

Solutions with a prescribed interval of positivity for differential systems with nonlocal conditions



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ABSTRACT

Based on fixed point index, the paper develops a theory of existence, localization and multiplicity of solutions to first-order differential systems subject to linear nonlocal conditions. The main features concern the role of the support of the nonlocal condition and the positivity of solutions which is only required on a prescribed subinterval. Several examples of problems admitting at least one, two, or sequences of such solutions are included, and numerical solutions are obtained using the Mathematica shooting program with starting initial conditions suggested by the theoretical localization results.

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1. Introduction

Many mathematical models arising from physics, chemistry, biology, economics etc. are given by first-order differential systems whose time-dependent variables stay for specific quantities, most often subject by their nature to the condition of positivity. In this connection, there is a huge literature on positive solutions to various classes of problems. However, there are cases where the positivity is only required on a subinterval of time. For instance, in case of the Lotka-Volterra predator-prey model, we may be interested that the size of the prey population $x(t)$ on a given time interval J - for example the reproduction season - be larger than a given threshold x_0 , and the size of predator population $y(t)$ on that time interval be less than y_0 . Under the substitutions $u = x - x_0$ and $v = y_0 - y$, the desired threshold conditions become the positivity conditions $u \geq 0$ and $v \geq 0$ on the prescribed time interval J . Analogous problems appear in economic models when the size of production of some goods in some periods of the year needs to be sufficiently larger to satisfy the market demand, or on the contrary, to be sufficiently smaller to avoid excessive stocks. Having in mind these examples as a main motivation, in this paper we seek solutions to differential systems on a given interval, which are necessarily positive only on a prescribed subinterval.

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Additionally in this paper, we treat differential systems with nonlocal conditions. In particular, they include the initial value condition, boundary value conditions, multi-point and integral constraints. As a motivating example, we have the integral condition $\int_{\Omega} u = 1$ which appears in many physical models where u is a probability distribution. The research on differential equations with nonlocal conditions began with the pioneer works of Cioranescu [5], Whyburn [17] and Conti [6], and have gained a special attention in the last decades motivated by concrete applications in different domains (see, e.g., [1–4,8–16,18]).

More exactly in this paper we study the existence, localization and multiplicity of solutions to the following problem:

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [0, 1] \\ g(u) = 0 \\ u \geq 0 & \text{on } [a, b] \end{cases} \quad (1.1)$$

where $[a, b]$ is a fixed subinterval of $[0, 1]$, $f : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, and $g : C([0, 1], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ a continuous linear mapping. We shall seek solutions $u \in C^1([0, 1], \mathbb{R}^n)$.

Taking $\alpha(u) := u(a) + g(u)$, the nonlocal condition $g(u) = 0$ can be replaced by

$$u(a) = \alpha(u).$$

Our approach to problem (1.1) is based on fixed point principles and fixed point index theory, and takes into consideration the support $[a, c]$ of α , namely the smallest subinterval $[a, c] \subseteq [a, b]$ such that $\alpha(u) = \alpha(v)$ for every $u, v \in C([0, 1], \mathbb{R}^n)$ having the same restriction to $[a, c]$.

Recall that for a closed convex subset D of a Banach X , a subset $U \subseteq D$ open in D , and a completely continuous operator $N : \bar{U} \rightarrow D$ such that $N(u) \neq u$ on the $\partial_D U = \bar{U} \setminus U$, one can consider the *fixed point index* of N over U with respect to D , which is denoted by $i(N, U, D)$ (see, e.g., [7, p. 238]) and has the following properties: $i(N, U, D) \neq 0$ implies that N has at least one fixed point in U (existence); $i(u_0, U, D) = 1$ for any constant operator $N \equiv u_0 \in U$ (normalization); the homotopy invariance; and additivity with respect to U . We have the following lemma that is used in the proof of our main result.

Lemma 1.1. *If $U \subseteq D$ is bounded, convex, open in D , and $N(\bar{U}) \subseteq U$, then $i(N, U, D) = 1$.*

Proof. Since $N(\bar{U}) \subseteq U$, one has $N(u) \neq u$ on the $\partial_D U$, so the index $i(N, U, D)$ is defined. Let $u_0 \in U$ be arbitrarily fixed. Then the homotopy $H_\lambda(u) = \lambda N(u) + (1 - \lambda)u_0$ ($\lambda \in [0, 1]$) is admissible since $H_\lambda(\bar{U}) \subseteq U$ for all $\lambda \in [0, 1]$, due to the convexity of U . Finally, by the homotopy invariance and the normalization properties of the index, we have

$$i(N, U, D) = i(H_1, U, D) = i(H_0, U, D) = i(u_0, U, D) = 1. \quad \square$$

Basic notations. We conclude the introduction by some nonstandard notations for vectors and vector-valued functions that will be used throughout the paper. First we make the convention that all vectors in $\mathbb{R}^n$ are identified to column matrices, and for two vectors $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, we let $x \leq y$ ($x < y$) if $x_i \leq y_i$ ($x_i < y_i$, respectively) for $i = 1, \dots, n$. Also by $\max\{x, y\}$ we mean the vector of components $\max\{x_i, y_i\}$, $i = 1, \dots, n$, and by the notation $x >_i y$ we mean that the i -th components x_i, y_i of the two vectors satisfy the inequality $x_i > y_i$.

We shall use the notation $\alpha[1]$, for the matrix whose columns are

$$\alpha((1, 0, \dots, 0)), \alpha((0, 1, \dots, 0)), \dots, \alpha((0, 0, \dots, 1)).$$

Thus, in virtue of the linearity of α , for any constant vector-valued function $c = (c_1, \dots, c_n)$, using the decomposition

$$c = c_1(1, 0, \dots, 0) + c_2(0, 1, \dots, 0) + \dots + c_n(0, 0, \dots, 1)$$

one has that $\alpha(c)$ is the product of matrix $\alpha[1]$ with the column vector c , i.e.,

$$\alpha(c) = \alpha[1]c. \quad (1.2)$$

Also the notation I will stay for the identity matrix of size n .

Finally, for a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, by $|x|$ we shall mean the column vector whose elements are $|x_1|, \dots, |x_n|$; for a scalar function $v \in C[0, 1]$, we shall denote by $|v|_\infty$ its supremum norm, and for a vector-valued function $u \in C([0, 1], \mathbb{R}^n)$ of scalar components $u_1, \dots, u_n$, we shall denote by $|u|_\infty$ the column vector of elements $|u_1|_\infty, \dots, |u_n|_\infty$. Also, for any subinterval $[a, b]$ of $[0, 1]$, by $\int_a^b u(t)dt$ we shall mean the column vector of the elements $\int_a^b u_1(t)dt, \dots, \int_a^b u_n(t)dt$. Thus, when referred to vectors or vector-valued functions, the equalities and inequalities from below have to be seen as equalities and inequalities between column vectors.

2. Main result

First, if the matrix $I - \alpha[1]$ is non-singular, then our differential system subject to the condition $u(a) = \alpha(u)$ is equivalent to the integral type equation in $C([0, 1], \mathbb{R}^n)$, namely

$$u(t) = (I - \alpha[1])^{-1} \alpha \left(\int_a^t f(s, u(s))ds \right) + \int_a^t f(s, u(s))ds, \quad t \in [0, 1]. \quad (2.1)$$

To prove this it suffices to integrate the differential equation from a to an arbitrary t , to obtain

$$u(t) = u(a) + \int_a^t f(s, u(s))ds, \quad (2.2)$$

then to apply the linear mapping α and use the condition $u(a) = \alpha(u)$, to obtain

$$\alpha(u) = \alpha(u(a)) + \alpha\left(\int_a^t f(s, u(s))ds\right) = \alpha[1]u(a) + \alpha\left(\int_a^t f(s, u(s))ds\right),$$

whence

$$u(a) = (I - \alpha[1])^{-1}\alpha\left(\int_a^t f(s, u(s))ds\right).$$

Now replacing in (2.2) gives (2.1).

The solutions $u \in C([0, 1], \mathbb{R}^n)$ of (2.1) are the fixed points of the operator $N : C([0, 1], \mathbb{R}^n) \rightarrow C([0, 1], \mathbb{R}^n)$ given by

$$N(u)(t) = (I - \alpha[1])^{-1}\alpha\left(\int_a^t f(s, u(s))ds\right) + \int_a^t f(s, u(s))ds \quad (t \in [0, 1]),$$

which is completely continuous due to the continuity of f and α .

Looking for solutions that are positive on the prescribed subinterval $[a, b]$ of $[0, 1]$, we are led to consider the set

$$K = \{u \in C([0, 1], \mathbb{R}^n) : u(t) \geq 0 \text{ for } t \in [a, b]\}.$$

Obviously, K is a wedge (i.e., closed, convex and with $\lambda K \subset K$ for every $\lambda \in \mathbb{R}_+$). In order to use fixed point principles and index, we need that the condition $N(K) \subset K$ is satisfied. It is easily seen that this happens if additionally the following positivity properties hold:

$$f(t, u) \geq 0 \text{ for all } t \in [a, b] \text{ and } u \in \mathbb{R}_+^n, \quad (2.3)$$

$$\alpha(u) \geq 0 \text{ for every } u \in C([0, 1], \mathbb{R}^n) \text{ with } u|_{[a, c]} \geq 0, \quad (2.4)$$

$$(I - \alpha[1])^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+), \quad (2.5)$$

where $[a, c]$ is the support of α . Thus $f(t, \cdot)$ is not necessarily positive on $\mathbb{R}_+^n$ for $t \in [0, 1] \setminus [a, b]$, but $\alpha(u)$ is positive even for functions u which are not positive on the whole interval $[0, 1]$. Also note that under condition (2.4), one has $\alpha[1] \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ and then the hypothesis (2.5) is equivalent to the fact that the eigenvalues of the matrix $\alpha[1]$ are located inside the unit circle.

Our main result is the following theorem on the existence, localization and multiplicity of solutions to problem (1.1).

For two vectors $x, y \in \mathbb{R}_+^n$ with $y \leq x$, and $i = 1, \dots, n$, we denote:

$$\underline{f}_i^x(s) := \max_{u \in [0, x_1] \times \dots \times [0, x_n]} f_i(s, u) \quad (s \in [a, b]),$$

$$\bar{f}_i^x(s) := \max_{u \in [-x_1, x_1] \times \dots \times [-x_n, x_n]} |f_i(s, u)| \quad (s \in [0, 1] \setminus (a, b)),$$

$$\underline{f}_i^{y,x}(s) := \min_{u \in [y_1, x_1] \times \dots \times [y_n, x_n]} f_i(s, u) \quad (s \in [a, b]).$$

Also $\underline{f}^x(s)$, $\bar{f}^x(s)$ and $\underline{f}^{y,x}(s)$ are the vector-valued functions of the corresponding scalar functions from the above table. For instance,

$$\underline{f}^x(s) = (\underline{f}_1^x(s), \dots, \underline{f}_n^x(s)),$$

which is looked as a column vector.

Theorem 2.1. *Let the conditions (2.3)–(2.5) hold.*

(1⁰) *If for a vector $R \in \mathbb{R}_+^n$, the following condition*

$$(I - \alpha[1])^{-1}\alpha\left(\int_a^t \underline{f}^R(s)ds\right) + \max \left\{ \int_0^a \bar{f}^R(s)ds, \int_a^b \underline{f}^R(s)ds + \int_b^1 \bar{f}^R(s)ds \right\} \leq R \quad (2.6)$$

holds, then problem (1.1) has a solution $u \in K$ such that

$$|u|_\infty \leq R.$$

(2⁰) If in addition to (2.6) there is a vector $r \in \mathbb{R}_+^n$, such that $0 \neq r < R$ and

$$(I - \alpha[1])^{-1} \alpha \left(\int_a^\cdot \underline{f}^{r,R}(s) ds \right) >_i r \quad \text{for all } i \text{ with } r_i > 0, \quad (2.7)$$

then problem (1.1) has a solution $u \in K$ such that

$$|u|_\infty \leq R \quad \text{and} \quad \min_{t \in [a,b]} u_i(t) \geq r_i \quad \text{for } i = 1, \dots, n. \quad (2.8)$$

(3⁰) If in addition to (2.6) and (2.7) there is a nonempty set of indices $\Lambda \subseteq \{1, \dots, n\}$ and a vector $\tau \in (0, +\infty)^n$ such that $\tau_i < r_i$ for all $i \in \Lambda$, $\tau_i = R_i$ for $i \notin \Lambda$, and

$$(I - \alpha[1])^{-1} \alpha \left(\int_a^\cdot \underline{f}^\tau(s) ds \right) + \max \left\{ \int_0^a \bar{f}^\tau(s) ds, \int_a^b \underline{f}^\tau(s) ds + \int_b^1 \bar{f}^\tau(s) ds \right\} <_i \tau \quad \text{for } i \in \Lambda, \quad (2.9)$$

then problem (1.1) has at least three solutions $u, v, w \in K$ such that

$$\begin{aligned} |u|_\infty &\leq R, \quad |v|_\infty \leq \tau, \quad |w|_\infty \leq R, \\ \min_{t \in [a,b]} u_i(t) &\geq r_i \quad \text{for } i = 1, \dots, n, \\ \min_{t \in [a,b]} w_i(t) &< r_i \quad \text{for at least one } i \in \{1, \dots, n\}, \\ |w_i|_\infty &> \tau_i \quad \text{for at least one } i \in \Lambda. \end{aligned}$$

Proof. Consider the bounded closed convex subset of $C([0, 1], \mathbb{R}^n)$,

$$D := \{u \in K : |u|_\infty \leq R\}.$$

We use the fixed point index of the operator N over a number of open sets in D and with respect to D .

(1⁰) First we prove that under condition (2.6), one has $N(D) \subseteq D$, which, according to Lemma 1.1 guarantees that the fixed point index of N over D with respect to D is equal to one, i.e., $i(N, D, D) = 1$. Hence N has at least one fixed point in D , that is a solution as wished. Notice that the existence of a fixed point also follows from Schauder's fixed point theorem. For doing estimations on N , it is convenient to represent N as sum of two operators, $N = N^1 + N^2$, where

$$\begin{aligned} N^1(u) &:= (I - \alpha[1])^{-1} \alpha \left(\int_a^\cdot f(s, u(s)) ds \right), \\ N^2(u)(t) &:= \int_a^t f(s, u(s)) ds \quad (t \in [0, 1]), \end{aligned}$$

and to recall the notations introduced before the statement of the theorem.

First we estimate from above $N^1(u)$. For $u \in D$ and $t \in [a, c]$ we have that $0 \leq u(s) \leq R$ for every $s \in [a, t]$, and so

$$\int_a^t f(s, u(s)) ds \leq \int_a^t \underline{f}^R(s) ds.$$

Then, using (2.4) and the linearity of α , we deduce that

$$\alpha \left(\int_a^\cdot f(s, u(s)) ds \right) \leq \alpha \left(\int_a^\cdot \underline{f}^R(s) ds \right),$$

which together with (2.5) yields

$$|N^1(u)| = N^1(u) \leq (I - \alpha[1])^{-1} \alpha \left(\int_a^\cdot \underline{f}^R(s) ds \right). \quad (2.10)$$

Next we estimate from above $N^2(u)$ for $u \in D$. For $t \in [a, b]$, we still have $0 \leq u(s) \leq R$ for every $s \in [a, t]$, and so

$$\left| \int_a^t f(s, u(s)) ds \right| = \int_a^t f(s, u(s)) ds \leq \int_a^b \underline{f}^R(s) ds \quad (t \in [a, b]). \quad (2.11)$$

For $t \in [0, 1] \setminus [a, b]$, since $u(t)$ is not necessarily positive, we only have $-R \leq u(s) \leq R$. Hence

$$\left| \int_a^t f(s, u(s)) ds \right| \leq \int_0^a \bar{f}^R(s) ds \quad (t \in [0, a]), \quad (2.12)$$

$$\begin{aligned} \left| \int_a^t f(s, u(s)) ds \right| &\leq \left| \int_a^b f(s, u(s)) ds \right| + \left| \int_b^1 f(s, u(s)) ds \right| \\ &\leq \int_a^b \underline{f}^R(s) ds + \int_b^1 \bar{f}^R(s) ds \quad (t \in [b, 1]). \end{aligned} \quad (2.13)$$

Now (2.11), (2.12) and (2.13) give

$$|N^2(u)(t)|_\infty \leq \max \left\{ \int_0^a \bar{f}^R(s) ds, \int_a^b \underline{f}^R(s) ds + \int_b^1 \bar{f}^R(s) ds \right\} \quad \text{for all } t \in [0, 1]. \quad (2.14)$$

From (2.10) and (2.14) we deduce the final estimate from above

$$|N(u)|_\infty \leq (I - \alpha[1])^{-1} \alpha \left(\int_a^b \underline{f}^R(s) ds \right) + \max \left\{ \int_0^a \bar{f}^R(s) ds, \int_a^b \underline{f}^R(s) ds + \int_b^1 \bar{f}^R(s) ds \right\},$$

which together with condition (2.6) shows that $N(D) \subset D$. Thus the proof of (1⁰) is finished.

(2⁰) Let

$$U := \left\{ u \in D : \min_{t \in [a, b]} u_i(t) > r_i \text{ for all } i \text{ with } r_i > 0 \right\}.$$

The set U is open in D . We prove that $N(\bar{U}) \subseteq U$, which implies $i(N, U, D) = 1$, whence the conclusion. To this aim we need to estimate $N(u)$ from below. Thus, for $u \in \bar{U}$,

$$N^1(u) \geq (I - \alpha[1])^{-1} \alpha \left(\int_a^b \underline{f}^{r, R}(s) ds \right),$$

and for $t \in [a, b]$, $N^2(u)(t) \geq 0$. Hence, from (2.7) we have that $N(\bar{U}) \subseteq U$ as desired.

(3⁰) Now consider the set

$$V := \{u \in D : |u_i|_\infty < \tau_i \text{ for } i \in \Lambda\}.$$

Clearly, the set V is open in D and the sets $\bar{U}$ and V are disjoint. Since $\tau \leq R$, we have $\underline{f}^\tau(s) \leq \underline{f}^R(s)$ for $s \in [a, b]$, and $\bar{f}^\tau(s) \leq \bar{f}^R(s)$ for $s \in [0, 1] \setminus (a, b)$. Then from (2.9) and (2.6), we see that

$$(I - \alpha[1])^{-1} \alpha \left(\int_a^b \underline{f}^\tau(s) ds \right) + \max \left\{ \int_0^a \bar{f}^\tau(s) ds, \int_a^b \underline{f}^\tau(s) ds + \int_b^1 \bar{f}^\tau(s) ds \right\} \leq \tau.$$

Then, as at case (1⁰), we may infer that $N(\bar{V}) \subset V$ and so $i(N, V, D) = 1$. Hence a second fixed point v exists in V .

In the end, from

$$i(N, D \setminus (\bar{U} \cup \bar{V}), D) = i(N, D, D) - i(N, U, D) - i(N, V, D) = 1 - 1 - 1 = -1 \neq 0,$$

we find a third fixed point w in $D \setminus (\bar{U} \cup \bar{V})$. The localization properties of the three solutions come from the definitions of the sets U , V and $D \setminus (\bar{U} \cup \bar{V})$. $\square$

Remark 2.1. Under the assumptions of Theorem 2.1 (3⁰), the solutions u and w are different from zero. If in addition $f(\cdot, 0) \neq 0$, then the solution v is also different from zero.

Remark 2.2. For only one autonomous equation ($n = 1$), conditions (2.6), (2.7) and (2.9) read as:

$$\frac{\alpha(t-a)}{1-\alpha(1)} \max_{[0, R]} f + \max \left\{ a \max_{[-R, R]} |f|, (b-a) \max_{[0, R]} f + (1-b) \max_{[-R, R]} |f| \right\} \leq R,$$

$$\frac{\alpha(t-a)}{1-\alpha(1)} \min_{[r, R]} f > r,$$

$$\frac{\alpha(t-a)}{1-\alpha(1)} \max_{[0, \tau]} f + \max \left\{ a \max_{[-\tau, \tau]} |f|, (b-a) \max_{[0, \tau]} f + (1-b) \max_{[-\tau, \tau]} |f| \right\} < \tau.$$

In particular, if

$$\max_{[-R, R]} |f| = \max_{[0, R]} f = f(R), \max_{[-\tau, \tau]} |f| = \max_{[0, \tau]} f = f(\tau) \text{ and } \min_{[r, R]} f = f(r), \quad (2.15)$$

the above inequalities reduce to

$$\frac{f(R)}{R} \leq \frac{1-\alpha(1)}{\alpha(t-a) + (1-\alpha(1)) \max\{a, 1-a\}} =: A, \quad (2.16)$$

$$\frac{f(r)}{r} > \frac{1-\alpha(1)}{\alpha(t-a)} =: B, \quad (2.17)$$

$$\frac{f(\tau)}{\tau} < \frac{1-\alpha(1)}{\alpha(t-a) + (1-\alpha(1)) \max\{a, 1-a\}} = A \quad (2.18)$$

showing the oscillation of the function $f(u)/u$.

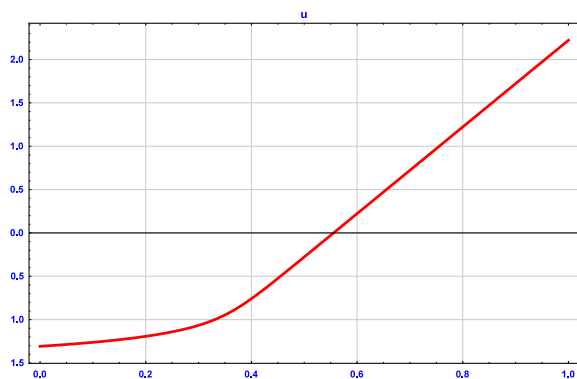


Fig. 1. Solution to the problem in Example 2.1.

Notice the dependence of the numbers A and B on the support $[a, c]$ of the mapping α through the values $\alpha(1)$ and $\alpha(t - a)$. Also note a sufficient condition for (2.15) to hold, namely that f is nonnegative and increasing on $[0, R]$ and $f - \lambda$ is odd or even on $[-R, R]$ for some $\lambda \geq 0$.

Example 2.1. The problem

$$\begin{cases} u' = \frac{5}{1 + e^{-8(u+1)}}, & t \in [0, 1] \\ u\left(\frac{2}{3}\right) = \frac{1}{4}u(1) \\ u \geq 0 \quad \text{on } \left[\frac{2}{3}, 1\right] \end{cases}$$

has a solution (see Fig. 1).

It is easy to show that the function $f(t, u) = f(u) = 5/(1 + e^{-8(u+1)})$ is positive and increasing on $\mathbb{R}$. Hence (2.15) holds whatever would be r and R . Such numbers r, R with $0 < r < R$ and satisfying (2.16) and (2.17) exist since

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = +\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = 0.$$

The result then follows from Theorem 2.1 (2⁰). We underline the fact that the solution is not positive on the whole interval $[0, 1]$, but, as predicted by our theory, it is positive on the subinterval $[2/3, 1]$, the support of the nonlocal condition.

Example 2.2. Consider the problem

$$\begin{cases} u' = f(u), & t \in [0, 1] \\ u\left(\frac{1}{2}\right) = \frac{2}{5}u(1) \\ u \geq 0 \quad \text{on } \left[\frac{1}{2}, 1\right]. \end{cases} \quad (2.19)$$

For

$$f(u) = \frac{|u| + \lambda_1}{|u| + \lambda_2} \quad \text{and} \quad 0 < \lambda_1 < \lambda_2,$$

the problem has at least one solution u with $u > 0$ on $[1/2, 1]$.

Indeed, the function $f(t, u) = f(u)$ is even on $\mathbb{R}$ and nonnegative, increasing on $\mathbb{R}_+$ since $0 < \lambda_1 < \lambda_2$, hence (2.15) holds whatever will be r and R . We have $\alpha(u) = (2/5)u(1)$ and then $\alpha(1) = 2/5$ and $\alpha(t - 1/2) = (2/5)(1 - 1/2) = 1/5$. Also $A = 6/5$ and $B = 3$. Looking for a number $r > 0$ with $f(r)/r > B$, we are led to the inequality $3r^2 + (3\lambda_2 - 1)r - \lambda_1 < 0$, which has positive solutions for $\lambda_1 > 0$. Finally, a number $R > r$ can be found since $f(x)/x \rightarrow 0$ as $x \rightarrow +\infty$. The conclusion follows from Theorem 2.1 (2⁰).

Example 2.3. For

$$f(u) = \frac{5u^2 + \lambda_1}{u^2 + \lambda_2}$$

$\lambda_2 \in (0, 25/36)$ and sufficiently small $\lambda_1 \in (0, \lambda_2)$, problem (2.19) has at least three nonzero solutions. As in the previous example, the function f is even on $\mathbb{R}$ and nonnegative, increasing on $\mathbb{R}_+$, hence (2.15) holds whatever would be τ, r and

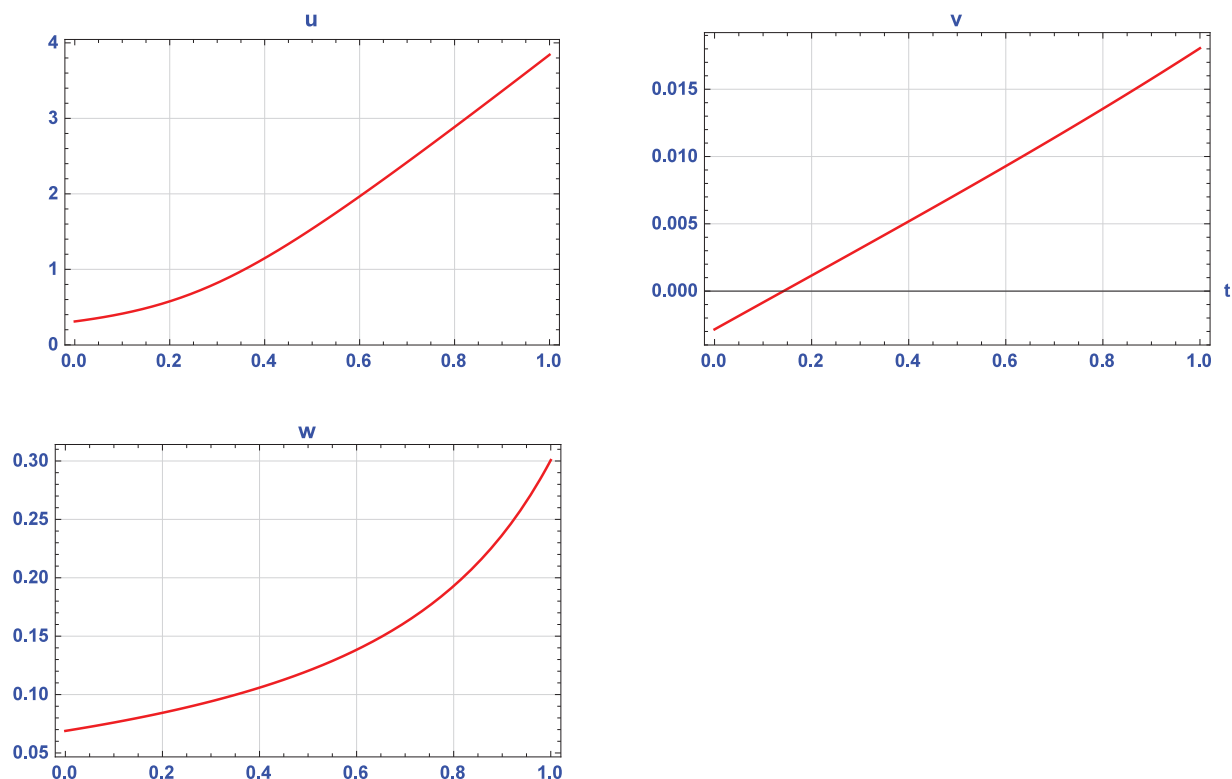


Fig. 2. Three solutions to the problem in Example 2.3, where $\lambda_1 = 0.01$ and $\lambda_2 = 0.5$.

R. Also $A = 6/5$ and $B = 3$. First we prove that there are two numbers r and τ with $0 < \tau < r$ such that $f(\tau)/\tau < A$ and $f(r)/r > B$, that is

$$\begin{aligned} 6\tau^3 - 25\tau^2 + 6\lambda_2\tau - 5\lambda_1 &> 0, \\ 3r^3 - 5r^2 + 3\lambda_2r - \lambda_1 &< 0. \end{aligned} \quad (2.20)$$

Since $\lambda_2 < 1/36$, there is $r > 0$ with $3r^3 - 5r^2 + 3\lambda_2r < 0$, which makes true the second inequality in (2.20) for every $\lambda_1 > 0$. Next we choose $\tau \in (0, r)$ such that $6\tau^2 - 25\tau + 6\lambda_2 > 0$, and then $\lambda_1 \in (0, \lambda_2)$ such that $5\lambda_1 < (6\tau^2 - 25\tau + 6\lambda_2)\tau$. Thus conditions (2.20) are fulfilled. Finally, since $f(x)/x \rightarrow 0$ as $x \rightarrow +\infty$, there exists $R > r$ such that $f(R)/R \leq A$. From Theorem 2.1 (3^o), the problem has three solutions u, v, w with

$$\min_{[1/2, 1]} u \geq r, \quad |u|_\infty \leq R; \quad |v|_\infty \leq \tau; \quad |w|_\infty > \tau, \quad \min_{[1/2, 1]} w < r. \quad (2.21)$$

These solutions are nonzero since $f(0) \neq 0$. To test our theoretical result, we consider $\lambda_1 = 0.01$ and $\lambda_2 = 0.5$. Then the conditions on the three radii R, r and τ are fulfilled by the values $R = 4.2$, $r = 1$, $\tau = 0.1$, and using the Shooting program of Mathematica, we obtain three numerical solutions u, v, w satisfying (2.21), as shown in Fig. 2.

Example 2.4. For

$$f(u) = \lambda + \frac{21}{10}u - \frac{19}{20}u \sin\left(\frac{3}{5} \ln u^2\right) \text{ for } u \neq 0 \text{ and } f(0) = \lambda,$$

and any $\lambda \in \mathbb{R}_+$, problem (2.19) has a sequence of solutions u_k with $|u_k|_\infty \rightarrow +\infty$. If $\lambda = 0$, then in addition there exists a sequence of nonzero solutions v_k with $v_k \rightarrow 0$.

The function f is nonnegative and increasing on $\mathbb{R}_+$, and $f - \lambda$ is odd on $\mathbb{R}$. Hence (2.15) holds. As in the previous two examples, one have $A = 6/5$ and $B = 3$. Since

$$\liminf_{x \rightarrow +\infty} \frac{f(x)}{x} = \frac{23}{20} < A \quad \text{and} \quad \limsup_{x \rightarrow +\infty} \frac{f(x)}{x} = \frac{61}{20} > B,$$

there exist sequences r_k and R_k tending to $+\infty$ and such that

$$r_k < R_k < r_{k+1} < R_{k+1}, \quad \frac{f(R_k)}{R_k} < A \quad \text{and} \quad \frac{f(r_k)}{r_k} > B$$

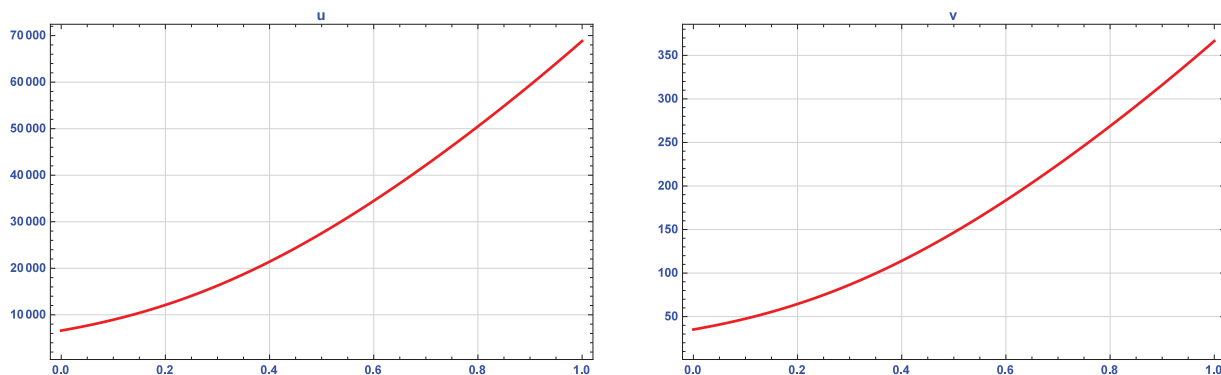


Fig. 3. Two functions from the sequence of solutions to the problem in Example 2.4, where $\lambda = 0.1$.

for all k . Hence, from Theorem 2.1 (2^0), for each k , the problem has a solution u_k with $|u_k|_\infty \leq R_k$ and $\min_{[1/2, 1]} u_k \geq r_k$. From

$$|u_k|_\infty \geq \min_{[1/2, 1]} u_k \geq r_k > R_{k-1}$$

and $R_k \rightarrow +\infty$, we find that $|u_k|_\infty \rightarrow +\infty$. For $\lambda = 0.1$, two such solutions are numerically obtained with the Shooting program of Mathematica, and are represented in Fig. 3.

If $\lambda = 0$, then we also have

$$\liminf_{x \rightarrow 0} \frac{f(x)}{x} = \frac{23}{20} < A \quad \text{and} \quad \limsup_{x \rightarrow 0} \frac{f(x)}{x} = \frac{61}{20} > B.$$

Consequently, there exist sequences r_k and R_k tending to 0 and such that

$$r_{k+1} < R_{k+1} < r_k < R_k, \quad \frac{f(R_k)}{R_k} < A \quad \text{and} \quad \frac{f(r_k)}{r_k} > B$$

for all k . Then for each k , there is a solution v_k satisfying $|v_k|_\infty \leq R_k$ and $\min_{[1/2, 1]} v_k \geq r_k$. Clearly from $|v_k|_\infty \leq R_k$ and $R_k \rightarrow 0$, we have that $v_k \rightarrow 0$.

Example 2.5. Consider the following problem related to a bidimensional system

$$\begin{cases} u' = f(u), & t \in [0, 1] \\ u\left(\frac{1}{2}\right) = \frac{2}{5}u(1) \\ u \geq 0 \quad \text{on } \left[\frac{1}{2}, 1\right]. \end{cases} \quad (2.22)$$

If

$$f_1(u) = \frac{u_1^2 + u_2^2}{u_1^2 + u_2^2 + \lambda_1}, \quad f_2(u) = \frac{|u_1| + |u_2|}{|u_1| + |u_2| + \lambda_2},$$

$\lambda_1 \in (0, 1/18)$ and $\lambda_2 \in (0, 1/3]$, the problem has at least one solution satisfying $u > 0$ on $[1/2, 1]$. In this case we have

$$\alpha[1] = \begin{bmatrix} \alpha_1(1) & 0 \\ 0 & \alpha_2(1) \end{bmatrix}, \quad (I - \alpha[1])^{-1} = \begin{bmatrix} \frac{1}{1 - \alpha_1(1)} & 0 \\ 0 & \frac{1}{1 - \alpha_2(1)} \end{bmatrix},$$

$\alpha_1 = \alpha_2$ and $\alpha_1(1) = \alpha_2(1) = 2/5$. Also, taking $R_1 = R_2 =: R_0$ and $r_1 = r_2 =: r_0$, the vector conditions in Theorem 2.1 (2^0) are

$$(I - \alpha[1])^{-1} \alpha \left(\left(t - \frac{1}{2} \right) f(R) \right) + \frac{1}{2} f(R) \leq R,$$

$$(I - \alpha[1])^{-1} \alpha \left(\left(t - \frac{1}{2} \right) f(r) \right) > r.$$

Explicitly, the first condition gives

$$\gamma f_1(R_0, R_0) \leq R_0, \quad \gamma f_2(R_0, R_0) \leq R_0, \quad (2.23)$$

where $\gamma = \alpha_1(t - 1/2)/(1 - \alpha_1(1)) + 1/2 = 1/3 + 1/2 = 5/6$. Since for both $i = 1, 2$, one has $f_i(x, x)/x \rightarrow 0$ as $x \rightarrow +\infty$, such a number R_0 exists sufficiently large. The second vector condition gives

$$\eta f_1(r_0, r_0) > r_0, \quad \eta f_2(r_0, r_0) > r_0, \quad (2.24)$$

where $\eta = \alpha_1(t - 1/2)/(1 - \alpha_1(1)) = 1/3$. These inequalities yield to the system

$$6r_0^2 - 2r_0 + 3\lambda_1 < 0, \quad 6r_0 < 2 - 3\lambda_2$$

which has a positive solution if $\lambda_1 \in (0, 1/18)$ and $\lambda_2 \in (0, 1/3]$. Thus Theorem 2.1 (2^0) applies and gives the conclusion.

Example 2.6. Consider problem (2.22) where

$$f_1(u) = \lambda_1 \frac{u_1^2(|u_2| + 1)}{(u_1^2 + 1)(|u_2| + 2)}, \quad f_2(u) = \lambda_2 \frac{|u_1| + |u_2|}{|u_1| + |u_2| + 1}.$$

For each number $r_0 > 0$ and any large enough $\lambda_1, \lambda_2 > 0$, the problem has at least three nonzero solutions. Indeed, with the notations from the previous example, the conditions (2.24) are satisfied for sufficiently large λ_1 and λ_2 . Next conditions (2.23) allow us to find $R_0 > r_0$ large enough since for both $i = 1, 2$, one has $f_i(x, x)/x \rightarrow 0$ as $x \rightarrow +\infty$. Now we put $\Lambda = \{1\}$, so $\tau_2 = R_2(= R_0)$, and we look for $\tau_1 \in (0, r_0)$ such that the additional condition in Theorem 2.1 (3^0) is satisfied. Thus, we need that

$$\gamma f_1(\tau_1, R_0) < \tau_1.$$

Such a number τ_1 exists since $f_1(x, R_0)/x \rightarrow 0$ as $x \rightarrow 0$. Now the conclusion follows from Theorem 2.1 (3^0).

In the above examples we have considered autonomous equations. We emphasize the more general applicability of Theorem 2.1 to nonautonomous equations, as the following example shows.

Example 2.7. In problem (2.19), consider the t -depending function

$$f(t, u) = \frac{5u^2 + \lambda_1(t + 1)}{u^2 + \lambda_2}.$$

As in Example 2.3, if $\lambda_2 \in (0, 25/36)$ and $\lambda_1 \in (0, \lambda_2)$ is sufficiently small, there exists at least three nonzero solutions. Indeed, by direct computation involving maximization and minimization of $f(t, u)$ also with respect to t , one can see that conditions (2.9) and (2.7) of Theorem 2.1 are satisfied providing that

$$\begin{aligned} 6\tau^3 - 25\tau^2 + 6\lambda_2\tau - 10\lambda_1 &> 0, \\ 3r^3 - 5r^2 + 3\lambda_2r - \lambda_1 &< 0. \end{aligned} \quad (2.25)$$

Similarly to Example 2.3, these inequalities make possible the choice of r and τ . In addition, condition (2.6) is fulfilled by a sufficiently large R .

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