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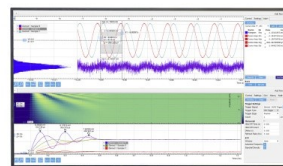
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Spectral Collocation Based on Quasi-Classical Orthogonal Polynomials Applied to Solve a Singular BVP from Hydrodynamics.

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Abstract. It is well established that spectral collocation methods based on classical orthogonal polynomials, in spite of their high order accuracy, use bad conditioned differentiation matrices, i.e., fully populated, rather non-normal and badly conditioned with respect to inversion. The aim of this essay is to try to find other orthogonal polynomials, with respect to more sophisticated measures, which could generate better differentiation matrices and consequently more accurate collocation methods. We are mainly interested in solving boundary value problems on unbounded intervals.

Keywords: spectral collocation, orthogonal polynomial, non standard, sinc function, boundary value problem, unbounded domain

PACS: 02.60.-x, 02.70.Hm, 47.11.Kb, 47.27.er, 47.85.Dh, 02.10.Yn

INTRODUCTION

As we are mainly interested in solving boundary value problems on unbounded intervals, we analyse orthogonal polynomials with respect to the generalized Laguerre measure and the logistic measure (see [2, chap. 1]).

The capabilities of some non-standard orthogonal polynomials in obtaining accurate quadrature formulas has been documented in the above quoted monograph. Unfortunately, few attempts have been made to use them in collocation methods (see [7] as an exception). We want to fill this gap as much as possible.

The benchmark problem we take into account is the so-called *density profile equation* which is a genuinely nonlinear and defined on the half line with singularities at both ends (see for instance [5] and [1] for analytical results). The spectral collocation methods considered here, cast the attached boundary value problem into a nonlinear system of algebraic equations. The combined effect of singularity and nonlinearity increases when a physical parameter approaches one. We elaborate on the numerical methods to solve the nonlinear system analysing the outputs of some optimization algorithms such as first order optimality, elapsed CPU time, number of iterations, function counts, etc.

The best and more reliable results are obtained by *sinc collocation* (see additionally our contributions [3] and [4, chap. 2]). The search for better collocation methods based on non standard orthogonal polynomials is still an open issue. The first attempts are rather less encouraging.

A SHORT HISTORICAL REMARK

We have observed that spectral collocation based on Jacobi, Chebyshev, Laguerre, Hermite polynomials have the following drawbacks:

- the differential collocation matrices are fully populated;
- they are highly non-normal, i.e.,

$$\mathbf{A}\mathbf{A}^* - \mathbf{A}^*\mathbf{A} \neq \mathbf{0}$$

and moreover the Henrici's number, defined as

$$Hen(\mathbf{A}) := \left(\|\mathbf{A}\mathbf{A}^* - \mathbf{A}^*\mathbf{A}\|^{1/2} \right) / \|\mathbf{A}\|, \quad (1)$$

where $\|\cdot\|$ stands for the Frobenius' norm of $\mathbf{A}$ and $\mathbf{A}^*$ is the conjugate transpose of $\mathbf{A}$, tends to $\sqrt[4]{2} \simeq 1.18920711$;

- they exhibit no symmetries except *Fourier* and *sinc* cases ;

TABLE 1. The values of the Henrici's number for the first three Laguerre differentiation matrices $\mathbf{D}^{(i)}$, $i = 1, 2, 3$ when scaling factor $b := 2$.

	$a = 0$	$a = 1$	$a = 2$
$\mathbf{D}^{(1)}$	1.098784	0.639221	0.575467
$\mathbf{D}^{(2)}$	1.181160	0.409238	0.378240
$\mathbf{D}^{(3)}$	1.187628	0.981121	0.828393

- they are badly conditioned.

Thus, we search for *non-classical orthogonal polynomials such that the corresponding collocation differentiation matrices are better conditioned, more normal, exhibiting symmetries, etc., than those corresponding to the classical polynomials*. We are mainly interested (and restrict ourselves) in problems formulated on unbounded domains.

Some non classical orthogonal polynomials on the half line

The generalised Laguerre polynomials are defined by the measure and the weight

$$d\lambda(t) := w(t) dt, \quad w(t) := t^a \exp(-t), a > 0. \quad (2)$$

It is important to underline that the normality of the differentiation matrices decrease as a increases (see Table 1).

SPECTRAL COLLOCATION SOLUTIONS TO DENSITY PROFILE EQUATION

We now investigate the *monotonously increasing solutions* of the *nonlinear and singular* problem (D P Eq.)

$$\begin{cases} \rho''(r) + \frac{N_f - 1}{r} \rho'(r) = 4\lambda(\rho(r) + 1)\rho(r)(\rho(r) - \xi), & 0 < r < \infty, \\ \rho'(0) = 0, \rho(\infty) = \xi, \end{cases} \quad (3)$$

where $\rho(r)$ is the density of a fluid. The parameters are $\lambda = 1$ without restriction of generality, $N_f = 3$ which corresponds to a physically meaningful case and $\xi \in [0, 1]$ such as to reflect different physical situations.

A problem to embed in

Negative solutions to a second-order singular ODE

$$u(r) := \rho(r) - \xi, \quad (4)$$

$$(P_\infty) \begin{cases} u'' + \frac{k}{r} u' = 4\lambda^2[(\xi + \xi^2)u + h(u)], \\ u'(0) = 0, u \rightarrow 0, r \rightarrow \infty, \end{cases} \quad (5)$$

where

$$k := N_f - 1, h(u) := u^3 + (1 + 2\xi)u^2; \quad (6)$$

and

$$H(u) := \int_0^u h(s) ds; \quad (7)$$

satisfies

$$H(u) - \theta u h(u) = (1 + 2\xi) \frac{u^3}{12} \leq 0, u \leq 0, \text{ when } \theta = 1/4. \quad (8)$$

Moreover, it is assumed that

$$\forall u \leq 0, |h(u)| < Lu^q, q \in (1, (k+3)/(k-1)), \text{ for } L > 0, \text{ and } q = 4. \quad (9)$$

For this problem, in [1], the authors show that (P_∞) has at least one negative solution !

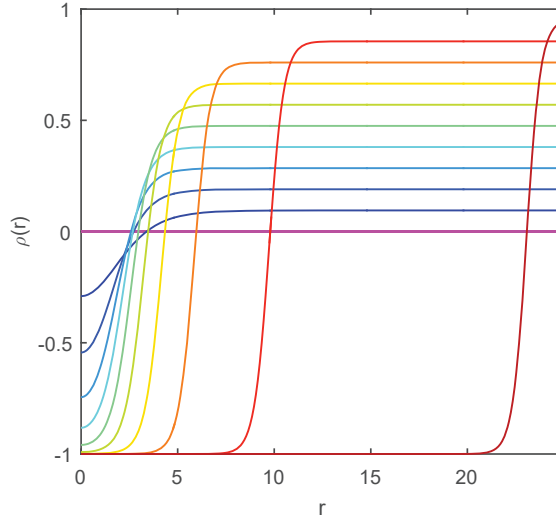


FIGURE 1. Sinc collocation solutions to problem (3) for various $\xi = 0.095, 0.19, \dots, 0.95$ when $N = 500$ and $h := 0.1$

The discrete form of collocation

The nonlinear algebraic system of equations provided by spectral collocation reads

$$\mathbf{F}(\mathbf{U}) = 0, \quad (10)$$

where

$$\mathbf{F}(\mathbf{U}) := \mathbf{D}_m^{(2)}\mathbf{U} + \frac{N_f - 1}{\mathbf{X}}\mathbf{D}_m^{(1)}\mathbf{U} - 4(\mathbf{U}^3 + (1 + 2\xi)\mathbf{U}^2 + (\xi + \xi^2)\mathbf{U}). \quad (11)$$

Here,

- $\mathbf{D}_m^{(i)}$ stand for the i -th derivative of method m , $\mathbf{X}$ and $\mathbf{U}$ stand for the vector of nodes and that of unknowns respectively;
- we start with $u_0(x) := -\sinh(x)$;
- use *continuation* with respect to parameter ξ ;
- we provide an analytic Jacobian $\mathbf{F}'(\mathbf{U})$;
- use the Levenberg-Marquardt algorithm implemented by MATLAB built in `fsolve`.

In Fig. 1 we depict some fairly smooth sinc collocation solutions to problem (3). Sinc collocation outputs are displayed in Table 2. Some logistic solutions to D P Eq. are depicted in Fig. 2 and they are by far less accurate than the sinc collocation solutions.

In order to reduce the effect of singularity in origin we have *multiplied all the members of the nonlinear algebraic system (10) by $\mathbf{X}$* . Only in this way we could reach the value $\xi = 0.95$ when solving this system.

Up to our knowledge, it is the largest value of ξ for which a reasonable numerical solution is reported (see for instance [5] and [6]).

SHORT CONCLUSIONS

To sum all this up, we observe that Laguerre and logistic collocation work faster than sinc collocation but with less accurate results. In the presence of the sharp layer for $\xi = 0.9$, the Laguerre and logistic collocation algorithm failed to produce any useful results at all. However, comparisons with a method previously presented in the literature support the statement that the above sinc collocation code is comparably straightforward to use.

TABLE 2. The number of iterations and of the function count used by `fsolve`, the bubble radius R and the values of $\rho(0)$ for various ξ when $N := 500$.

	Iterations	Function count	Bubble radius R	Boundary value $\rho(0)$
$\xi = 0.095$	6	6	3.40	-2.901225391219628e-01
0.19	6	6	2.70	-5.435789996461979e-01
0.285	10	13	2.60	-7.443415892459432e-01
0.38	10	13	2.70	-8.820293258426984e-01
0.475	10	13	3.00	-9.591019008236007e-01
0.57	9	11	3.50	-9.911159910827078e-01
0.665	33	48	4.30	-9.991997205557132e-01
0.76	109	163	6.00	-9.999900685626580e-01
0.855	374	559	9.80	-9.99999996974022e-01
$\xi = 0.95$	2504	3756	23.10	-1.000002291299574e+00

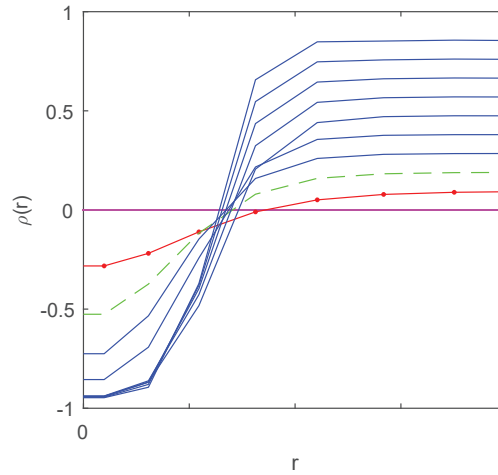


FIGURE 2. Logistic collocation solutions to problem (3) for various $\xi = 0.095, 0.19, \dots, 0.855$ when $N := 80$ and $b := 2$.

So far we have not found a non classical orthogonal polynomial to produce more accurate results than sinc collocation. This is due to the poor conditioning of their differentiation matrices. However, our research goes on.

REFERENCES

1. D. Bonheure, J. M. Gomes and L. Sanchez, *Nonlinear Analysis* **61**, 1383–1399 (2005).
2. W. Gautschi, *Orthogonal Polynomials in MATLAB. Exercises and Solutions*, SIAM Philadelphia, 2016.
3. C.I. Gheorghiu, *Numer. Algor.* **68**, 1–14 (2015).
4. C.I. Gheorghiu, *Spectral Collocation Solutions to Problems on Unbounded Domains.*, Casa Cărții de Știință Publishing House, Cluj-Napoca, 2018.
5. G. Kitchofer, O. Koch, P. Lima and E. Weinmüller, *J. Sci. Comput.* **32**, 411–424 (2007).
6. P.M. Lima, N.B. Konyukhov, A.I. Sukov and N.V. Chemetov, *J. Comput. Appl. Math.* **189**, 260–273 (2006).
7. J.A.C. Weideman, *International Series of Numerical Mathematics* **131**, 239–251 (1999).