obtain

$$(L_{m,0,r}^T e_2)(x) = x^2 + \frac{x(1-x)}{m^2} \left\{ (m+r-1) \left[1 + \frac{\varphi_{m-r}(1)}{\varphi_{m-r+1}(1)} \right] + (r-1)^2 \right\}.$$

4 A representation of the remainder

It is easy to see that for $p = 0$ the operator $L_{m,0,r}^Q$ is interpolatory at both ends of the interval $[0, 1]$ and the approximation formula

$$(9) \quad f(x) = \left(L_{m,0,r}^Q f\right)(x) + \left(R_{m,0,r}^Q f\right)(x)$$

has the degree of exactness 1.

If $f \in C^2[0, 1]$ we can apply the well known theorem of Peano and we obtain that the remainder in the above approximation formula can be represented in the form

$$(10) \quad \left(R_{m,0,r}^Q f\right)(x) = \int_0^1 G_{m,r}^Q(t; x) f''(t) dt,$$

where $G_{m,r}^Q(t; x)$ is the Peano kernel given by

$$G_{m,r}^Q(t; x) = \left(R_{m,0,r}^Q f\right)(x)$$

and $\varphi_x(t) = (x - t)_+ = \frac{x-t+|x-t|}{2}$, the operator acting on $\varphi_x(t)$ as a function of x .

Because $G_{m,r}^Q(t; x) \leq 0$ on the square $[0, 1] \times [0, 1]$ it follows that we can apply the mean value theorem to the integral in (10) and we obtain that exists ξ in $[0, 1]$ such that

$$\left(R_{m,0,r}^Q f\right)(x) = f''(\xi) \int_0^1 G_{m,r}^Q(t; x) dt.$$

If in the approximation formula $f(x) = \left(L_{m,0,r}^Q f\right)(x) + f''(\xi) \int_0^1 G_{m,r}^Q(t; x) dt$ we take $f(x) = x^2$ we obtain

$$\int_0^1 G_{m,r}^Q(t; x) dt = \frac{1}{2} \left(R_{m,0,r}^Q e_2\right)(x).$$

Taking into account that

$$\left(L_{m,0,r}^Q e_2\right)(x) = x^2 + x(1-x) \left[d_{m-r+1}^Q \left(1 - \left(\frac{r-1}{m}\right)^2\right) + \left(\frac{r-1}{m}\right)^2\right]$$

we finally arrive at

$$(11) \quad \left(R_{m,0,r}^Q f\right)(x) = \frac{1}{2} x(x-1) \left[d_{m-r+1}^Q \left(1 - \left(\frac{r-1}{m}\right)^2\right) + \left(\frac{r-1}{m}\right)^2\right] f''(\xi).$$

For the Touchard delta operator T the above relation becomes

$$\left(R_{m,0,r}^T f\right)(x) = \frac{1}{2m^2} x(x-1) \left\{ (m+r-1) \left[1 + \frac{\varphi_{m-r}(1)}{\varphi_{m-r+1}(1)} \right] + (r-1)^2 \right\}.$$

5 Estimates involving the first and the second order modulus of continuity

In order to obtain quantitative evaluation of the order of approximation by using the moduli of continuity we need the following results established by O. Shisha and B. Mond ([22]) and by H.H. Gonska and R.K. Kovacheva ([8])

Lemma 8 [22] *If L is a positive operator mapping $C[a, b]$ into $C[c, d]$ such that $[c, d] \subseteq [a, b]$, then for all $f \in C[a, b]$, $x \in [c, d]$ and $\delta > 0$, the following estimate:*

$$|f(x) - (Lf)(x)| \leq |f(x)| |(Le_0)(x) - 1| + \left[(Le_0)(x) + \delta^{-2} L((t-x)^2; x) \right] \omega_1(f; \delta)$$

holds, where $\omega_1(f; \delta)$ is the first order modulus of continuity.

Lemma 9 [8] *Let $[a, b]$ denote a compact interval of the real axis. If $L : C[a, b] \rightarrow C[c, d]$ (where $[c, d] \subseteq [a, b]$) is a positive linear operator such that $Le_0 = e_0$ then for all $f \in C[a, b]$, all $x \in [c, d]$ and $0 < \delta \leq (b-a)/2$ the following inequality holds:*

$$|f(x) - (Lf)(x)| \leq \frac{2}{\delta} L((t-x); x) \omega_1(f; \delta) + \frac{3}{2} \left\{ 1 + \frac{1}{\delta} L((t-x); x) + \frac{1}{2\delta^2} L((t-x)^2; x) \right\} \omega_2(f; \delta),$$

where $\omega_2(f; \delta)$ is the second modulus of continuity.

From Lemma 8 and the relation (8) with $p = 0$ we obtain the following

Theorem 10 *For $f \in C[0, 1]$, $x \in [0, 1]$ and $\delta > 0$ we have the estimation*

$$|f(x) - (L_{m,0,r}^Q f)(x)| \leq \left\{ 1 + \frac{x(1-x)}{\delta^2} \left[d_{m-r+1}^Q \left(1 - \left(\frac{r-1}{m} \right)^2 \right) + \left(\frac{r-1}{m} \right)^2 \right] \right\} \omega_1(f; \delta).$$

Taking into account that $x(1-x) \leq \frac{1}{4}$ we can write

$$\|f - L_{m,0,r}^Q f\| \leq \left\{ 1 + \frac{1}{4\delta^2} \left[d_{m-r+1}^Q \left(1 - \left(\frac{r-1}{m} \right)^2 \right) + \left(\frac{r-1}{m} \right)^2 \right] \right\} \omega_1(f; \delta).$$

For $Q = D$ and $\delta = \frac{1}{\sqrt{m}}$ the above relation reduces to

$$\|f - L_{m,0,r}^D f\| \leq \left(\frac{5}{4} + \frac{1}{4} \frac{r(r-1)}{m} \right) \omega_1\left(f; \frac{1}{\sqrt{m}}\right)$$

while for $Q = \frac{\nabla_\alpha}{\alpha}$ and $\delta = \frac{1}{m} \sqrt{\frac{m(1+m\alpha)+r(r-1)}{1+\alpha}}$ one obtains

$$\|f - L_{m,0,r}^{\frac{\nabla_\alpha}{\alpha}} f\| \leq \frac{5}{4} \omega_1\left(f; \frac{1}{m} \sqrt{\frac{m(1+m\alpha)+r(r-1)}{1+\alpha}}\right).$$

Using Lemma 9 and the relation (8) for $p = 0$ we can state the following

Theorem 11 If $f \in C[0, 1]$, $x \in [0, 1]$ and $0 < \delta \leq (b-a)/2$ the next estimation

$$\left| f(x) - \left(L_{m,0,r}^Q f \right) \right| \leq \left\{ \frac{3}{2} + \frac{3}{4\delta^2} x(1-x) \left[d_{m-r+1}^Q \left(1 - \left(\frac{r-1}{m} \right)^2 \right) + \left(\frac{r-1}{m} \right)^2 \right] \right\} \omega_2(f; \delta)$$

holds.

For $Q = D$ and $\delta = \frac{1}{\sqrt{m}}$ this leads us to the inequality

$$\|f - L_{m,0,r}^D f\| \leq \frac{3}{16} \left[9 + \frac{r(r-1)}{m} \right] \omega_2 \left(f; \frac{1}{\sqrt{m}} \right)$$

while for $Q = \frac{\nabla \alpha}{\alpha}$ and $\delta = \frac{1}{m} \sqrt{\frac{m(1+m\alpha)+r(r-1)}{1+\alpha}}$ we can write

$$\|f - L_{m,0,r}^{\frac{\nabla \alpha}{\alpha}} f\| \leq \frac{27}{16} \omega_2 \left(f; \frac{1}{m} \sqrt{\frac{m(1+m\alpha)+r(r-1)}{1+\alpha}} \right).$$

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